

Deriving Thébault's Configuration Properties and Sangaku Equal Circles through Geometry of Conic Sections

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Abstract

Circle configurations in triangles are examined through the lens of conics. Conic properties are applied to derive concurrency and tangency relations that recover and extend Thebault's configuration. Identities for inradii and exradii, together with sangaku-style equal circle results, are obtained as corollaries.

Theorem 1. *Let $\triangle A_1BC$ be a triangle, and let X be a point on the segment $[BC]$. Let c_1 and c_2 be the two circles tangent to the segment $[BC]$, the line A_1X , and the arc BC of the circumcircle of $\triangle A_1BC$ that does not contain A_1 . Let A_2 be the reflection of A_1 across the perpendicular bisector of the segment BC . Then, the following properties hold:*

1. *There exists a common tangent to the circles c_1 and c_2 that passes through A_2 .*
2. *The intersection point of this tangent with the line A_1X lies on an ellipse whose foci are A_1 and A_2 , and the line joining the centers of the circles c_1 and c_2 is tangent to this ellipse at that point.*

Proof.

Remark 1. The first part corresponds to the parallel tangent lemma in reference [1]. We will use the same proof technique to establish the second part as well.

Let D be the internal homothetic center of the circles c_1 and c_2 .

From point D , draw the other common tangent (distinct from A_1X), and let its intersection with the arc BC of the circumcircle that contains A_1 be point A' . We aim to prove that $A_2 = A'$.

Let N be the midpoint of the arc BC that contains A_1 . Define $Y = A'D \cap BC$. Let T_1 and T_2 be the points of tangency of line A_1X with circles c_1 and c_2 , respectively. Similarly, let K_1 and K_2 be the points of tangency of line $A'Y$ with circles c_1 and c_2 , respectively.

The power of point N with respect to both circles c_1 and c_2 is equal and given by:

$$NB^2 = NC^2 = t^2$$

Applying Casey's Theorem to the quadruple N, A_1, c_1, A' , we obtain:

$$NA_1 \cdot A'K_2 + NA' \cdot A_1T_1 = t \cdot A_1A'$$

Applying Casey's Theorem to N, A_1, c_2, A' , we obtain:

$$NA_1 \cdot A'K_1 + NA' \cdot A_1T_2 = t \cdot A_1A'$$

Subtracting the two equations yields:

$$NA_1 \cdot K_1K_2 = NA' \cdot T_1T_2$$

We have $K_1K_2 = T_1T_2$, which implies $NA_1 = NA'$, and hence $A' = A_2$. Substituting into the first equation, we obtain:

$$NA_1(A_2K_2 + A_1T_1) = t \cdot A_1A_2$$

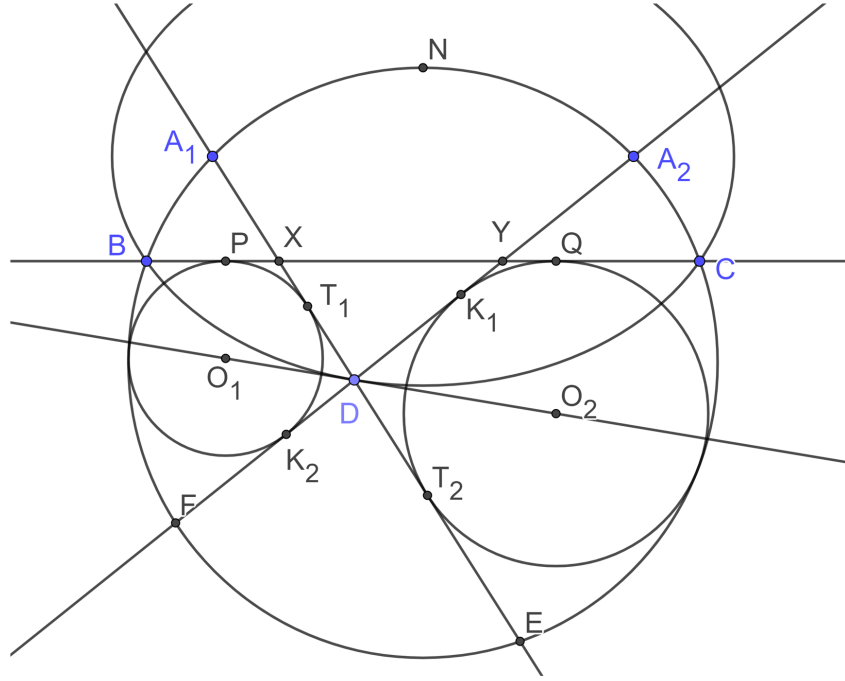


Figure 1: Geometric configuration for Theorem 1.

In addition, $A_2K_2 + A_1T_1 = A_1D + A_2D$, since $DK_2 = DT_1$ and $DK_1 = DT_2$. Therefore, the quantity is constant and equals:

$$A_1D + A_2D = \frac{t \cdot A_1A_2}{NA_1} = \frac{NB \cdot A_1A_2}{NA_1}$$

Since this quantity is constant, we compute it when the two circles c_1 and c_2 degenerate to point B (or C), yielding:

$$A_1D + A_2D = \frac{NB \cdot A_1A_2}{NA_1} = A_1B + A_2B = A_1C$$

We conclude that the locus of point D is an ellipse with foci A_1 and A_2 , passing through points B and C . Moreover, the line O_1O_2 is tangent to this ellipse at D , as it satisfies the optical property of tangents:

$$\angle A_1DO_1 = \angle O_2DA_2$$

□

Theorem 2. Let $\triangle ABC$ be a triangle. Let D be a point on the segment $[BC]$. Let X, Y, Z be the touch points of the incircle of $\triangle ABC$ with sides BC, CA, AB , respectively. Let G, F, R be the touch points of the incircle of $\triangle ADC$ with sides DC, CA, AD , respectively. Let E, H, L be the touch points of the incircle of $\triangle ABD$ with sides BD, DA, AB , respectively. Let I, J, K be the incenters of triangles $\triangle ABC$, $\triangle ABD$, and $\triangle ADC$, respectively.

Then the line JK is tangent to the hyperbola with foci X and A and passes through B and C .

Proof.

Remark 2. Problems of this type appear in Akopyan's *Geometry in Figures* [2], specifically 4.5.35 [3] and 4.5.36 [4].

Define ω_1 and ω_2 as the incircles of $\triangle ABD$ and $\triangle ADC$, respectively. Let Ω be the circle centered at A and passing through Y and Z .

Remark 3. The construction admits two distinct configurations depending on the the conic ω :

- If the conic satisfies the optical property with respect to the internal bisector of $\angle MAN$, then the point I lies on the internal angle bisector, and the corresponding projections E', F' reflect that geometry.
- If the conic satisfies the optical property with respect to the external angle bisector of $\angle MAN$, then I lies on the external bisector, and the projections E', F' shift accordingly.

Thus, the involution φ on line MN may arise from either configuration, and the fixed location of E' and F' depends on which angle property the conic admits.

Theorem 4. Let Ω and ω be the circumcircle and incircle of triangle $\triangle ABC$ respectively. Let X, Y, Z be the touch points of ω with sides BC, CA, AB . Let A' be the reflection of A across the perpendicular bisector of BC . Let $D \in BC$ and E be the second intersection point of AD with Ω other than A . Let ω_1 and ω_2 be the incircles of triangles $\triangle ABD$ and $\triangle ADC$ respectively. Let c_1 and c_2 be circles tangent to AE and BC and the arc BC of Ω not containing A , such that ω_1 and c_1 lie on the same side with respect to AD . Let I, J, K, O_1, O_2 be the centers of $\omega, \omega_1, \omega_2, c_1, c_2$ respectively. Then lines JK, O_1O_2 , and BC are concurrent.

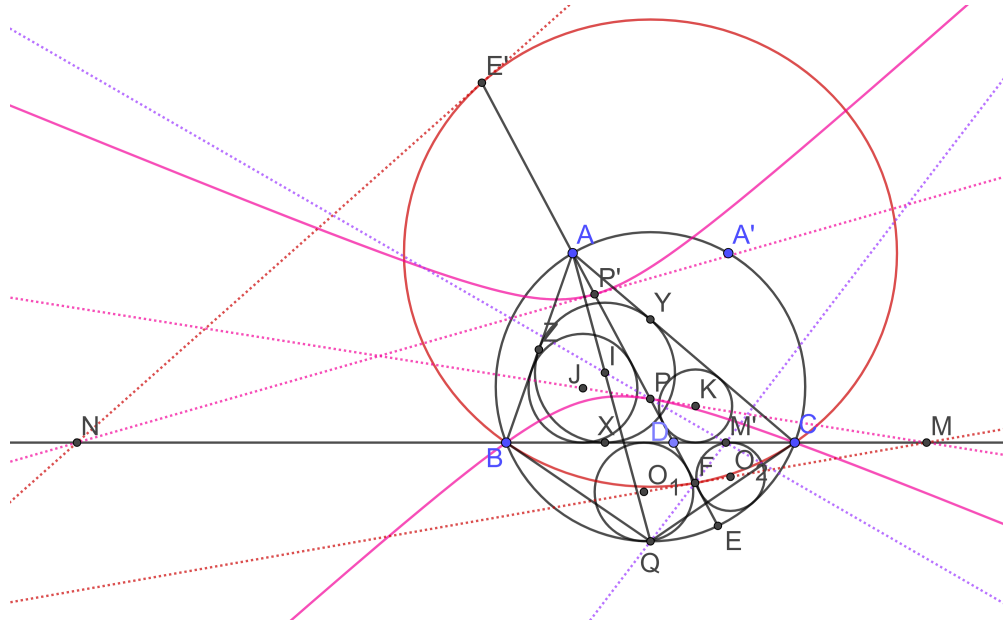


Figure 4: Configuration for Theorem 4.

Proof. Let $P = JK \cap AD$ and $F = O_1O_2 \cap AD$. By Theorem 1, F lies on an ellipse e with foci A and A' passing through B and C , and JK is tangent to e at F . By Theorem 2, P lies on the hyperbola h with foci A and X passing through B and C , and O_1O_2 is tangent to h at F . By Theorem 3, lines JK, O_1O_2 , and BC concur at a point M .

Remark 4. To ensure that both e and h satisfy the optical property with the internal angle bisector of $\angle BAC$ we note that I (incenter of $\triangle ABC$) is the polar of BC with respect to h , since BI and CI are tangents to h (reflection of X across BI and CI are Z and Y respectively). In addition, let Q the second intersection of AI with Ω then Q is the polar of BC with respect to e , since both QB and QC are tangents to e as they satisfy the optical property of the tangent ($\angle(QB, BA') = \angle(AB, QB)$ and similarly ($\angle(QC, CA) = \angle(A'C, QC)$). $\square$

Corollary 5. Assume the same notation as in Theorem 4 and Figure 4. Lines PI, QF and BC are concurrent.

Proof. PI is the polar of M with respect to h and QF is the polar of M with respect to e then the lines PI, QF and BC concurs at M' the harmonic conjugate of M with respect to the pair of points B and C . $\square$

Theorem 6. Assume the same notation as in Theorem 4 and Figure 4. Let r_1, r_2, ρ_1, ρ_2 be the radii of circles $\omega_1, \omega_2, c_1, c_2$ respectively. Then

$$\frac{1}{r_1} + \frac{1}{\rho_2} = \frac{1}{\rho_1} + \frac{1}{r_2}.$$

Remark 5. This identity appears as Theorem 3.2 in Rabinowitz [5], where its correctness was verified using a computer algebra system.

Proof. By Theorem 4, lines JK, O_1O_2 , and BC are concurrent. Let u, v be the reflections of BC across JK and O_1O_2 respectively. Define $U = u \cap AD$ and $V = v \cap AD$. Then ω_1 and ω_2 are the incircle and excircle of $\triangle MUD$, while c_1 and c_2 are the incircle and excircle of $\triangle MDV$. By Theorem 6.2 in Rabinowitz [6], we obtain

$$\frac{1}{r_1} + \frac{1}{\rho_2} = \frac{1}{\rho_1} + \frac{1}{r_2}.$$

This follows immediately from the known identity relating inradius r , exradius r_a , and altitude h_a in a triangle $\triangle ABC$ [6]:

$$\frac{1}{r} - \frac{1}{r_a} = \frac{2}{h_a}.$$

□

Corollary 7. As a corollary to Theorem 6, we obtain a sangaku-style construction of two equal incircles (cf. Fukagawa–Pedoe [7]). Draw a circle centered at A with radius $\frac{AB+AC}{2}$. Let F be the intersection of this circle with the perpendicular bisector of BC , chosen so that F and A lie on opposite sides of BC . Define D as the intersection of AP with PC . Then the two incircles of triangles $\triangle ABD$ and $\triangle ADC$ have equal radii.

Proof. Assume the same notation as in Theorems 4 and 6. By Theorem 1, and symmetry of e with respect to the perpendicular bisector of $[BC]$ the tangent O_1O_2 is parallel to BC then $\rho_1 = \rho_2$ which implies $r_1 = r_2$. □

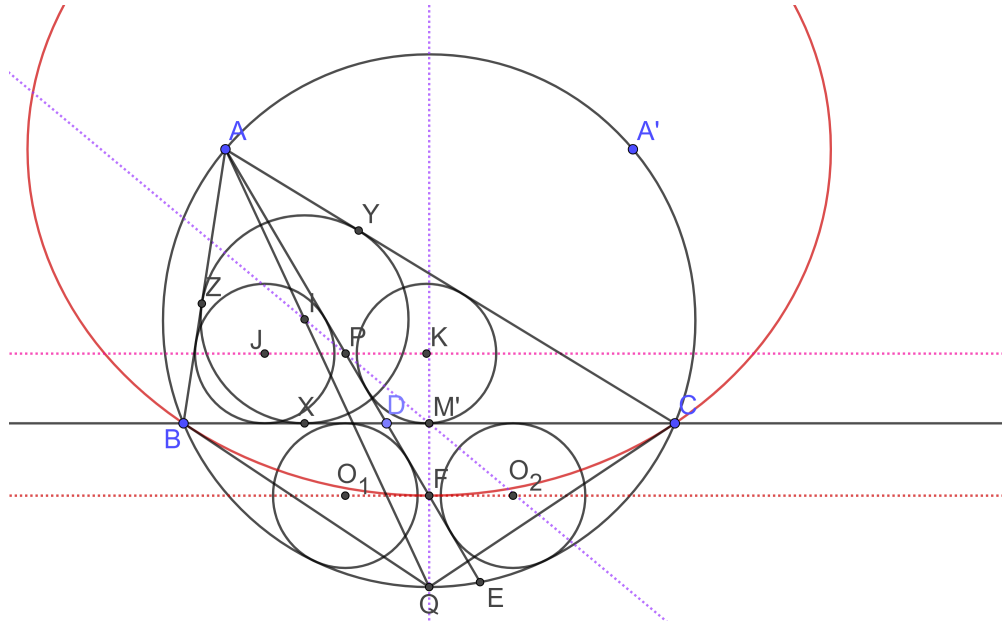


Figure 5: Configuration for Corollary 7.

Theorem 8. Let $\triangle ABC$ be a triangle with circumcircle Ω . Let D be a point on $[BC]$. Line AD meets Ω again at E . Let c_1 and c_2 be the circles tangents to AE and BC and the arc $\widehat{BC}$ of Ω that does not contain

Proof. This result was presented on Jean-Louis Ayme site which is not available right now. A proof is presented here based on Taiwan TST3 2014 P3 [8]. Consider the inversion τ with respect to the circle centered at M such that the circle Ω is invariant under τ . Then circle c_1 is mapped to c_2 by τ . Thus we have:

Let ω be the incircle of $\triangle EBC$ with center S . Let W be the touch point of ω with BC . By Sawayama and Thebault's theorem [9] lines UU' , VV' and O_1O_2 concurs at S . Since lines UU' and VV' are parallel to the internal and external angle bisectors of $\angle EDC$ respectively, then the two lines parallel to AD from U' and V' are tangents to ω . Let ∞_{AD} be the infinity point at line AD . By Desargues' Dual Involution Theorem on the degenerate quadrilateral $EBWC$, there is an involution swapping $(\infty_{AD}B, \infty_{AD}C)$, $(\infty_{AD}E, \infty_{AD}W)$ and $(\infty_{AD}U', \infty_{AD}V')$. Projecting this involution on line BC from ∞_{AD} we get that there is an involution on line BC swapping (B, C) , (D, W) and (U', V') . Hence the involution corresponds to the inversion τ

Then $TADW$ is cyclic. Let line TW meets Ω again at F . By Reim's theorem $EF \parallel BC$ then $\angle BTE = \angle WTC$ which is a well-known characteristic angular property of the touch point of the mixtilinear incircle with the circumcircle. $\square$

Proof. From the angular property of the mixtilinear incircle we have $\angle BEA = \angle TEC$ then $AT \parallel BC$. Thus, point M will be the infinity point on line BC and $O_1O_2 \parallel BC$. $\square$

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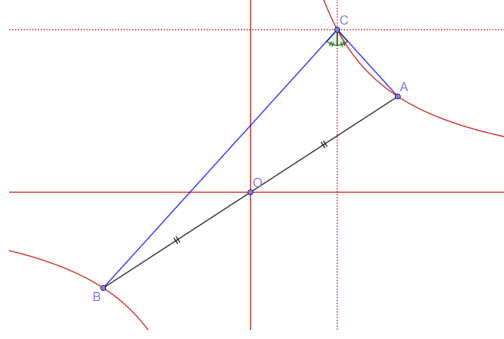


Figure 7: Configuration for Lemma 10.

Proof. We use the rectangular hyperbola $xy = 1$. The asymptotes of this rectangular hyperbola are the x -axis and y -axis respectively. Let $A = (a, \frac{1}{a})$ and $C = (c, \frac{1}{c})$. Then $B = (-a, -\frac{1}{a})$. The slope of AC

$$s_{AC} = \frac{\frac{1}{c} - \frac{1}{a}}{c - a} = -\frac{1}{ac}$$

The slope of BC

$$s_{BC} = \frac{\frac{1}{c} + \frac{1}{a}}{c + a} = \frac{1}{ac}$$

Thus, $s_{AC} = -s_{BC}$ which leads to the required result. $\square$

Theorem 11. Let $\triangle ABC$ be a triangle with Ω as the circumcircle. Let D be a point on side BC . Line AD meets Ω again at E . Let A' be the reflection of A across the perpendicular bisector of BC . Let c_1 and c_2 be the circles tangents to AE and BC and the arc $\widehat{BC}$ of Ω that does not contain A such that c_1 and B are on the same side with respect to line AD . Let U and V be the touch points of circles c_1 and c_2 with line AD respectively. Then

1.

$$AD \cdot AE = AU \cdot AV$$

2. The ratio

$$\frac{DU}{DV} = \frac{AB + AC - AA'}{AB + AC + AA'}$$

stays fixed as D moves on BC .

Proof of part 1. To prove part 1 we will use a rectangular hyperbola see Figure 8. Let S, S_e, S_b and S_c be incenter and excenters of triangle $\triangle EBC$. Points S, S_e, S_b and S_c forms an orthocentric system, thus, any circumconic that passes through these point is a rectangular hyperbola with center on the nine point circle which is the circle Ω . In particular, let's consider the rectangular hyperbola h centered at A . Define E' as the intersection of $S_b S_c$ with BC then we have $(S_b, S_c; E, E') = -1$ is a harmonic bundle. Also Define N the midpoint of arc BC that contains A , then E, S, N and S_e are collinear and $NS = NS_e = NB = NC = m$. In addition, if E'' is the intersection of ES and BC , then $m^2 = NE'' \cdot NE$, which implies $(S_e, S; E, E'') = -1$ is a harmonic bundle. Hence Line BC is the polar of E with respect to h . Define X_1 and X_2 the antipodes on h on line AD . Thus the tangents to h at X_1 and X_2 are parallel to BC . By Lemma 11 in its degenerate case X_1, X_1, X_2 we conclude that the asymptotes of the hyperbola are parallel to the angle bisectors of the angle formed by the tangent at X_1 and line AD , i.e., the asymptotes of h are parallel to the internal and external angle bisector of $\angle EDC$. By Sawayama and Thebault's theorem SU and SV are also parallel to the internal and external angle bisector of $\angle EDC$. Using Lemma 11 again on triangle SX_1X_2 we conclude that $(X_1, X_2; U, V) = -1$ is a harmonic bundle formed by the internal and external angle bisectors of $\angle X_1SX_2$. In addition we have $(X_1, X_2; D, E) = -1$ is also a harmonic bundle because BC is the polar of E with

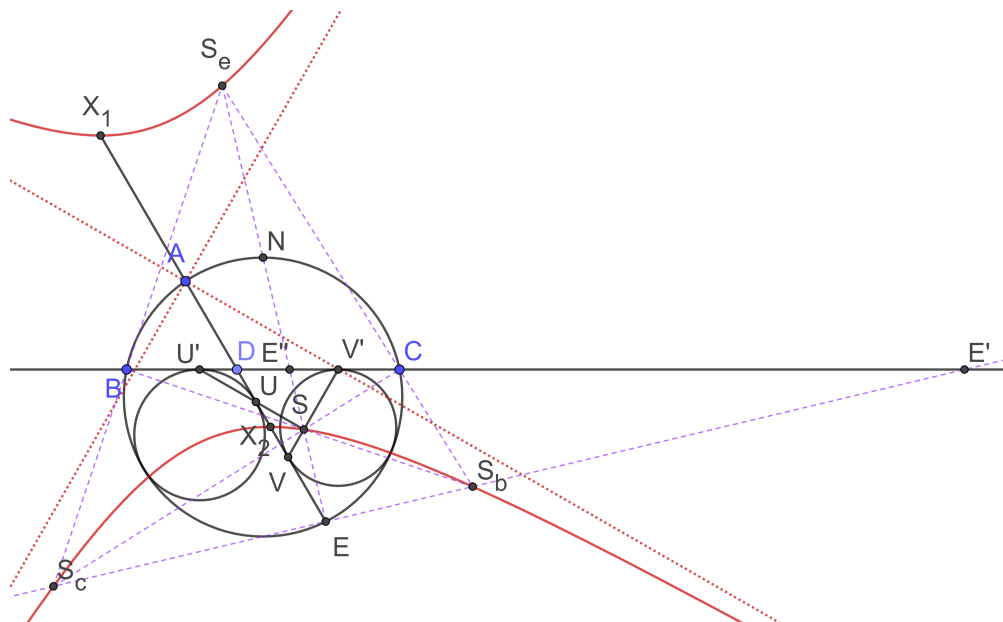


Figure 8: Configuration for Theorem 11 part 1.

respect to h . Then we conclude that the involution that swaps (D, E) and (U, V) on line AD corresponds to the inversion with respect to the circle centered at A with radius $AX_1 = AX_2$.

$$AX_1^2 = AX_2^2 = AD \cdot AE = AU \cdot AV$$

□

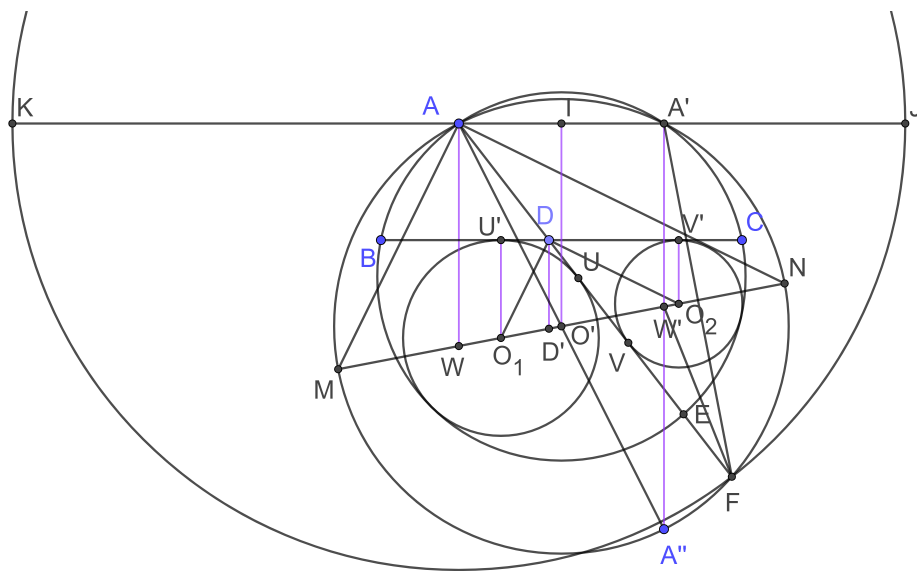


Figure 9: Configuration for Theorem 11 part 2.

Proof of Part 2. Let O_1 and O_2 the centers of circles c_1 and c_2 respectively. Let U' and V' the touch points of circles c_1 and c_2 respectively with line BC . Let F the reflection of A' across line O_1O_2 . By theorem 1 F lies on line AD such that $AF = AB + AC$. Let Γ be the circumcircle of triangle FAA' . Let N and M be the

points on Γ where the internal and external angle bisector of $\angle FAA'$ respectively. Then MN is a diameter of Γ and points O_1 and O_2 are on the diameter MN . The perpendicular lines to AA' meet the line MN at points W and W' respectively. In addition Let D' on MN such that $DD' \parallel AW$. Let O' be the midpoint of MN i.e., the circumcenter of Γ . Let I the projection of O' on AA' then I is the midpoint of AA' which implies that O' is the midpoint of WW' . In addition we have

$$\frac{DU}{DV} = \frac{DU'}{DV'} = \frac{D'O_1}{D'O_2} = \frac{WM}{WN} = \frac{W'N}{W'M}$$

Let K and J be the antipodes on the circle centered at A with radius $R = AB + AC$, thus, point F is on this circle. $\angle FMN = \frac{\angle FAA'}{2} = \angle FJK$. Similarly, $\angle FNM = \angle FJK$. Hence, the triangles $\triangle FJK$ and $\triangle FMN$ are directly similar. Thus, there exists a spiral similarity s centered at F such that $s(J) = N$, $s(K) = M$ and $s(A) = O'$. Let $W'' = s(A')$, then, $\angle W''A'F = \angle O'AF$. Hence lines $A'W''$ and AO' meet at the antipode of A with respect to Γ and $AA' \perp A'W''$. Since $A' \in KJ$ then $W'' \in MN$ which implies that $W'' = W'$. Hence we conclude that

$$\frac{DU}{DV} = \frac{W'N}{W'M} = \frac{A'J}{A'K} = \frac{AB + AC - AA'}{AB + AC + AA'}$$

Thus the ratio is fixed when D varies on line BC . □

Corollary 12. Let $\triangle ABC$ be a triangle, and let Ω denote its circumcircle. Consider a circle ω tangent to BC at X_2 and tangent to the arc BC of Ω not containing A at T . From A , draw the tangents to ω ; they intersect BC at D_1 and D_2 , with the points ordered as B, D_1, D_2, C . The lines AD_1 and AD_2 meet Ω again at E_1 and E_2 , respectively.

Define c_1 as the circle tangent to D_1E_1 , tangent to BD_1 at U_1 , and tangent internally to the arc BE_1 of Ω not containing other vertices. Similarly, let c_2 be the circle tangent to D_2E_2 , tangent to BD_2 at U_2 , and tangent internally to the arc CE_2 of Ω not containing other vertices.

Let c denote the incircle of $\triangle AD_1D_2$, with center N and radius r . Denote by O_1 and r_1 the center and radius of c_1 , and by O_2 and r_2 the center and radius of c_2 . Then

$$r^2 = r_1 \cdot r_2.$$

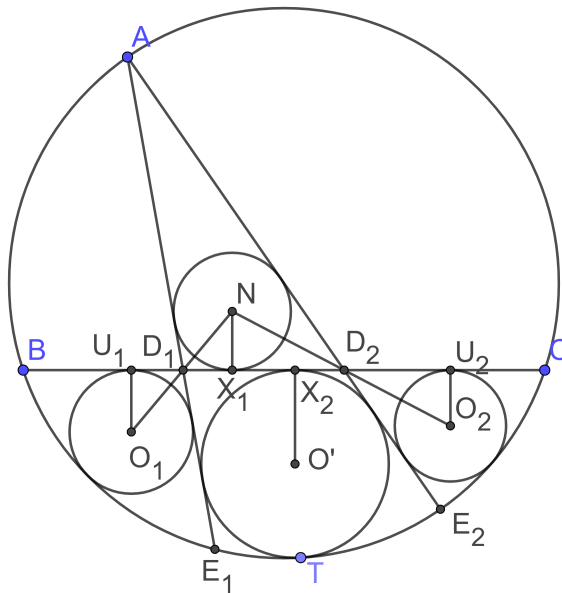


Figure 10: Configuration for Corollary 12.

