

Junior Problems

J739. Solve in real numbers the system of equations

$$\begin{aligned}x^6 + y^2 &= z^8 + 1, \\y^6 + z^2 &= x^8 + 1, \\z^6 + x^2 &= y^8 + 1.\end{aligned}$$

Proposed by Alessandro Ventullo, Milan, Italy

J740. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 \geq 3$. Prove that

$$\sqrt{\frac{2ab}{a^2 + b^2}} + \sqrt{\frac{2bc}{b^2 + c^2}} + \sqrt{\frac{2ca}{c^2 + a^2}} \leq a + b + c.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J741. Solve in nonzero complex numbers the system of equations

$$\begin{aligned}x + y + \frac{xy}{z} &= \frac{6}{z^2}, \\y + z - \frac{2yz}{x} &= \frac{3}{x^2}, \\z + x + \frac{4zx}{y} &= -\frac{9}{y^2}.\end{aligned}$$

Proposed by Adrian Andreescu, Dallas, USA

J742. Let a and b be two odd natural numbers. If (a, b) is the greatest common divisor of the numbers a and b , show that

$$\min \{(a, b + 2), (a + 2, b)\} \leq -1 + \sqrt{a + b + 3}.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

J743. Let $ABCD$ be a trirectangular tetrahedron with apex D , that is, the edges AD , BD , and CD are pairwise perpendicular. Let h be the length of the altitude from D to the face ABC . Prove that

$$\frac{1}{(AD^2[BCD])^2} + \frac{1}{(BD^2[CAD])^2} + \frac{1}{(CD^2[ABD])^2} = \frac{1}{(h^2[ABC])^2},$$

where $[XYZ]$ denotes the area of triangle XYZ .

Proposed by Titu Andreescu, Dallas, USA

J744. Let Γ_1, Γ_2 be two circles with centers O_1, O_2 , tangent at E . On the line perpendicular to O_1O_2 that passes through E let B and D be points above and below point E , respectively. The tangents from B and D to Γ_1 are BN and DP respectively, and the tangents from B and D to Γ_2 are BM and DQ respectively. If H_1, H_2 are the orthocenters of triangles MNQ and NQP , respectively, show that $\angle PMH_1 = \angle PH_2H_1$.

Proposed by Mihaela Berindeanu, Bucharest, Romania

Senior Problems

S739. Solve in real numbers the equation

$$13\{x\} = x + 2[x] + 3(x),$$

where $[x]$ is the integer part of x , $\{x\}$ is the fractional part of x and (x) is the closest integer to x (half-integers are always rounded to even numbers).

Proposed by Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania

S740. If n and k are positive integers such that $n^2 + k^2(8k + 3)$ is prime, prove that $n + 3k + 1$ is not a perfect cube.

Proposed by Alessandro Ventullo, Milan, Italy

S741. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq 1 + \frac{\frac{a^4}{b+c} + \frac{b^4}{c+a} + \frac{c^4}{a+b}}{a^3 + b^3 + c^3}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S742. Let ABC be a non-right triangle with $AB \neq AC$ and let H be its orthocenter, O be its circumcenter. Let M_a be the midpoint of BC and let D be the midpoint of AM_a . Let E, F be the feet of the perpendiculars drawn from A, M_a respectively to the line OH . Knowing that $HDFM_a$ is a cyclic quadrilateral, prove that $EDOM_a$ is a cyclic quadrilateral too.

Proposed by Todor Zaharinov, Sofia, Bulgaria

S743. Let $a \geq 1 \geq b \geq c \geq 0$ such that $a + b + c = 3$. Prove that

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} + \frac{abc}{ab+bc+ca} \geq \frac{11}{6}.$$

Proposed by Marius Stănean, Zalău, Romania

S744. Prove that $\frac{4}{3}$ is the largest positive value of k such that

$$\frac{1}{(a+b)^2 + k} + \frac{1}{(b+c)^2 + k} + \frac{1}{(c+a)^2 + k} \geq \frac{3}{4+k}$$

for any nonnegative real numbers a, b, c with at most one of them larger than 1 and $ab + bc + ca = 3$.

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania

Undergraduate Problems

U739. Evaluate

$$\int \frac{\tan 5x}{\cos^2 x} dx.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U740. Let $A, B \in \mathcal{M}_2(\mathbb{R})$ be two matrices with the properties $\det(A - B) = 0$ and $A^2 + BA + B^2 = AB$. Determine $A^{2026} + B^{2026}$.