Proof. By Theorem 11 we have

$$\frac{D_1U_1}{D_1X_2} = \frac{D_2X_2}{D_2U_2}$$

However, since c and ω are the incircle and A-excircle of triangle ΔAD_1D_2 respectively. Then $D_1X_1 = D_2X_2$ and $D_1X_2 = D_2X_1$. Hence,

$$\frac{D_1U_1}{D_2X_1} = \frac{D_1X_1}{D_2U_2} \Rightarrow \frac{D_1U_1}{D_1X_1} = \frac{D_2X_1}{D_2U_2} \Rightarrow \frac{r_1}{r} = \frac{r}{r_2} \Rightarrow r^2 = r_1 \cdot r_2$$

□

We conclude this article by a beautiful sangaku having a lot of circles of same radii (see Figure 11)

Corollary 13. Assuming the same notation in Corollary 12. If T is the midpoint of arc BC . In addition, let i_1 and e_1 be the incircle and A-excircle of triangle ΔABD_1 . Let i_2 and e_2 be the incircle and A-excircle of triangle ΔAD_2C . Let ω_1 and ω_2 be the incircles of triangles ΔABD_2 and ΔAD_1C . Then the circles c_1 , c_2 , and c have equal radii. Similarly, the circles i_1 and i_2 have equal radii. Moreover, the circles ω_1 , ω_2 , e_1 , and e_2 have equal radii.

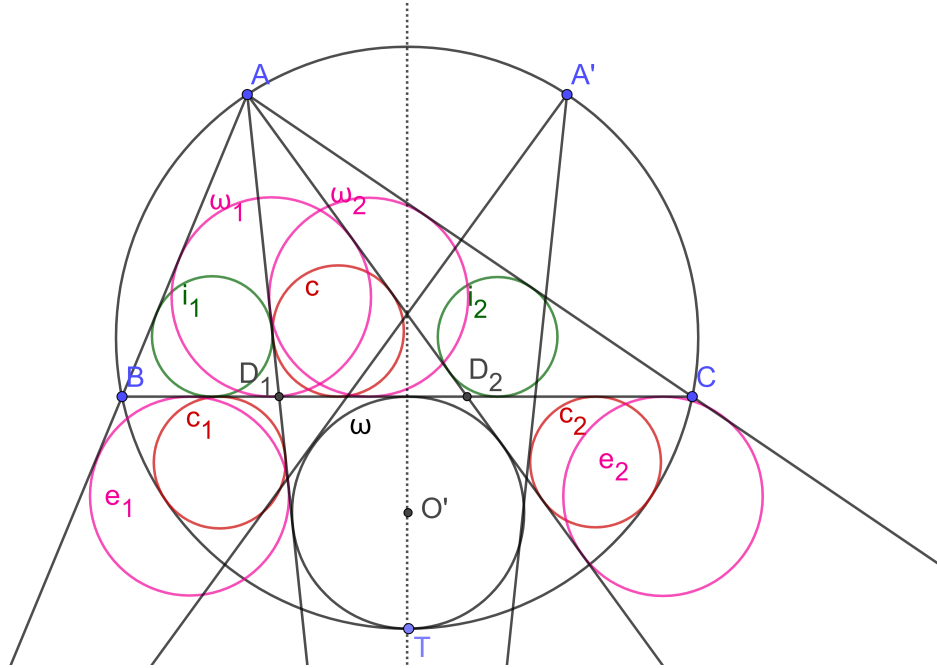


Figure 11: Configuration for Corollary 13.

Proof. Let p be the perpendicular bisector of BC . Define A' as the reflection of A across p . By Theorem 1 and symmetry with respect with p , the reflection of AD_1 across p is a common tangent to ω and c_2 . Also the reflection of AD_2 across p is the common tangent to ω and c_1 . By symmetry with respect to p , we have c_1 and c_2 are congruent. By Corollary 12 the circle c share with c_1 and c_2 the same radius r . Let ρ_1 , ρ_2 , and R denote the radii of the circles i_1 , i_2 , and ω , respectively. Similarly, let p_1 and p_2 denote the radii of the circles ω_1 and ω_2 . By Theorem 6 we have

$$\frac{1}{p} + \frac{1}{r} = \frac{1}{\rho} + \frac{1}{R}$$

Let k_1 and k_2 be the radii of the circles e_1 and e_2 respectively. By Theorem 6.2 in Rabinowitz [6], we obtain $k_1 = k_2 = k$ and

$$\frac{1}{k} + \frac{1}{r} = \frac{1}{\rho} + \frac{1}{R}$$

Then $p = k$

□

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