Proposed by Mihaela Berindeanu, Bucharest, Romania

U741. Let $\{a_n\}_{n \geq 1}$ be a sequence such that $|a_{n+1}| \leq c|a_n|$, $c > 0$. If

$$\sum_{n=1}^{\infty} |a_n(\cos(3nx_0) + 7\cos(nx_0) - 2\cos(2nx_0) - 2)| < +\infty.$$

for some $x_0 \in [0, 2\pi)$, prove that $\sum_{n=1}^{\infty} |a_n| < +\infty$.

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

U742. Let $f \in C^3([0, 1])$ satisfy $f'(0) = f'(1) = 0$ and $f'''(x) > 0$ for all $x \in (0, 1)$. Prove that there exists exactly one point $c \in (0, 1)$ such that $f(c) - f(0) = cf'(c)$ and $f''(c) > 0$.

Proposed by Alessandro Ventullo, Milan, Italy

U743. Let $n \geq 3$ and $1 \leq k \leq n - 1$. If $a_1, a_2, \dots, a_n$ are positive real numbers such that

$$a_1 \geq a_2 \geq \dots \geq a_k \geq 1 \geq a_{k+1} \geq \dots \geq a_n$$

and $a_1 a_2 \dots a_n = 1$, prove that

$$\frac{1}{ka_1 + 1} + \frac{1}{ka_2 + 1} + \dots + \frac{1}{ka_n + 1} \geq \frac{n}{k + 1}.$$

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania

U744. Let $x > 0$. Evaluate

$$\frac{1 \cdot \binom{2n}{0}}{x} - \frac{2^2 \cdot \binom{2n}{1}}{x+1} + \frac{3^2 \cdot \binom{2n}{2}}{x+2} - \frac{4^2 \cdot \binom{2n}{3}}{x+3} + \dots + \frac{(2n+1)^2 \cdot \binom{2n}{2n}}{x+2n}.$$

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

Olympiad Problems

- O739.** Let $(a_n)_{n \in \mathbb{N}}$ be a sequence defined recursively as follows: $a_1 = 1$ and $a_{n+1} = a_n + \left\lfloor \sqrt{a_n} + \frac{1}{2} \right\rfloor$. Find $n \in \mathbb{N}$ for which both solutions of the equation $x^2 + 2x - a_n = 0$ are integers.

Proposed by Mihaela Berindeanu, Bucharest, Romania

- O740.** Find all pairs of positive integers (m, n) satisfying

$$3^m - 1 = 2^n \cdot m.$$

Proposed by Alessandro Ventullo, Milan, Italy

- O741.** Let $a_1, \dots, a_n$ be pairwise relatively prime nonzero integers, and let c be an integer. For every $1 \leq i \leq n$, let b_i be the product of all the a_j with $j \in \{1, \dots, n\} \setminus \{i\}$. Prove that

$$f(x_1, \dots, x_n) := \left\lfloor \frac{x_1}{a_1} \right\rfloor + \dots + \left\lfloor \frac{x_n}{a_n} \right\rfloor$$

is constant for all the n -tuples $(x_1, \dots, x_n)$ of integers satisfying the equation

$$b_1 x_1 + \dots + b_n x_n = c.$$

Proposed by Titu Andreescu, USA and Marian Tetiva, Romania

- O742.** Let ABC be a non-right triangle with $AB \neq AC$ and let H be its orthocenter, I be its incenter and M be the midpoint of BC . Let D be the reflection of I in the line BC and let E be the reflection of A in H . Knowing that AI is parallel to DE , prove that the points H, I, M are collinear.

Proposed by Todor Zaharinov, Sofia, Bulgaria

- O743.** Let $x_1, x_2, \dots, x_n$ be positive real numbers such that $x_1 + x_2 + \dots + x_n = n$, $n \geq 3$. Prove that

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} - n \geq \frac{27(n-1)^2}{(2n^2 + n - 1)^2} (x_1^3 + x_2^3 + \dots + x_n^3 - n).$$

Proposed by Marius Stănean, Zalău, Romania

- O744.** Let a, b, c, d be positive real numbers such that $a \geq b \geq 1 \geq c \geq d$ and $abcd = 1$. Prove that

$$\frac{1}{a^2 + a + 1} + \frac{1}{b^2 + b + 1} + \frac{1}{c^2 + c + 1} + \frac{1}{d^2 + d + 1} \geq \frac{4}{3}.$$

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania