

Junior problems

J739. Solve in real numbers the system of equations

$$\begin{aligned}x^6 + y^2 &= z^8 + 1, \\y^6 + z^2 &= x^8 + 1, \\z^6 + x^2 &= y^8 + 1.\end{aligned}$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution 1 by the author

Adding all the three equations side by side, we get

$$(x^8 - x^6 - x^2 + 1) + (y^8 - y^6 - y^2 + 1) + (z^8 - z^6 - z^2 + 1) = 0,$$

i.e.

$$(x^6 - 1)(x^2 - 1) + (y^6 - 1)(y^2 - 1) + (z^6 - 1)(z^2 - 1) = 0.$$

So,

$$(x^2 - 1)^2(x^4 + x^2 + 1) + (y^2 - 1)^2(y^4 + y^2 + 1) + (z^2 - 1)^2(z^4 + z^2 + 1) = 0.$$

Since $(t^2 - 1)^2(t^4 + t^2 + 1) \geq 0$ for all $t \in \mathbb{R}$, then

$$(x^2 - 1)^2(x^4 + x^2 + 1) = (y^2 - 1)^2(y^4 + y^2 + 1) = (z^2 - 1)^2(z^4 + z^2 + 1) = 0$$

i.e. $x^2 = y^2 = z^2 = 1$. So,

$$\begin{aligned}(x, y, z) \in \{ & (1, 1, 1), (1, 1, -1), (1, -1, 1), (-1, 1, 1), \\ & (-1, -1, 1), (-1, 1, -1), (1, -1, -1), (-1, -1, -1) \}.\end{aligned}$$

An easy check shows that all these triples are solutions to the given system of equations.

Solution 2 by Adam John Frederickson, Utah Valley University, UT, USA

Set $(a, b, c) = (x^2, y^2, z^2)$. Then $a, b, c \geq 0$, and the system becomes

$$\begin{cases} a^3 + b = c^4 + 1, \\ b^3 + c = a^4 + 1, \\ c^3 + a = b^4 + 1. \end{cases}$$

We claim that $a = b = c$. Suppose otherwise. By cyclic symmetry, we may assume that $c = \max\{a, b, c\}$. Then $a \leq c$ and $b \leq c$, with at least one of these inequalities being strict. Hence

$$c^4 + 1 = a^3 + b < c^3 + c,$$

which gives

$$c^4 - c^3 - c + 1 < 0.$$

But

$$c^4 - c^3 - c + 1 = (c - 1)(c^3 - 1) = (c - 1)^2(c^2 + c + 1) \geq 0,$$

a contradiction. Therefore $a = b = c$. Let $a = b = c = t$. Then any equation of the system gives

$$t^3 + t = t^4 + 1 \iff (t - 1)^2(t^2 + t + 1) = 0$$

Since $t^2 + t + 1 > 0$, we must have $t = 1$. Thus $a = b = c = 1$, and consequently

$$x^2 = y^2 = z^2 = 1.$$

Hence $x, y, z \in \{-1, 1\}$.

Conversely, every independent choice of signs satisfies the original system. Therefore the system has exactly eight real solutions $(x, y, z) \in \{-1, 1\}^3$.

Editor's Comment. Mihály Bencze generalized the problem and proved that the system of equations

$$\begin{cases} ax^6 + by^2 = cz^8 + a + b - c, \\ by^6 + cz^2 = ax^8 + b + c - a, \\ cz^6 + ax^2 = by^8 + c + a - b, \end{cases}$$

where $a, b, c > 0$, has eight solutions in real numbers, that is, $(x, y, z) \in \{-1, 1\}^3$. For $a = b = c = 1$ we get the original problem and the solution is exactly the same.

Also solved by Amanda Goodwin, SUNY Brockport, NY, USA; Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Arkady Alt, San Jose, California, USA; Braden Mauerman, SUNY Brockport, NY, USA; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Cortland Greindl, SUNY Brockport, NY, USA; Dow Lee, St. George's School, Vancouver, Canada; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Jayden Choi, Bergen Catholic High School in Oradell, New Jersey, USA; Jinu Kong, Marriotts Ridge High School in Ellicott City, MD, USA; José Luis Díaz-Barrero, Barcelona, Spain; Kousik Sett, India; Mihály Bencze, Braşov, Romania; Mikhail Balepin, California, USA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Sundaresh H. R., Shivamogga, Karnataka, India; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Kristine Lee, Mount Juliet High School, TN, USA; Murad Safarli, Physics Mathematics and Informatics Lyceum, Baku, Azerbaijan; Turan Musayev, Shaki City Lyceum with Chemistry and Biology Specialisation, Shaki, Azerbaijan; Nguyen Tuan Minh, Polyhedra, Polk State College, FL, USA; Saarang Agarwal, AMMOC; Teodor Cristian Petrescu, Romania; Vidyut Sundar, Dhirubhai Ambani International School, Mumbai, India; Yeonwoo Amelia Tae, Northfield Mount Hermon School, Gill, Massachusetts, USA.

J740. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 \geq 3$. Prove that

$$\sqrt{\frac{2ab}{a^2 + b^2}} + \sqrt{\frac{2bc}{b^2 + c^2}} + \sqrt{\frac{2ca}{c^2 + a^2}} \leq a + b + c.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution 1 by the author

The Cauchy-Schwarz inequality and the inequality $\frac{2ab}{a^2+b^2} \leq \frac{2ab+c^2}{a^2+b^2+c^2}$ (which is equivalent to $c^2(a-b)^2 \geq 0$, obviously true), gives us

$$\begin{aligned} \sum_{\text{cyc}} \sqrt{\frac{2ab}{a^2 + b^2}} &\leq \sqrt{3 \sum_{\text{cyc}} \frac{2ab}{a^2 + b^2}} \\ &\leq \sqrt{3 \sum_{\text{cyc}} \frac{2ab + c^2}{a^2 + b^2 + c^2}} \\ &= \sqrt{\frac{3(a+b+c)^2}{a^2 + b^2 + c^2}} \\ &\leq a + b + c. \end{aligned}$$

The last inequality is true because $a^2 + b^2 + c^2 \geq 3$. The proof is completed.

Solution 2 by Polyhedra, Polk State College, FL, USA

Since $\frac{2ab}{a^2+b^2} \leq 1 = \frac{c^2}{c^2}$,

$$\frac{2ab}{a^2 + b^2} \leq \frac{2ab + c^2}{a^2 + b^2 + c^2} \leq \frac{2ab + c^2}{3}.$$

Therefore, by the concavity of $\sqrt{x}$ (or the Cauchy-Schwarz inequality),

$$\sqrt{\frac{2ab}{a^2 + b^2}} + \sqrt{\frac{2bc}{b^2 + c^2}} + \sqrt{\frac{2ca}{c^2 + a^2}} \leq 3\sqrt{\frac{1}{3} \left(\frac{2ab + c^2 + 2bc + a^2 + 2ca + b^2}{3} \right)} = a + b + c.$$

Solution 3 by Murad Safarli, Physics Mathematics and Informatics Lyceum, Baku, Azerbaijan and Turan Musayev, Shaki City Lyceum with Chemistry and Biology Specialisation, Shaki, Azerbaijan

Write $q = a^2 + b^2 + c^2$ and $s = a + b + c$. Since $a^2 + b^2 \leq q$, and similarly for the other pairs,

$$\begin{aligned} \frac{2ab}{a^2 + b^2} + \frac{2bc}{b^2 + c^2} + \frac{2ca}{c^2 + a^2} &= 3 - \frac{(a-b)^2}{a^2 + b^2} - \frac{(b-c)^2}{b^2 + c^2} - \frac{(c-a)^2}{c^2 + a^2} \\ &\leq 3 - \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{q} \\ &= \frac{s^2}{q}. \end{aligned}$$

Consequently, the Cauchy-Schwarz inequality and $q \geq 3$ give

$$\sqrt{\frac{2ab}{a^2 + b^2}} + \sqrt{\frac{2bc}{b^2 + c^2}} + \sqrt{\frac{2ca}{c^2 + a^2}} \leq \sqrt{\frac{3s^2}{q}} \leq s.$$

Because all three variables are positive, equality in the first estimate forces $a = b = c$; equality in the last estimate also requires $q = 3$. Thus equality holds if and only if $a = b = c = 1$.

Also solved by Arkady Alt, San Jose, California, USA; Brenna Meagher, SUNY Brockport, NY, USA; Arkan Manva, IIT Bombay, Mumbai, India; Cao Minh Quang, Nguyen Binh Khiem high school for the

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J741. Solve in nonzero complex numbers the system of equations

$$\begin{aligned}x + y + \frac{xy}{z} &= \frac{6}{z^2}, \\y + z - \frac{2yz}{x} &= \frac{3}{x^2}, \\z + x + \frac{4zx}{y} &= -\frac{9}{y^2}.\end{aligned}$$

Proposed by Adrian Andreescu, Dallas, USA

Solution 1 by Yeonwoo Amelia Tae, Northfield Mount Hermon School, Gill, Massachusetts, USA

Multiplying the three equations by z^2, x^2, y^2 , respectively, gives

$$xz^2 + yz^2 + xyz = 6, \tag{1}$$

$$x^2y + x^2z - 2xyz = 3, \tag{2}$$

$$y^2z + xy^2 + 4xyz = -9. \tag{3}$$

Adding these three equations, we obtain

$$(x + y + z)(xy + yz + zx) = 0.$$

On the other hand, equation (1) can be written as

$$z(xy + yz + zx) = 6.$$

Since $z \neq 0$, it follows that $xy + yz + zx \neq 0$, and therefore $x + y + z = 0$. Set $r = \frac{x}{z}$. Then

$$\frac{y}{z} = -1 - r.$$

Substituting $x = rz$ and $y = -(1 + r)z$ into (1) and (2), respectively, gives

$$-z^3(r^2 + r + 1) = 6$$

and

$$z^3(-r^3 + 2r^2 + 2r) = 3.$$

Eliminating z^3 , we get

$$2r^3 - 5r^2 - 5r - 1 = 0 \iff (2r + 1)(r^2 - 3r - 1) = 0.$$

Hence

$$r = -\frac{1}{2} \quad \text{or} \quad r = \frac{3 + \varepsilon\sqrt{13}}{2}, \quad \varepsilon \in \{-1, 1\}.$$

If $r = -\frac{1}{2}$, then

$$-z^3\left(\frac{1}{4} - \frac{1}{2} + 1\right) = 6,$$

so $z^3 = -8$. Thus, writing $\omega^3 = 1$,

$$z = -2\omega, \quad x = y = \omega,$$

and we obtain

$$(x, y, z) = \omega(1, 1, -2), \quad \omega^3 = 1.$$

Now let $r = \frac{3 + \varepsilon\sqrt{13}}{2}$, where $\varepsilon = \pm 1$. Since $r^2 = 3r + 1$, we have

$$r^2 + r + 1 = 4r + 2 = 2(4 + \varepsilon\sqrt{13}),$$

and therefore

$$z^3 = -\frac{6}{r^2 + r + 1} = -\frac{3}{4 + \varepsilon\sqrt{13}} = \varepsilon\sqrt{13} - 4.$$

Moreover,

$$\frac{x}{z} = r = \frac{3 + \varepsilon\sqrt{13}}{2}, \quad \frac{y}{z} = -1 - r = -\frac{5 + \varepsilon\sqrt{13}}{2}.$$

Hence the remaining solutions are

$$(x, y, z) = t \left(\frac{3 + \varepsilon\sqrt{13}}{2}, -\frac{5 + \varepsilon\sqrt{13}}{2}, 1 \right),$$

where

$$t^3 = \varepsilon\sqrt{13} - 4, \quad \varepsilon = \pm 1.$$

Conversely, each of the triples obtained above satisfies the original system. Since each of the three nonzero values of t^3 has exactly three complex cube roots, the system has exactly nine solutions.

Thus the complete solution set is $(x, y, z) = \omega(1, 1, -2)$, $\omega^3 = 1$, together with

$$(x, y, z) = t \left(\frac{3 + \varepsilon\sqrt{13}}{2}, -\frac{5 + \varepsilon\sqrt{13}}{2}, 1 \right), \quad t^3 = \varepsilon\sqrt{13} - 4, \quad \varepsilon = \pm 1.$$

Solution 2 by Corneliu Mănescu-Avram, Ploiești, Romania

Set

$$p = x + y + z, \quad q = xy + yz + zx, \quad r = xyz.$$

After clearing denominators, the three equations can be written as

$$\begin{cases} zq = 6, \\ x(q - 3yz) = 3, \\ y(q + 3zx) = -9. \end{cases}$$

Adding them gives

$$pq = 0.$$

But the first equation yields

$$q = \frac{6}{z} \neq 0,$$

so necessarily

$$p = x + y + z = 0.$$

From $zq = 6$ and $x + y = -z$, we obtain

$$6 = zq = z(xy + yz + zx) = xyz + z^2(x + y) = r - z^3.$$

Hence

$$r = z^3 + 6.$$

We now express x and y in terms of z . From the second equation,

$$xq - 3r = 3,$$

so

$$x = \frac{3r + 3}{q}.$$

Similarly, from the third equation,

$$yq + 3r = -9,$$

hence

$$y = -\frac{3r+9}{q}.$$

Using

$$q = \frac{6}{z}, \quad r = z^3 + 6,$$

we get

$$x = \frac{z(z^3+7)}{2}, \quad y = -\frac{z(z^3+9)}{2}.$$

Since $r = xyz = z^3 + 6$, it follows that

$$\frac{z(z^3+7)}{2} \left(-\frac{z(z^3+9)}{2} \right) z = z^3 + 6.$$

Let

$$u = z^3.$$

Then

$$-\frac{u(u+7)(u+9)}{4} = u+6,$$

or equivalently

$$u(u+7)(u+9) + 4(u+6) = 0.$$

Thus

$$u^3 + 16u^2 + 67u + 24 = 0,$$

and this factors as

$$(u+8)(u^2+8u+3) = 0.$$

Therefore

$$u = z^3 \in \{-8, -4 + \sqrt{13}, -4 - \sqrt{13}\}.$$

If $z^3 = -8$, then

$$x = -\frac{z}{2}, \quad y = -\frac{z}{2}.$$

Thus one family of solutions is

$$\left(-\frac{z}{2}, -\frac{z}{2}, z \right), \quad z^3 = -8.$$

If $z^3 = -4 + \sqrt{13}$, then

$$x = \frac{3 + \sqrt{13}}{2} z, \quad y = -\frac{5 + \sqrt{13}}{2} z,$$

so we obtain

$$\left(\frac{3 + \sqrt{13}}{2} z, -\frac{5 + \sqrt{13}}{2} z, z \right), \quad z^3 = -4 + \sqrt{13}.$$

Finally, if $z^3 = -4 - \sqrt{13}$, then

$$x = \frac{3 - \sqrt{13}}{2} z, \quad y = -\frac{5 - \sqrt{13}}{2} z,$$

giving

$$\left(\frac{3 - \sqrt{13}}{2} z, -\frac{5 - \sqrt{13}}{2} z, z \right), \quad z^3 = -4 - \sqrt{13}.$$

Each of the three nonzero values of z^3 has exactly three complex cube roots. Therefore the system has exactly 9 solutions.

Also solved by Dow Lee, St. George's School, Vancouver, Canada; Jayden Choi, Bergen Catholic High School in Oradell, New Jersey, USA; José Luis Díaz-Barrero, Barcelona, Spain; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Rousen Pirgulyev, Sumqayit, Azerbaijan; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Adam John Frederickson, Utah Valley University, UT, USA; Nguyen Tuan Minh; Polyahedra, Polk State College, FL, USA; Vidyut Sundar, Dhirubhai Ambani International School, Mumbai, India.

J742. Let a and b be two odd natural numbers. If (a, b) is the greatest common divisor of the numbers a and b , show that

$$\min\{(a, b+2), (a+2, b)\} \leq -1 + \sqrt{a+b+3}.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution 1 by Polyhedra, Polk State College, FL, USA

The inequality is clearly true for $a = b = 1$. Now suppose $a + b \geq 4$ and we establish in this case the stronger upper bound of $M = -1 + \sqrt{\frac{a+b}{2} + 2}$. Let $c = (a, b+2)$ and $d = (a+2, b)$. We may assume $c \geq d$. If $e \mid c$ and $e \mid d$, then $e \mid a$ and $e \mid (a+2)$, so $e \mid 2$. Since both c and d are odd, then e is odd, so $e = 1$, which means that $(c, d) = 1$. If $c = d$, then $d = 1 \leq M$. Consider $c > d$. Then $c \geq d + 2$. Since both c and d divide the even number $a + b + 2$,

$$a + b + 2 \geq 2cd \geq 2(d+2)d = 2(d+1)^2 - 2,$$

thus $d \leq M$.

Solution 2 by Shilong Zhuang, California, USA

Let $p = (a, b+2)$ and $q = (a+2, b)$. Since $p \mid a$, $p \mid b+2$, $q \mid a+2$, $q \mid b$, we have $pq \mid ab$ and $pq \mid (a+2)(b+2) = ab + 2(a+b) + 4$. It follows that $pq \mid 2(a+b) + 4$. Notice that p and q must be odd numbers because a and b are odd numbers. Hence, $(pq, 2) = 1$ and $pq \mid a + b + 2$.

If $p = q$, we have $p \mid a$ and $p \mid a+2$, it follows that $p \mid 2$, then $p = q = 1$. It is clear that the inequality holds because

$$\min\{(a, b+2), (a+2, b)\} = \min\{p, q\} = 1 < -1 + \sqrt{5} \leq -1 + \sqrt{a+b+3}.$$

If $p \neq q$, without loss of generality, we may assume $p > q$. Then $p \geq q + 2$. Clearly, $a + b + 2 \geq pq$. It follows that

$$\min\{(a, b+2), (a+2, b)\} = q = -1 + \sqrt{(q+2)q+1} \leq -1 + \sqrt{pq+1} \leq -1 + \sqrt{a+b+3}.$$

Also solved by Arkan Manva, IIT Bombay, Mumbai, India; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Corneliu Mănescu-Avram, Ploiești, Romania; José Luis Díaz-Barrero, Barcelona, Spain; Theo Koupelis, Clark College, Washington, USA; Vidyut Sundar, Dhirubhai Ambani International School, Mumbai, India.

J743. Let $ABCD$ be a trirectangular tetrahedron with apex D , that is, the edges AD , BD , and CD are pairwise perpendicular. Let h be the length of the altitude from D to the face ABC . Prove that

$$\frac{1}{(AD^2[BCD])^2} + \frac{1}{(BD^2[CAD])^2} + \frac{1}{(CD^2[ABD])^2} = \frac{1}{(h^2[ABC])^2},$$

where $[XYZ]$ denotes the area of triangle XYZ .

Proposed by Titu Andreescu, Dallas, USA

Solution 1 by Polyhedra, Polk State College, FL, USA

Let $a = AD$, $b = BD$, $c = CD$ and let V denote the volume of the tetrahedron $ABCD$.

Since AD, BD, CD are pairwise perpendicular, the three faces meeting at D are right triangles. Hence

$$[BCD] = \frac{bc}{2}, \quad [CAD] = \frac{ca}{2}, \quad [ABD] = \frac{ab}{2}.$$

By de Gua's theorem for a trirectangular tetrahedron,

$$[ABC]^2 = [BCD]^2 + [CAD]^2 + [ABD]^2.$$

Using the expressions above for the three right-triangle areas,

$$[ABC]^2 = \frac{b^2c^2}{4} + \frac{c^2a^2}{4} + \frac{a^2b^2}{4} = \frac{a^2b^2 + b^2c^2 + c^2a^2}{4}.$$

Therefore

$$\begin{aligned} \frac{1}{(AD^2[BCD])^2} + \frac{1}{(BD^2[CAD])^2} + \frac{1}{(CD^2[ABD])^2} &= \frac{4}{a^4b^2c^2} + \frac{4}{a^2b^4c^2} + \frac{4}{a^2b^2c^4} \\ &= \frac{4(a^2b^2 + b^2c^2 + c^2a^2)}{a^4b^4c^4} \\ &= \frac{16[ABC]^2}{a^4b^4c^4}. \end{aligned}$$

It remains to express abc in terms of h and $[ABC]$. Since the three edges AD, BD, CD are mutually perpendicular,

$$V = \frac{abc}{6}.$$

On the other hand, taking ABC as the base and h as the corresponding altitude,

$$V = \frac{h[ABC]}{3}.$$

Therefore

$$\frac{abc}{6} = \frac{h[ABC]}{3} \iff abc = 2h[ABC] \iff a^4b^4c^4 = 16h^4[ABC]^4.$$

Consequently,

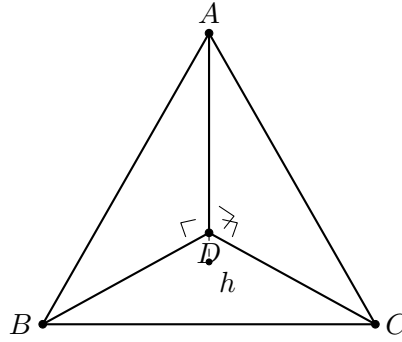
$$\frac{16[ABC]^2}{a^4b^4c^4} = \frac{16[ABC]^2}{16h^4[ABC]^4} = \frac{1}{h^4[ABC]^2}.$$

Therefore

$$\frac{1}{(AD^2[BCD])^2} + \frac{1}{(BD^2[CAD])^2} + \frac{1}{(CD^2[ABD])^2} = \frac{1}{(h^2[ABC])^2},$$

as required.

Solution 2 by Vidhyut Sundar, Dhurubhai Ambani International School, Mumbai, India



Place D at the origin with $A = (p, 0, 0)$, $B = (0, q, 0)$, $C = (0, 0, r)$, so $AD = p$, $BD = q$, $CD = r$.

Face areas. $[BCD] = \frac{qr}{2}$, $[CAD] = \frac{rp}{2}$, $[ABD] = \frac{pq}{2}$. Also $\vec{AB} \times \vec{AC} = (qr, pr, pq)$, so

$$[ABC] = \frac{1}{2} \sqrt{p^2 q^2 + q^2 r^2 + r^2 p^2} = \frac{1}{2} \sqrt{\Sigma}, \quad \Sigma := p^2 q^2 + q^2 r^2 + r^2 p^2.$$

Altitude. Plane ABC : $qr x + pr y + pq z = pqr$, so the distance from the origin is $h = \frac{pqr}{\sqrt{\Sigma}}$.

Two identities.

$$h \cdot [ABC] = \frac{pqr}{\sqrt{\Sigma}} \cdot \frac{\sqrt{\Sigma}}{2} = \frac{pqr}{2} \implies p^2 q^2 r^2 = 4h^2 [ABC]^2, \quad \frac{1}{h^2} = \frac{\Sigma}{p^2 q^2 r^2} = \frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}.$$

Computation.

$$\frac{1}{(AD^2 [BCD])^2} = \frac{1}{(p^2 \cdot \frac{qr}{2})^2} = \frac{4}{p^4 q^2 r^2},$$

and cyclically. Summing,

$$\sum \frac{1}{(AD^2 [BCD])^2} = \frac{4}{p^2 q^2 r^2} \left(\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2} \right) = \frac{4}{p^2 q^2 r^2} \cdot \frac{1}{h^2} = \frac{4}{4h^2 [ABC]^2} \cdot \frac{1}{h^2} = \frac{1}{(h^2 [ABC])^2}.$$

Solution 3 by Theo Koupelis, Clark College, Washington, USA

Let $DA = a$, $DB = b$, and $DC = c$. The triangles ABD , BCD , CAD are right triangles, and thus $[ABD] = ab/2$, $[BCD] = bc/2$, $[CAD] = ac/2$, $AB = \sqrt{a^2 + b^2} := x$, $BC = \sqrt{b^2 + c^2} := y$, and $CA = \sqrt{a^2 + c^2} := z$. Using the above and Heron's formula we get

$$[ABC] = \frac{1}{4} \sqrt{2(x^2 y^2 + y^2 z^2 + z^2 x^2) - x^4 - y^4 - z^4} = \frac{1}{2} \cdot \sqrt{a^2 b^2 + b^2 c^2 + c^2 a^2}.$$

Evaluating the volume V of the tetrahedron in two different ways we get

$$V = \frac{[ABD] \cdot CD}{3} = \frac{[ABC] \cdot h}{3} \implies h[ABC] = \frac{abc}{2} \implies (h^2 [ABC])^2 = \frac{(abc)^4}{16[ABC]^2}.$$

Using the above we get

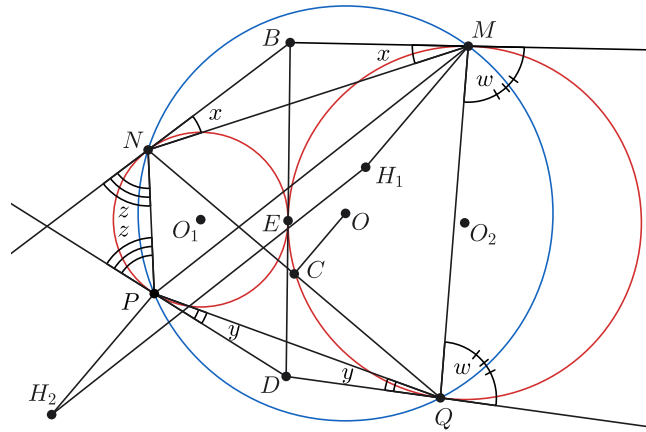
$$\begin{aligned} \frac{1}{(AD^2 [BCD])^2} + \frac{1}{(BD^2 [CAD])^2} + \frac{1}{(CD^2 [ABD])^2} &= \frac{4}{(abc)^2} \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \\ &= \frac{4(a^2 b^2 + b^2 c^2 + c^2 a^2)}{(abc)^4} = \frac{16[ABC]^2}{(abc)^4} = \frac{1}{(h^2 [ABC])^2}. \end{aligned}$$

Also solved by Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; José Luis Díaz-Barrero, Barcelona, Spain; Kousik Sett, India; Miquel Amengual Covas, Cala Figuera, Mallorca, Spain; Telemachus Baltasvias, Kerameies Junior High School, Kefalonia, Greece; Kristine Lee, Mount Juliet High School, TN, USA; Yeonwoo Amelia Tae, Northfield Mount Hermon School, Gill, Massachusetts, USA.

J744. Let Γ_1, Γ_2 be two circles with centers O_1, O_2 , tangent at E . On the line perpendicular to O_1O_2 that passes through E let B and D be points above and below point E , respectively. The tangents from B and D to Γ_1 are BN and DP respectively, and the tangents from B and D to Γ_2 are BM and DQ respectively. If H_1, H_2 are the orthocenters of triangles MNQ and NQP , respectively, show that $\angle PMH_1 = \angle PH_2H_1$.

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution 1 by Polyhedra, Polk State College, FL, USA



Since BE and BN are tangent segments from B to the first circle, while BE and BM are tangent segments from B to the second circle, we have $BM = BE = BN$. Thus $\triangle BMN$ is isosceles, and hence

$$\angle BMN = \angle BNM.$$

Likewise, $DP = DE = DQ$, so $\triangle DPQ$ is isosceles and

$$\angle DPQ = \angle DQP.$$

Using also the tangent-chord theorem at M, N, P, Q , and the angle notation w, x, y, z indicated in the figure, we obtain

$$\angle NMQ + \angle NPQ = 360^\circ - w - x - y - z = \angle MNP + \angle MQP.$$

The two expressions are the sums of the two pairs of opposite angles of quadrilateral $MNPQ$. Since the sum of its four angles is 360° , each of the above sums is 180° . Therefore

$$\angle NMQ + \angle NPQ = 180^\circ,$$

and hence M, N, P, Q are concyclic.

Let O be the center of their common circumcircle, and let C be the midpoint of NQ . We shall use the following elementary fact.

If X, N, Q lie on a circle with center O , C is the midpoint of NQ , and H is the orthocenter of $\triangle XNQ$, then

$$XH \parallel OC \quad \text{and} \quad XH = 2OC.$$

Indeed, let X' be the point antipodal to X on the circumcircle. Since XX' is a diameter,

$$NX' \perp NX, \quad QX' \perp QX.$$

On the other hand, since H is the orthocenter of $\triangle XNQ$,

$$QH \perp NX, \quad NH \perp QX.$$

Consequently,

$$QH \parallel NX', \quad NH \parallel QX',$$

so $NHQX'$ is a parallelogram. Its diagonals NQ and HX' bisect each other. Since C is the midpoint of NQ , it follows that C is also the midpoint of HX' .

Furthermore, O is the midpoint of XX' . Thus, in triangle XHX' , the segment OC joins the midpoints of XX' and HX' . By the midpoint theorem,

$$OC \parallel XH \quad \text{and} \quad OC = \frac{1}{2}XH,$$

which proves the claim.

We now apply this fact to the two triangles MNQ and PNQ , which have the same circumcircle and the same side NQ . Since H_1 and H_2 are their respective orthocenters, we obtain

$$MH_1 \parallel OC, \quad MH_1 = 2OC,$$

and

$$PH_2 \parallel OC, \quad PH_2 = 2OC.$$

Hence

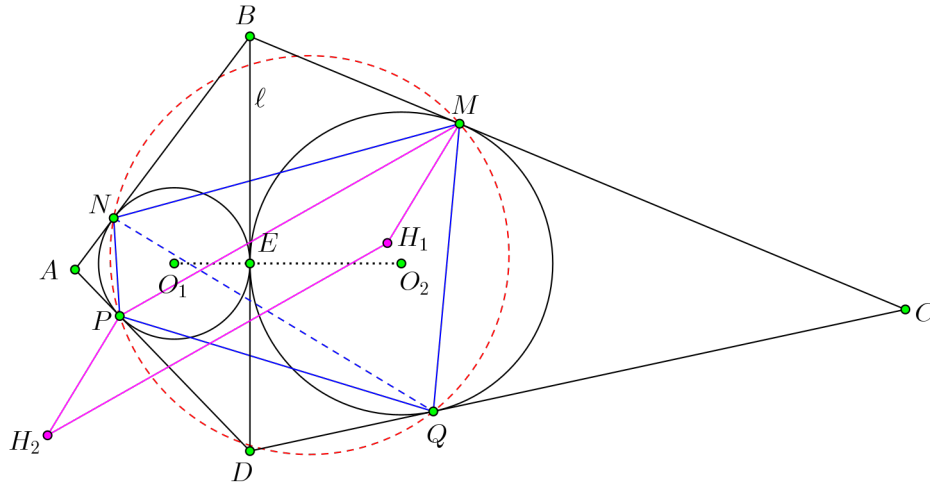
$$MH_1 \parallel PH_2 \quad \text{and} \quad MH_1 = PH_2.$$

Therefore PMH_1H_2 is a parallelogram. Its opposite angles are equal, so $\angle PMH_1 = \angle PH_2H_1$.

Solution 2 by Kousik Sett, India

Let ℓ be the line passing through E perpendicular to the line of centers O_1O_2 . Since Γ_1 and Γ_2 are tangent at E , the line ℓ is the common internal tangent (and the radical axis) of Γ_1 and Γ_2 .

Because B and D lie on ℓ , both B and D have equal power with respect to both circles. So $BM^2 = BN^2$ and $DP^2 = DQ^2$, that is, $BM = BN$ and $DP = DQ$.



Let $A = BN \cap DP$ and $C = BM \cap DQ$. Then $AN = AP$ and $CM = CQ$. In quadrilateral $ABCD$, let $\angle DAB = 2\alpha$, $\angle ABC = 2\beta$, $\angle BCD = 2\gamma$, and $\angle CDA = 2\delta$. So, $2\alpha + 2\beta + 2\gamma + 2\delta = 360^\circ$, that is, $\alpha + \beta + \gamma + \delta = 180^\circ$.

Since $AN = AP$, we have $\angle ANP = \angle APN = 90^\circ - \alpha$. Similarly, $\angle BNM = \angle BMN = 90^\circ - \beta$, $\angle CMQ = \angle CQM = 90^\circ - \gamma$, and $\angle DQP = \angle DPQ = 90^\circ - \delta$.

Hence it follows that

$$\begin{aligned} \angle PNM + \angle PQM &= 180^\circ - \angle ANP - \angle BNM + 180^\circ - \angle CQM - \angle DQP \\ &= 360^\circ - (90^\circ - \alpha) - (90^\circ - \beta) - (90^\circ - \gamma) - (90^\circ - \delta) \\ &= \alpha + \beta + \gamma + \delta = 180^\circ. \end{aligned}$$

Therefore, four points M, N, P, Q lie on a single circle, and let its center be O . Let $\vec{H_1}$ and $\vec{H_2}$ be the position vector (with respect to O) of the orthocenter of $\triangle MNQ$ and $\triangle NQP$, respectively. Then

$$\vec{H_1} = \vec{M} + \vec{N} + \vec{Q} \quad \text{and} \quad \vec{H_2} = \vec{N} + \vec{Q} + \vec{P}.$$

Subtracting these two vector equations yields

$$\vec{H_2} - \vec{H_1} = (\vec{N} + \vec{Q} + \vec{P}) - (\vec{M} + \vec{N} + \vec{Q}) = \vec{P} - \vec{M}.$$

Therefore, we have

$$\overrightarrow{H_1 H_2} = \overrightarrow{MP},$$

which implies

$$H_1 H_2 \parallel MP \quad \text{and} \quad H_1 H_2 = MP.$$

Consequently, the quadrilateral MH_1H_2P is a parallelogram and therefore, $\angle PMH_1 = \angle PH_2H_1$.

Also solved by José Luis Díaz-Barrero, Barcelona, Spain; Theo Koupelis, Clark College, Washington, USA; Tiberiu Popa, San Francisco, CA, USA; Vidyut Sundar, Dhirubhai Ambani International School, Mumbai, India.

Senior problems

S739. Solve in real numbers the equation

$$13\{x\} = x + 2[x] + 3(x),$$

where $[x]$ is the integer part of x , $\{x\}$ is the fractional part of x and (x) is the closest integer to x (half-integers are always rounded to even numbers).

Proposed by Nicușor Zlota, Traian Vuia Technical College, Focșani, Romania

Solution 1 by Theo Koupelis, Clark College, Washington, USA

Let $x = n + r$, where n is an integer and $0 \leq r < 1$.

(i) Let $r = 1/2$. If n is even, then the given equation becomes $\frac{13}{2} = n + \frac{1}{2} + 2n + 3n \implies n = 1$, which contradicts the assumption. If n is odd, then the given equation becomes $\frac{13}{2} = n + \frac{1}{2} + 2n + 3(n+1) \implies n = \frac{1}{2}$, which is not possible.

(ii) Let $\frac{1}{2} < r < 1$. The given equation becomes $13r = n + r + 2n + 3(n+1) \implies 4r = 2n + 1$. But $2 < 4r < 4$, and thus $2n + 1 = 3$. Therefore $n = 1$ and $r = \frac{3}{4}$. The solution $x = 7/4$ is an acceptable solution.

(iii) Let $0 \leq r < \frac{1}{2}$. The given equation becomes $13r = n + r + 2n + 3n \implies 2r = n$. But $0 \leq 2r < 1$, and thus we must have $n = r = 0$. The solution $x = 0$ is an acceptable solution.

Solution 2 by Yeonwoo Amelia Tae, Northfield Mount Hermon School, Gill, Massachusetts, USA

Let $[x] = n$ and $\{x\} = f$, where $n \in \mathbb{Z}$ and $0 \leq f < 1$. Then $x = n + f$. Also write $(x) = n + e$, where $e \in \{0, 1\}$. More precisely,

$$e = \begin{cases} 0, & f < \frac{1}{2}, \\ 1, & f > \frac{1}{2}, \end{cases}$$

while for $f = \frac{1}{2}$, the value of e is determined by the rule that half-integers are rounded to the nearest even integer.

The given equation becomes

$$13f = (n + f) + 2n + 3(n + e) \iff 12f = 6n + 3e \iff 4f = 2n + e.$$

Since $2n + e$ is an integer, $4f$ is an integer. As $0 \leq f < 1$, we must have

$$f \in \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}.$$

Moreover, since

$$2n + e = 4f \geq 0$$

and $e \in \{0, 1\}$, we necessarily have $n \geq 0$.

We now consider the four possible values of f .

Case 1: $f = 0$.

Then $2n + e = 0$. Thus $n = 0$ and $e = 0$, and therefore $x = n + f = 0$. This is indeed a solution.

Case 2: $f = \frac{1}{4}$.

Then $2n + e = 1$. Hence $n = 0$ and $e = 1$. This would give $x = \frac{1}{4}$. However, $(\frac{1}{4}) = 0$, so $e = (x) - n = 0$, contradicting $e = 1$. Thus there is no solution in this case.

Case 3: $f = \frac{1}{2}$.

Then $2n + e = 2$. The only possibility is $n = 1$ and $e = 0$. Thus $x = \frac{3}{2}$. But, by the convention that half-integers are rounded to the even integer, $\left(\frac{3}{2}\right) = 2$. Hence

$$e = (x) - n = 2 - 1 = 1,$$

contradicting $e = 0$. Therefore there is no solution in this case.

Case 4: $f = \frac{3}{4}$.

Then $2n + e = 3$, so $n = 1$ and $e = 1$. Therefore $x = 1 + \frac{3}{4} = \frac{7}{4}$. Since $\left(\frac{7}{4}\right) = 2$, this case is valid.

Hence the equation has exactly two real solutions: $x = 0$ or $x = \frac{7}{4}$.

Also solved by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Cortland Greindl, SUNY Brockport, NY, USA; José Luis Díaz-Barrero, Barcelona, Spain; Mohammed Aassila, Strasbourg, France; Rousen Pirgulyev, Sumqayit, Azerbaijan; Shrivathsan Krishnan, AMMOC; Sundares H. R., Shivamogga, Karnataka, India; The Ring Lords, Baden-Württemberg, Germany; Adam John Frederickson, Utah Valley University, UT, USA; Nguyen Tuan Minh; Saarang Agarwal, AMMOC; Teodor Cristian Petrescu, Romania; Vidyut Sundar, Dhirubhai Ambani International School, Mumbai, India.

S740. If n and k are positive integers such that $n^2 + k^2(8k + 3)$ is prime, prove that $n + 3k + 1$ is not a perfect cube.

Proposed by Alessandro Ventullo, Milan, Italy

Solution 1 by Murad Safarli, Physics Mathematics and Informatics Lyceum, Baku, Azerbaijan and Turan Musayev, Shaki City Lyceum with Chemistry and Biology Specialisation, Shaki, Azerbaijan

Suppose, to the contrary, that

$$n + 3k + 1 = t^3$$

for some positive integer t . Since $n, k \geq 1$, we have $t \geq 2$. Substituting $n = t^3 - 3k - 1$, we obtain

$$\begin{aligned} n^2 + k^2(8k + 3) &= (t^3 - 3k - 1)^2 + k^2(8k + 3) \\ &= t^6 - (6k + 2)t^3 + (2k + 1)^3 \\ &= (t^2)^3 + t^3 + (2k + 1)^3 - 3t^2 \cdot t \cdot (2k + 1). \end{aligned}$$

By the identity

$$u^3 + v^3 + w^3 - 3uvw = (u + v + w)(u^2 + v^2 + w^2 - uv - vw - wu),$$

it follows that

$$n^2 + k^2(8k + 3) = (t^2 + t + 2k + 1)B,$$

where

$$\begin{aligned} B &= t^4 + t^2 + (2k + 1)^2 - t^3 - t(2k + 1) - t^2(2k + 1) \\ &= \frac{(t^2 - t)^2 + (t - 2k - 1)^2 + (2k + 1 - t^2)^2}{2}. \end{aligned}$$

Both factors are integers. Moreover,

$$t^2 + t + 2k + 1 > 1, \quad B \geq \frac{(t^2 - t)^2}{2} \geq 2,$$

because $t \geq 2$. Thus $n^2 + k^2(8k + 3)$ is composite, a contradiction.

Solution 2 by Theo Koupelis, Clark College, Washington, USA

Let $p = n^2 + k^2(8k + 3)$ be a prime number. We proceed by contradiction, assuming that $n + 3k + 1 = m^3$, where m is an integer. Substituting $n = m^3 - 3k - 1$ into the expression for p we get

$$\begin{aligned} p &= (m^3 - 3k - 1)^2 + k^2(8k + 3) \\ &= (m^2 + m + 1 + 2k)[4k^2 - 2k(m^2 + m - 2) + m^4 - m^3 - m + 1]. \quad (*) \end{aligned}$$

The discriminant of the quadratic in k in the expression in brackets in $(*)$ above is $\Delta = -12m^2(m - 1)^2 \leq 0$.

(i) If $m = 0$, then $p = (2k + 1)^3$, which is not a prime for any value of k .

(ii) If $m = 1$, then $p = 4k^2(2k + 3)$, which is not a prime for any value of k .

(iii) If $m(m - 1) \neq 0$, then $\Delta < 0$. The minimum value of the expression in brackets in $(*)$ is $3m^2(m - 1)^2/4 \geq 3$; thus, for p to be a prime we must have $m^2 + m + 1 + 2k = 1$, and therefore, $k = -m(m + 1)/2 < 0$, $n = m^3 - 1 + 3m(m + 1)/2$, and $p = 3m^4 + 3m^3 - 3m + 1$. Thus, for positive integers k the problem statement holds.

Note: If k can take negative integer values, then $p = n^2 + k^2(8k + 3) = 3m^4 + 3m^3 - 3m + 1$ is a prime for many values of m , while the quantity $n + 3k + 1 = m^3$ is a perfect cube. For example, for $m = 2, 6, 10, 14$ we

get the corresponding values $k = -3, -21, -55, -105$, $n = 16, 278, 1164, 3058$, $p = 67, 4519, 32971, 123439$, and $n + 3k + 1 = 2^3, 6^3, 10^3, 14^3$. While it seems that the sequence $m = 2 + 4\lambda$ might produce a prime for any nonnegative integer value of λ , for the next value in the sequence, namely $\lambda = 4$, we get $p = 332371 = 13 \cdot 37 \cdot 691$, which is not a prime.

Editor's Comment. If $k = 1$, we get Problem S393 of Mathematical Reflections 6/2016, proposed by Titu Andreescu.

Also solved by Adam Deli; Arkan Manva, IIT Bombay, Mumbai, India; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Corneliu Mănescu-Avram, Ploiești, Romania; Cortland Greindl, SUNY Brockport, NY, USA; José Luis Díaz-Barrero, Barcelona, Spain; Mihály Bencze, Brașov, Romania; Mohammed Aassila, Strasbourg, France; Rousen Pirguliyev, Sumqayit, Azerbaijan; The Ring Lords, Baden-Württemberg, Germany; Anderson Torres, Sao Paulo, Brazil; Nguyen Tuan Minh; Vidyut Sundar, Dhirubhai Ambani International School, Mumbai, India.

S741. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq 1 + \frac{\frac{a^4}{b+c} + \frac{b^4}{c+a} + \frac{c^4}{a+b}}{a^3 + b^3 + c^3}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution 1 by Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam

Set $S = a^3 + b^3 + c^3$. The desired inequality is equivalent to

$$S \left(\sum_{\text{cyc}} \frac{a}{b+c} - 1 \right) \leq \sum_{\text{cyc}} \frac{a^4}{b+c}.$$

Moving all terms to one side, it remains to prove

$$\sum_{\text{cyc}} \frac{a^4 - aS}{b+c} + S \geq 0.$$

Now

$$a^4 - aS = a^4 - a(a^3 + b^3 + c^3) = -a(b^3 + c^3).$$

Hence

$$\frac{a^4 - aS}{b+c} = -\frac{a(b^3 + c^3)}{b+c} = -a(b^2 - bc + c^2).$$

Therefore it is enough to prove

$$a^3 + b^3 + c^3 - \sum_{\text{cyc}} a(b^2 - bc + c^2) \geq 0.$$

Expanding, this becomes

$$a^3 + b^3 + c^3 + 3abc - \sum_{\text{sym}} a^2b \geq 0,$$

that is,

$$a^3 + b^3 + c^3 + 3abc \geq a^2b + a^2c + b^2a + b^2c + c^2a + c^2b.$$

But this is exactly Schur's inequality of degree 3:

$$a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) \geq 0.$$

Thus the required inequality follows. Equality holds if and only if $a = b = c$.

Solution 2 by Tiberiu Popa, San Francisco, CA, USA

We will prove the chain of inequalities:

$$\sum_{\text{cyc}} \frac{a}{b+c} \leq 1 + \frac{\sum_{\text{cyc}} a^3}{\sum_{\text{cyc}} a^2(b+c)} \leq 1 + \frac{\sum_{\text{cyc}} \frac{a^4}{b+c}}{\sum_{\text{cyc}} a^3}.$$

First, we establish the right inequality. This follows directly from the Cauchy-Schwarz inequality:

$$\left(\sum_{\text{cyc}} a^3 \right)^2 = \left(\sum_{\text{cyc}} \frac{a^2}{\sqrt{b+c}} \cdot a\sqrt{b+c} \right)^2 \leq \left(\sum_{\text{cyc}} \frac{a^4}{b+c} \right) \left(\sum_{\text{cyc}} a^2(b+c) \right).$$

Next, we prove the left inequality, which can be rearranged as:

$$\sum_{\text{cyc}} \frac{a}{b+c} - 1 \leq \frac{\sum_{\text{cyc}} a^3}{\sum_{\text{cyc}} a^2(b+c)}.$$

We simplify the left-hand side. Using the polynomial identities

$$\prod_{\text{cyc}}(a+b) = 2abc + \sum_{\text{cyc}} a^2(b+c)$$

and

$$\sum_{\text{cyc}} a(a+b)(a+c) = \sum_{\text{cyc}} a^3 + \sum_{\text{cyc}} a^2(b+c) + 3abc,$$

we obtain:

$$\sum_{\text{cyc}} \frac{a}{b+c} - 1 = \frac{\sum_{\text{cyc}} a^3 + abc}{\sum_{\text{cyc}} a^2(b+c) + 2abc}.$$

Thus, the left inequality is equivalent to

$$\frac{\sum_{\text{cyc}} a^3 + abc}{\sum_{\text{cyc}} a^2(b+c) + 2abc} \leq \frac{\sum_{\text{cyc}} a^3}{\sum_{\text{cyc}} a^2(b+c)}.$$

Cross-multiplying, which preserves the inequality since all variables are strictly positive, gives:

$$\left(\sum_{\text{cyc}} a^3 + abc \right) \left(\sum_{\text{cyc}} a^2(b+c) \right) \leq \left(\sum_{\text{cyc}} a^3 \right) \left(2abc + \sum_{\text{cyc}} a^2(b+c) \right).$$

Expanding both sides:

$$abc \sum_{\text{cyc}} a^2(b+c) \leq 2abc \sum_{\text{cyc}} a^3.$$

Dividing by $abc > 0$, we get

$$\begin{aligned} \sum_{\text{cyc}} a^2(b+c) &\leq 2 \sum_{\text{cyc}} a^3 \iff \sum_{\text{cyc}} (2a^3 - a^2b - a^2c) \geq 0 \\ &\iff \sum_{\text{cyc}} (a^3 - a^2b - ab^2 + b^3) \geq 0 \\ &\iff \sum_{\text{cyc}} (a+b)(a-b)^2 \geq 0 \end{aligned}$$

Because $a, b, c > 0$, this final inequality is clearly true. Thus, the original inequality holds, with equality if and only if $a = b = c$ (both sides are $\frac{3}{2}$).

Also solved by Arkady Alt, San Jose, California, USA; Arkan Manva, IIT Bombay, Mumbai, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Marin Chirciu, Colegiul National Zinca Golescu Pitești, Romania; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Rovsen Pirgulyev, Sumqayit, Azerbaijan; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Anderson Torres, Sao Paulo, Brazil; Daniel Pascuas, Barcelona, Spain; Etisha Sharma and Toyesh Prakash Sharma, Agra College, Agra, India; Nguyen Tuan Minh; Shilong Zhuang, California, USA; Teodor Cristian Petrescu, Romania; Vidyut Sundar, Dhirubhai Ambani International School, Mumbai, India; Yeon-woo Amelia Tae, Northfield Mount Hermon School, Gill, Massachusetts, USA.

S742. Let ABC be a non-right triangle with $AB \neq AC$ and let H be its orthocenter, O be its circumcenter. Let M_a be the midpoint of BC and let D be the midpoint of AM_a . Let E, F be the feet of the perpendiculars drawn from A, M_a respectively to the line OH . Knowing that $HDFM_a$ is a cyclic quadrilateral, prove that $EDOM_a$ is a cyclic quadrilateral too.

Proposed by Todor Zaharinov, Sofia, Bulgaria

Solution 1 by the author

Let denote with δ the circle $(HDFM_a)$ with center J . Since $\angle HFM_a = 90^\circ$ hence HM_a is a diameter of δ .

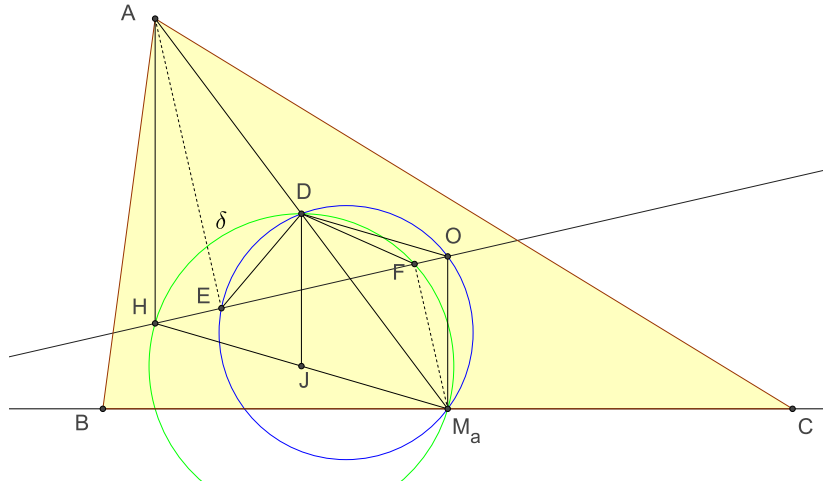


Figure 1:

In the triangle AHM_a , DJ is a midsegment, so $DJ \parallel AH$ or $DJ \perp BC$ and $DJ = \frac{1}{2}AH$. But $OM_a \perp BC$ and $OM_a = \frac{1}{2}AH = DJ$. Since J is center of δ , $DJ = JM_a$, hence DOM_aJ is a rhombus and $\angle DM_aO = \angle DM_aJ$.

Since $AE \perp EF$, $M_aF \perp EF$, $AD = DM_a$, hence D lies on the perpendicular bisector of EF and $DE = DF$. So from $\triangle DEF$, $\angle DEF = \angle DFE$.

$$\angle DM_aO \stackrel{DOM_aJ}{=} \angle DM_aJ \equiv \angle DM_aH \stackrel{(\delta)}{=} \angle DFH \equiv \angle DFE \stackrel{\triangle DEF}{=} \angle DEF \equiv \angle DEO$$

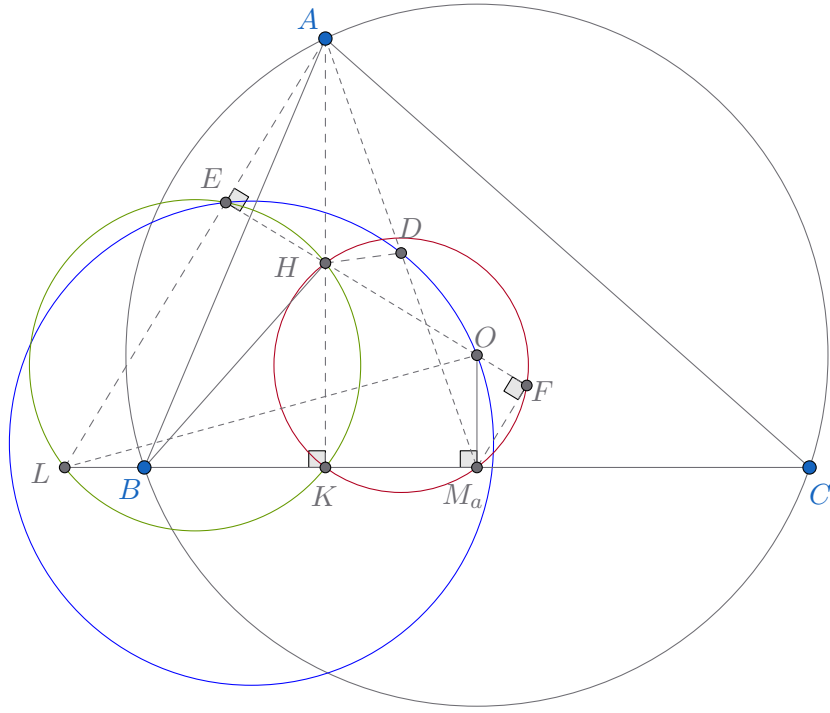
It follows that the points D, O, M_a, E are concyclic.

Solution 2 by Theo Koupelis, Clark College, Washington, USA

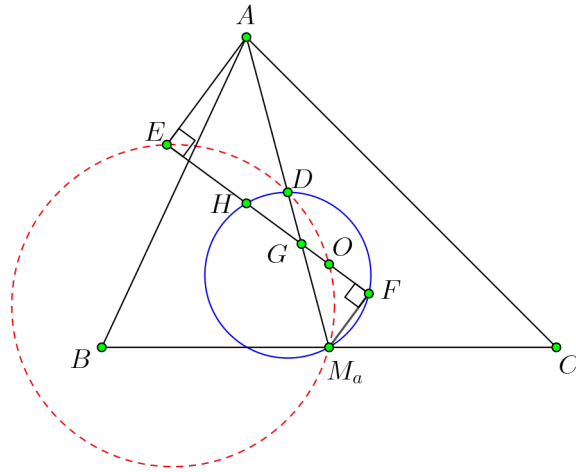
Let K be the intersection of the lines AH and BC , and let L be the intersection point of the lines AE and BC . The points H, K, M_a, F are concyclic because $\angle HKM_a = \angle HFM_a = 90^\circ$. Similarly, the points E, L, K, H are concyclic because $\angle LEH = \angle LKH = 90^\circ$, and thus $AH \cdot AK = AE \cdot AL$. Finally, the points E, L, M_a, O are concyclic because $\angle LEO = \angle LM_aO = 90^\circ$. Thus, if D is on the circle $(HFM_a) = (HKFM_a)$ then we must have $AD \cdot AM_a = AH \cdot AK = AE \cdot AL$, and therefore D is on the circle $(ELM_a) = (EOM_a)$.

Note: If D is to be on the circle (HFM_a) , then for a non-right, non-isosceles triangle, the angle A must be acute. Indeed, if $A > 90^\circ$, then H is on the line AK but outside the triangle, and the segment AM_a will be inside the circle (HKM_a) . In such a case, D is inside the circle (HFM_a) .

Let $BC = a$, $AC = b$, and $AB = c$. We have $AH = b \cos A / \sin B$, and $AK = c \sin B$. Thus, $AH \cdot AK = bc \cos A = (b^2 + c^2 - a^2)/2$. Also, $AD \cdot AM_a = AM_a^2/2 = [2(b^2 + c^2) - a^2]/8$. Therefore, when D is on the circle (HFM_a) we must have $3a^2 = 2(b^2 + c^2)$.



Solution 3 by Kousik Sett, India



Let G be the centroid of $\triangle ABC$. Since G lies on AM_a and the Euler line HO , $G = AM_a \cap HO$. Moreover, $AG : GM_a = 2 : 1 = HG : GO$. Thus, $GO = \frac{1}{2}HG$.

Since $\angle AEG = \angle M_aFG = 90^\circ$, $\angle AGE = \angle M_aGF$, we infer that $\triangle AEG \sim \triangle M_aFG$. Therefore, $EG : GF = AG : GM_a = 2 : 1$. Thus, $EG = 2GF$.

Since the four points H, D, F, M_a are concyclic, by the Power of a Point Theorem, we obtain

$$DG \cdot GM_a = HG \cdot GF.$$

Now, Using the above condition, we obtain

$$EG \cdot GO = 2GF \cdot \frac{1}{2}HG = GF \cdot HG = DG \cdot GM_a.$$

Hence, by the Converse of the Power of a Point Theorem, the four points E, D, O, M_a are concyclic. Therefore, $EDOM_a$ is a cyclic quadrilateral.

Also solved by Tiberiu Popa, San Francisco, CA, USA; Vidyut Sundar, Dhirubhai Ambani International School, Mumbai, India.

S743. Let $a \geq 1 \geq b \geq c \geq 0$ such that $a + b + c = 3$. Prove that

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} + \frac{abc}{ab+bc+ca} \geq \frac{11}{6}.$$

Proposed by Marius Stănean, Zalău, Romania

Solution by the author

If we denote $q = ab + bc + ca \in (0, 3]$ and $r = abc \in [0, 1]$ the inequality becomes

$$\frac{a^2 + b^2 + c^2 + 3(ab + bc + ca)}{(a+b)(b+c)(c+a)} + \frac{abc}{ab+bc+ca} \geq \frac{11}{6},$$

or

$$\frac{9+q}{3q-r} + \frac{r}{q} \geq \frac{11}{6},$$

or after clearing denominators

$$-6r^2 + 29qr + 27q(2-q) \geq 0.$$

Denote $a = 1 + x$, $b = 1 - y$, $c = 1 - z$. It follows that $x = y + z$, and $1 \geq z \geq y \geq 0$. We have

$$\begin{aligned} q &= (1+x)(1-y) + (1-y)(1-z) + (1-z)(1+x) \\ &= (1+y+z)(2-y-z) + (1-y)(1-z) \\ &= 3 + yz - (y+z)^2, \\ r &= (1+x)(1-y)(1-z) \\ &= (1+y+z)(1-y-z+yz) \\ &= 1 + yz - (y+z)^2 + yz(y+z), \end{aligned}$$

which means

$$r \geq q - 2.$$

Returning to the proposed inequality, we have two cases:

1. If $2 - q > 0$ the inequality is clearly true because by the AM-GM Inequality

$$(a+b+c)(ab+bc+ca) \geq 9abc$$

which means $q \geq 3r$ so $29qr - 6r^2 \geq 87r^2 - 6r^2 = 81r^2 \geq 0$.

2. If $2 - q \leq 0$ denote the left-hand side by $f(r)$, so that we want to prove $f(r) \geq 0$. Since f quadratic function of r and $\frac{29q}{12} > 1$, we see that $f(r)$ is an increasing function of r for $r \geq q - 2$. Hence it suffices to prove $f(q - 2) \geq 0$, that is

$$-6(q-2)^2 + 29q(q-2) + 27q(2-q) \geq 0,$$

or

$$4(q-2)(3-q) \geq 0,$$

obviously true.

The equality holds when $q = 3$ which means $a = b = c = 1$ or when $q = 2$ which means $r = 0$ so $a = 2$, $b = 1$, $c = 0$.

Also solved by Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Corneliu Mănescu-Avram, Ploiești, Romania; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Tiberiu Popa, San Francisco, CA, USA; Shilong Zhuang, California, USA; Teodor Cristian Petrescu, Romania; Vidyut Sundar, Dhirubhai Ambani International School, Mumbai, India.

S744. Prove that $\frac{4}{3}$ is the largest positive value of k such that

$$\frac{1}{(a+b)^2+k} + \frac{1}{(b+c)^2+k} + \frac{1}{(c+a)^2+k} \geq \frac{3}{4+k}$$

for any nonnegative real numbers a, b, c with at most one of them larger than 1 and $ab + bc + ca = 3$.

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania

Solution 1 by the author

For $b = 1$, $a + c = 2x$ and $ac = 3 - 2x$, where $x \in [1, 3/2]$, the constraints are satisfied and the inequality becomes as follows:

$$\begin{aligned} \frac{1}{(a+1)^2+k} + \frac{1}{(c+1)^2+k} + \frac{1}{(a+c)^2+k} &\geq \frac{3}{4+k}, \\ \frac{4x^2 - 2ac + 4x + 2k + 2}{(ac + 2x + 1)^2 + k(4x^2 - 2ac + 4x + 2) + k^2} + \frac{1}{4x^2 + k} &\geq \frac{3}{4+k}, \\ \frac{4x^2 + 8x + 2k - 4}{16 + k(4x^2 + 8x - 4) + k^2} &\geq \frac{12x^2 + 2k - 4}{(k+4)(4x^2 + k)}, \\ \frac{2x^2 + 4x + k - 2}{4kx^2 + 8kx + k^2 - 4k + 16} &\geq \frac{6x^2 + k - 2}{(k+4)(4x^2 + k)}, \\ 4(2-k)x^4 + 8(2-k)x^3 - (4-k)(8-k)x^2 + k(8-k)x + 2(2-k)^2 &\geq 0, \\ (x-1)f(x) \geq 0, \quad f(x) = 4(2-k)x^3 + 12(2-k)x^2 - (8+k^2)x - 2(2-k)^2. \end{aligned}$$

The inequality holds for $x \in [1, 3/2]$ if and only if $f(x) \geq 0$ for $x \in (1, 3/2]$. For the necessary condition $\lim_{x \rightarrow 1} f(x) \geq 0$, we get $k \leq \frac{4}{3}$. To show that $\frac{4}{3}$ is the largest value of k , we need to prove that $E \geq 0$ for $k = \frac{4}{3}$, where

$$E = \frac{1}{(a+b)^2+k} + \frac{1}{(b+c)^2+k} + \frac{1}{(c+a)^2+k} - \frac{3}{4+k}.$$

Assume $a \geq 1 \geq b \geq c$ and denote

$$X = (a+b)^2 + k, \quad Y = (a+c)^2 + k, \quad Z = (b+c)^2 + k.$$

For fixed a , assume that c and E are functions of b . By differentiating the equality constraint, we get $(a+b)c' + a + c = 0$. Therefore

$$\begin{aligned} \frac{E'(b)}{2} &= -\frac{a+b}{X^2} - \frac{b+c}{Z^2} - \left[\frac{a+c}{Y^2} + \frac{b+c}{Z^2} \right] c' \\ &= -\frac{a+b}{X^2} - \frac{b+c}{Z^2} + \left[\frac{a+c}{Y^2} + \frac{b+c}{Z^2} \right] \frac{a+c}{a+b} = -\frac{a+b}{X^2} + \frac{(a+c)^2}{(a+b)Y^2} - \frac{(b-c)(b+c)}{(a+b)Z^2}. \end{aligned}$$

We will show that $E'(b) \leq 0$, that is

$$\frac{(b-c)(b+c)}{Z^2} \geq \frac{(a+c)^2}{Y^2} - \frac{(a+b)^2}{X^2}, \quad \frac{(b-c)(b+c)}{Z^2} \geq \frac{AB}{X^2Y^2},$$

where

$$A = (a+c)X - (a+b)Y, \quad B = (a+c)X + (a+b)Y.$$

Since

$$A = (a+b)(a+c)(b-c) - k(b-c) = (b-c)(a^2 + 3 - k),$$

$$B = (a+b)(a+c)(2a+b+c) + k(2a+b+c) = (2a+b+c)(a^2+3+k),$$

and

$$XY \geq [(a+b)(a+c) + 2]^2 = (a^2+3+k)^2,$$

it suffices to show that

$$\frac{b+c}{Z^2} \geq \frac{(2a+b+c)(a^2+3-k)}{(a^2+3+k)^3},$$

which is equivalent to

$$\frac{x}{(a+x)(4x^2+k)^2} \geq \frac{a^2+3-k}{(a^2+3+k)^3},$$

where $x = \frac{b+c}{2} \leq 1$. From $3 = 2ax + bc \leq 2ax + x^2$, we get $\frac{a}{x} \leq \frac{a(2a+x)}{3} \leq \frac{a(2a+1)}{3}$. Since

$$\frac{x}{a+x} = \frac{1}{a/x+1} \geq \frac{1}{a(2a+1)/3+1} = \frac{3}{2a^2+a+3}$$

and $\frac{1}{(4x^2+k)^2} \geq \frac{1}{(4x+k)^2}$, it suffices to show that

$$\frac{3}{(2a^2+a+3)(4x+k)^2} \geq \frac{a^2+3-k}{(a^2+3+k)^3}.$$

From $(b-1)(c-1) \geq 0$, we get $bc \geq 2x-1$, and from $3 = 2ax + bc \geq 2ax + 2x - 1$, we get $x \leq \frac{2}{a+1}$, hence

$$4x+k \leq \frac{8}{a+1} + k = \frac{ka+8+k}{a+1}.$$

So, it suffices to show that

$$\frac{3(a+1)^2}{(2a^2+a+3)(ka+8+k)^2} \geq \frac{a^2+3-k}{(a^2+3+k)^3},$$

i.e.

$$3(a+1)^2(3a^2+13)^3 \geq 16(a+7)^2(3a^2+5)(2a^2+a+3).$$

We can get this inequality by multiplying the inequalities

$$(3a^2+13)^2 \geq 32(3a^2+5)$$

and

$$6(a+1)^2(3a^2+13) \geq (a+7)^2(2a^2+a+3).$$

We have

$$(3a^2+13)^2 - 32(3a^2+5) = 9(a^2-1)^2 \geq 0,$$

and

$$\begin{aligned} 6(a+1)^2(3a^2+13) - (a+7)^2(2a^2+a+3) &= 16a^4 + 7a^3 - 19a^2 + 65a - 69 \\ &= (a-1)(16a^3 + 23a^2 + 4a + 69) \geq 0. \end{aligned}$$

From $E'(b) \leq 0$, it follows that $E(b)$ is decreasing and has the minimum value when b is maximum (or c is minimum), hence when $b=1$ or $c=0$. So, it suffices to consider these cases.

Case 1: $b=1$. As shown at the beginning, it suffices to show that $f(x) \geq 0$ for $x \geq 1$, where

$$f(x) = 4(2-k)x^3 + 12(2-k)x^2 - (8+k^2)x - 2(2-k)^2,$$

Indeed,

$$f(x) = \frac{8(3x^3 + 9x^2 - 11x - 1)}{9} = \frac{8(x-1)(3x^2 + 12x + 1)}{9} \geq 0.$$

Case 2: $c = 0$. We need to show that $ab = 3$ implies

$$\frac{1}{(a+b)^2+k} + \frac{1}{a^2+k} + \frac{1}{b^2+k} \geq \frac{3}{4+k}$$

Let $x = a + b$. From $(a-1)(b-1) \leq 0$, we get $x \geq ab + 1 = 4$. Write the inequality as follows:

$$\frac{1}{(a+b)^2+k} + \frac{a^2+b^2+2k}{a^2b^2+k(a^2+b^2)+k^2} \geq \frac{3}{4+k},$$

$$\frac{1}{x^2+k} + \frac{x^2-6+2k}{9+k(x^2-6)+k^2} \geq \frac{3}{4+k},$$

$$\frac{1}{3x^2+4} + \frac{3x^2-10}{12x^2+25} \geq \frac{3}{16}.$$

Let $t = 3x^2$, $t \geq 48$. We need to show that

$$\frac{1}{t+4} + \frac{t-10}{4t+25} \geq \frac{3}{16},$$

i.e.

$$4t^2 - 155t - 540 \geq 0.$$

Indeed,

$$4t^2 - 155t - 540 > 4(t^2 - 42t - 135) = 4(t-45)(t+3) > 0.$$

The proof is completed. For $k = \frac{4}{3}$, the equality occurs when $a = b = c = 1$.

Solution 2 by Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam

We prove that the largest possible value is $\frac{4}{3}$.

First, we show that the inequality holds for $k = \frac{4}{3}$. In this case, it is enough to prove

$$\sum_{\text{cyc}} \frac{1}{3(a+b)^2+4} \geq \frac{3}{16}.$$

Set $s = a + b + c$ and $r = abc$. Since $ab + bc + ca = 3$, we have

$$s^2 \geq 3(ab + bc + ca) = 9 \iff s \geq 3.$$

Moreover, since at most one of a, b, c is larger than 1,

$$(a-1)(b-1)(c-1) \geq 0.$$

Expanding and using $ab + bc + ca = 3$, we obtain

$$abc + a + b + c - 1 - (ab + bc + ca) \geq 0,$$

that is,

$$r + s \geq 4.$$

Also, by AM-GM,

$$r = abc \leq \left(\frac{ab + bc + ca}{3} \right)^{3/2} = 1.$$

Thus

$$0 \leq r \leq 1, \quad s \geq 3, \quad r + s \geq 4.$$

Now put $u = a + b$, $v = b + c$, $w = c + a$. Then

$$u + v + w = 2s,$$

and

$$uv + vw + wu = a^2 + b^2 + c^2 + 3(ab + bc + ca) = s^2 + 3.$$

Furthermore,

$$uvw = (a + b)(b + c)(c + a) = (a + b + c)(ab + bc + ca) - abc = 3s - r.$$

Hence

$$u^2 + v^2 + w^2 = (u + v + w)^2 - 2(uv + vw + wu) = 2s^2 - 6,$$

and

$$\begin{aligned} u^2v^2 + v^2w^2 + w^2u^2 &= (uv + vw + wu)^2 - 2uvw(u + v + w) \\ &= (s^2 + 3)^2 - 4s(3s - r) \\ &= s^4 - 6s^2 + 4sr + 9. \end{aligned}$$

We now compute

$$\sum_{\text{cyc}} \frac{1}{3u^2 + 4} - \frac{3}{16}.$$

Putting everything over the common denominator gives

$$\sum_{\text{cyc}} \frac{1}{3u^2 + 4} - \frac{3}{16} = \frac{F(s, r)}{16(3u^2 + 4)(3v^2 + 4)(3w^2 + 4)},$$

where

$$F(s, r) = 36s^4 - 465s^2 + 630sr - 81r^2 - 540.$$

Indeed,

$$\begin{aligned} &16 \sum_{\text{cyc}} (3v^2 + 4)(3w^2 + 4) - 3(3u^2 + 4)(3v^2 + 4)(3w^2 + 4) \\ &= 36(u^2v^2 + v^2w^2 + w^2u^2) + 240(u^2 + v^2 + w^2) + 576 - 81u^2v^2w^2, \end{aligned}$$

and therefore, using the identities above,

$$\begin{aligned} F(s, r) &= 36(s^4 - 6s^2 + 4sr + 9) + 240(2s^2 - 6) + 576 - 81(3s - r)^2 \\ &= 36s^4 - 465s^2 + 630sr - 81r^2 - 540. \end{aligned}$$

Since the denominator is positive, it remains to prove

$$F(s, r) \geq 0.$$

For fixed $s \geq 3$, the function $F(s, r)$ is increasing in r on $0 \leq r \leq 1$. Indeed, if $r_2 \geq r_1$, then

$$F(s, r_2) - F(s, r_1) = (r_2 - r_1)(630s - 81(r_1 + r_2)) \geq 0,$$

because

$$630s - 81(r_1 + r_2) \geq 630 \cdot 3 - 81 \cdot 2 > 0.$$

We distinguish two cases.

If $3 \leq s \leq 4$, then $r + s \geq 4$ gives $r \geq 4 - s$. Hence

$$F(s, r) \geq F(s, 4 - s).$$

A direct simplification gives

$$F(s, 4 - s) = 12(s - 3)^2(3s^2 + 18s - 17) \geq 0.$$

If $s \geq 4$, then $r \geq 0$, so

$$F(s, r) \geq F(s, 0) = 36s^4 - 465s^2 - 540 > 0.$$

Thus

$$\sum_{\text{cyc}} \frac{1}{3(a+b)^2+4} \geq \frac{3}{16},$$

and therefore the required inequality holds for $k = \frac{4}{3}$.

It remains to prove that no larger value of k works.

Let $b = c = t$ and $a = \frac{3-t^2}{2t}$, where $0 < t < 1$. Then

$$ab + bc + ca = 2at + t^2 = 3,$$

and $b = c < 1 < a$, so the hypotheses are satisfied.

Set

$$\Delta = \sum_{\text{cyc}} \frac{1}{(a+b)^2+k} - \frac{3}{4+k}.$$

Substituting the above values of a, b, c , we obtain

$$\Delta = -\frac{6(t-1)^2(t+1)^2(3kt^2+2t^2-6)}{(k+4)(k+4t^2)(4kt^2+t^4+6t^2+9)}.$$

If $k > \frac{4}{3}$, then for $t < 1$ sufficiently close to 1,

$$3kt^2 + 2t^2 - 6 > 0.$$

Since the denominator is positive, this gives $\Delta < 0$, so the inequality fails.

Therefore no $k > \frac{4}{3}$ is admissible. Hence the largest possible value is $\frac{4}{3}$.

Also solved by Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Tiberiu Popa, San Francisco, CA, USA; Shilong Zhuang, California, USA; Teodor Cristian Petrescu, Romania; Vidyut Sundar, Dhirubhai Ambani International School, Mumbai, India.

Undergraduate problems

U739. Evaluate

$$\int \frac{\tan 5x}{\cos^2 x} dx.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution 1 by Murad Safarli, Physics Mathematics and Informatics Lyceum, Baku, Azerbaijan and Turan Musayev, Shaki City Lyceum with Chemistry and Biology Specialisation, Shaki, Azerbaijan

Put $t = \tan x$, so that $dt = dx/\cos^2 x$. The tangent multiple-angle formula gives

$$\tan 5x = \frac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1}.$$

Consequently, setting $u = t^2$ and then $v = u - 1$, we obtain

$$\begin{aligned} \int \frac{\tan 5x}{\cos^2 x} dx &= \int \frac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1} dt \\ &= \frac{1}{2} \int \frac{u^2 - 10u + 5}{5u^2 - 10u + 1} du \\ &= \int \left(\frac{1}{10} - \frac{4v}{5(v^2 - 4/5)} - \frac{8}{25(v^2 - 4/5)} \right) dv \\ &= \frac{v}{10} - \frac{2}{5} \log |v^2 - 4/5| - \frac{2\sqrt{5}}{25} \log \left| \frac{v - 2/\sqrt{5}}{v + 2/\sqrt{5}} \right| + C. \end{aligned}$$

Here we used

$$\int \frac{dv}{v^2 - a^2} = \frac{1}{2a} \log \left| \frac{v - a}{v + a} \right| + C \quad (a > 0).$$

Substituting $v = \tan^2 x - 1$ and combining logarithms gives

$$\begin{aligned} \int \frac{\tan 5x}{\cos^2 x} dx &= \frac{\tan^2 x}{10} \\ &\quad - \left(\frac{2}{5} + \frac{2\sqrt{5}}{25} \right) \log \left| \tan^2 x - 1 - \frac{2}{\sqrt{5}} \right| \\ &\quad - \left(\frac{2}{5} - \frac{2\sqrt{5}}{25} \right) \log \left| \tan^2 x - 1 + \frac{2}{\sqrt{5}} \right| + C. \end{aligned}$$

The formula holds on each interval where $\cos x \cos 5x \neq 0$. Although the substitution $u = t^2$ need not be one-to-one through $t = 0$, the resulting expression differentiates to the integrand there as well.

Solution 2 by Mohammed Aassila, Strasbourg, France

We first rewrite the integrand as

$$I = \int \tan(5x) \sec^2 x dx.$$

Since

$$\frac{d}{dx}(\tan x) = \sec^2 x,$$

the substitution

$$t = \tan x, \quad dt = \sec^2 x dx$$

gives

$$I = \int \tan(5 \arctan t) dt.$$

Now use the quintuple-angle identity

$$\tan(5\theta) = \frac{5u - 10u^3 + u^5}{1 - 10u^2 + 5u^4}, \quad u = \tan \theta.$$

Hence

$$I = \int \frac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1} dt.$$

We perform polynomial division:

$$\frac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1} = \frac{1}{5}t + \frac{-8t^3 + \frac{24}{5}t}{5t^4 - 10t^2 + 1}.$$

Next, factor the denominator:

$$5t^4 - 10t^2 + 1 = 5(t^2 - a)(t^2 - b),$$

where

$$a = 1 + \frac{2\sqrt{5}}{5}, \quad b = 1 - \frac{2\sqrt{5}}{5}.$$

A partial-fraction decomposition yields

$$\frac{-8t^3 + \frac{24}{5}t}{5(t^2 - a)(t^2 - b)} = \frac{At}{t^2 - a} + \frac{Bt}{t^2 - b},$$

where

$$A = -\frac{4}{5} - \frac{4\sqrt{5}}{25}, \quad B = -\frac{4}{5} + \frac{4\sqrt{5}}{25}.$$

Therefore,

$$I = \int \left(\frac{1}{5}t + A \frac{t}{t^2 - a} + B \frac{t}{t^2 - b} \right) dt = \frac{1}{10}t^2 + \frac{A}{2} \ln |t^2 - a| + \frac{B}{2} \ln |t^2 - b| + C.$$

Substituting back $t = \tan x$, we obtain

$$\begin{aligned} \int \frac{\tan(5x)}{\cos^2 x} dx &= \frac{1}{10} \tan^2 x - \left(\frac{10 + 2\sqrt{5}}{25} \right) \ln \left| \tan^2 x - 1 - \frac{2\sqrt{5}}{5} \right| \\ &\quad - \left(\frac{10 - 2\sqrt{5}}{25} \right) \ln \left| \tan^2 x - 1 + \frac{2\sqrt{5}}{5} \right| + C. \end{aligned}$$

Also solved by Arkady Alt, San Jose, California, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Cortland Greindl, SUNY Brockport, NY, USA; José Luis Díaz-Barrero, Barcelona, Spain; Mihály Bencze, Brașov, Romania; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Seán M. Stewart, King Abdullah University of Science and Technology, Saudi Arabia; Sundares H. R., Shivamogga, Karnataka, India; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Daniel Pascuas, Barcelona, Spain; Etisha Sharma and Toyesh Prakash Sharma, Agra College, Agra, India; Teodor Cristian Petrescu, Romania.

U740. Let $A, B \in \mathcal{M}_2(\mathbb{R})$ be two matrices with the properties $\det(A - B) = 0$ and $A^2 + BA + B^2 = AB$. Determine $A^{2026} + B^{2026}$.

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution 1 by Theo Koupelis, Clark College, Washington, USA

The following identities are well-known for any matrices $X, Y \in \mathcal{M}_2(\mathbb{R})$:

$$\det(X + Y) + \det(X - Y) = 2\det X + 2\det Y, \quad (1)$$

$$\det X^2 = (\det X)^2, \quad (2)$$

$$\det(2XY) = 4\det X \cdot \det Y, \quad (3)$$

$$\operatorname{tr}(XY) = \operatorname{tr}(YX), \quad (4)$$

$$\operatorname{tr}(X^2) = (\operatorname{tr}X)^2 - 2 \cdot \det X, \quad (5)$$

$$X^2 - (\operatorname{tr}X) \cdot X + (\det X)I = O \quad (\text{Cayley-Hamilton theorem}). \quad (6)$$

Using (1) and the given condition $\det(A - B) = 0$ we get

$$\det(A + B) = 2(\det A + \det B). \quad (7)$$

Using (2) and (7) we get

$$\det(A + B)^2 = (\det(A + B))^2 = 4(\det A + \det B)^2. \quad (8)$$

Using the given condition $A^2 + BA + B^2 = AB$ we get

$$(A + B)^2 = A^2 + AB + BA + B^2 = 2AB,$$

and thus, from (3) and (8) we get $4(\det A + \det B)^2 = \det(2AB) = 4\det A \det B$; expanding and simplifying we get $(\det A)^2 + (\det A)(\det B) + (\det B)^2 = 0$, which is possible for the real numbers $\det A$ and $\det B$ only if

$$\det A = \det B = 0. \quad (9)$$

Using (4) and the given condition $A^2 + B^2 = AB - BA$ we get

$$\operatorname{tr}(A^2 + B^2) = \operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = 0; \quad (10)$$

using (5) and (9) we now get $\operatorname{tr}(A^2) = (\operatorname{tr}A)^2 - 2\det A = (\operatorname{tr}A)^2$, and similarly $\operatorname{tr}(B^2) = (\operatorname{tr}B)^2$. Thus, substituting the above into (10) we get

$$\operatorname{tr}(A^2 + B^2) = (\operatorname{tr}A)^2 + (\operatorname{tr}B)^2 = 0 \implies \operatorname{tr}A = \operatorname{tr}B = 0. \quad (11)$$

Using (6), (9), and (11), we get $A^2 = B^2 = O$, and therefore $A^{2026} + B^{2026} = O$, where O is the zero matrix.

Solution 2 by Teodor Cristian Petrescu, Romania

We want to prove that $\det(A) + \det(B) = 0$.

The second relation in the hypothesis is equivalent to:

$$A^2 + BA = AB - B^2,$$

$$(A + B)A = (A - B)B,$$

$$\det[(A + B)A] = \det[(A - B)B],$$

$$\det(A + B)\det(A) = \det(A - B)\det(B) = 0,$$

so $\det(A) = 0$ or $\det(A + B) = 0$.

If $\det(A + B) = 0$, knowing that $\det(A + B) + \det(A - B) = 2(\det(A) + \det(B))$, we would have that $\det(A) + \det(B) = 0$.

If $\det(A) = 0$, we go to the 2nd relation in the hypothesis yet again and we have

$$BA + B^2 = AB - A^2,$$

$$B(A + B) = A(B - A),$$

$$\det[B(A + B)] = \det[A(B - A)],$$

$$\det(B) \det(A + B) = \det(A) \det(B - A) = 0$$

So, $\det(B) = 0$ or $\det(A + B) = 0$. If $\det(B) = 0$, we indeed get that $\det(A) + \det(B) = 0 + 0 = 0$ and if $\det(A + B) = 0$, we already presented above that we would obtain $\det(A) + \det(B) = 0$. So, in the end, we indeed get our desired outcome, that $\det(A) + \det(B) = 0$, $\det(A) = -\det(B)$

We go back to the second relation in the hypothesis yet again and we obtain

$$A^2 + B^2 = AB - BA$$

$$\text{Tr}(A^2 + B^2) = \text{Tr}(AB - BA) = 0,$$

$$\text{Tr}(A^2) + \text{Tr}(B^2) = 0.$$

Knowing that $\text{Tr}(A^2) = \text{Tr}^2(A) - 2\det(A)$, $\text{Tr}(B^2) = \text{Tr}^2(B) - 2\det(B)$, we have

$$\text{Tr}^2(A) + \text{Tr}^2(B) - 2\det(A) - 2\det(B) = 0,$$

$$\text{Tr}^2(A) + \text{Tr}^2(B) = 0,$$

and since $A, B \in \mathcal{M}_2(\mathbb{R})$, we would get that the traces of those two matrices are real numbers and using the hypothesis above we get $\text{Tr}(A) = \text{Tr}(B) = 0$.

Using the Hamilton-Cayley theorem we obtain:

$$A^2 - \text{Tr}(A)A + \det(A)I_2 = 0,$$

$$B^2 - \text{Tr}(B)B + \det(B)I_2 = 0$$

,so
 $A^2 = -\det(A)I_2$ and $B^2 = -\det(B)I_2$. Since $\det(A) = -\det(B)$, we get that $A^2 = -B^2$. Raising to the 1013th power we obtain $A^{2026} = -B^{2026}$, so $A^{2026} + B^{2026} = 0$.

Solution 3 by Mihály Bencze, Braşov, Romania

Set

$$C = A - B \quad \text{and} \quad t = \text{tr } C.$$

Rearranging the second hypothesis and using the definition of C gives

$$\begin{aligned} 0 &= A^2 + BA + B^2 - AB \\ &= (A^2 - AB - BA + B^2) + 2BA \\ &= (A - B)^2 + 2BA. \end{aligned}$$

Thus

$$C^2 + 2BA = 0. \tag{1}$$

Since C is a 2×2 matrix, the Cayley–Hamilton theorem says

$$C^2 - (\text{tr } C)C + (\det C)I_2 = 0.$$

The hypothesis $\det C = 0$ therefore yields

$$C^2 = tC. \quad (2)$$

Combining (1) and (2), we obtain

$$BA = -\frac{t}{2}C = -\frac{t}{2}(A - B). \quad (3)$$

We first prove that $t = 0$. Taking determinants in (3) and using $\det C = 0$, we get

$$\det A \det B = \det(BA) = \det\left(-\frac{t}{2}C\right) = 0.$$

Hence at least one of A and B is singular.

Suppose, for a contradiction, that $t \neq 0$. Equation (3) can be written in each of the following two forms:

$$\left(B + \frac{t}{2}I_2\right)A = \frac{t}{2}B, \quad (4)$$

$$B\left(A - \frac{t}{2}I_2\right) = -\frac{t}{2}A. \quad (5)$$

If $\det A = 0$, taking determinants in (4) gives

$$0 = \left(\frac{t}{2}\right)^2 \det B,$$

so $\det B = 0$. If instead $\det B = 0$, taking determinants in (5) similarly gives $\det A = 0$. Consequently, under the assumption $t \neq 0$, both matrices are singular:

$$\det A = \det B = 0. \quad (6)$$

Now take traces in the original matrix equation. Since $\text{tr}(BA) = \text{tr}(AB)$, the mixed terms cancel, and hence

$$\text{tr}(A^2) + \text{tr}(B^2) = 0. \quad (7)$$

For every 2×2 matrix X ,

$$\text{tr}(X^2) = (\text{tr } X)^2 - 2 \det X.$$

Using (6) in (7), we find

$$(\text{tr } A)^2 + (\text{tr } B)^2 = 0.$$

Both traces are real, so they must both be zero. This would give

$$t = \text{tr}(A - B) = \text{tr } A - \text{tr } B = 0,$$

contrary to $t \neq 0$. Therefore

$$t = 0. \quad (8)$$

Equations (2), (3), and (8) now show that

$$C^2 = 0 \quad \text{and} \quad BA = 0. \quad (9)$$

We next show that both A and B are singular. If A were invertible, then $BA = 0$ would imply $B = 0$. Thus $C = A$, and $C^2 = 0$ would imply $A^2 = 0$, which is impossible for an invertible matrix. Therefore A is singular. Likewise, if B were invertible, then $BA = 0$ would imply $A = 0$; hence $C = -B$ and $C^2 = B^2 = 0$, again a contradiction. Thus

$$\det A = \det B = 0. \quad (10)$$

Applying (7) once more and using (10), we get

$$(\text{tr } A)^2 + (\text{tr } B)^2 = 0.$$

Therefore

$$\operatorname{tr} A = \operatorname{tr} B = 0.$$

The characteristic polynomials of both A and B are consequently x^2 . By the Cayley–Hamilton theorem,

$$A^2 = 0 \quad \text{and} \quad B^2 = 0.$$

In particular, every power of either matrix with exponent at least 2 vanishes. Since $2026 \geq 2$, the required value is

$$\boxed{A^{2026} + B^{2026} = 0_{2 \times 2}}.$$

Generalization and complete characterization

The preceding argument gives considerably more than the value of the single power appearing in the problem.

Generalized theorem. For $A, B \in M_2(\mathbb{R})$, the following statements are equivalent:

1.

$$\det(A - B) = 0 \quad \text{and} \quad A^2 + BA + B^2 = AB;$$

2. there exist $a, b \in \mathbb{R}$ and a matrix $N \in M_2(\mathbb{R})$ satisfying $N^2 = 0$ such that

$$A = aN, \quad B = bN.$$

Consequently,

$$A^n + B^n = 0_{2 \times 2} \quad \text{for every integer } n \geq 2. \quad (11)$$

Thus the exponent 2026 can be replaced by *any* integer at least 2.

Proof of the characterization. We have already proved from statement 1 that

$$A^2 = B^2 = BA = 0.$$

Substitution into the original equation also gives

$$AB = 0. \quad (12)$$

If one of A, B is zero, the desired representation is immediate: use the other matrix as N . We may therefore suppose that both are nonzero.

A nonzero square-zero endomorphism of the two-dimensional space $\mathbb{R}^2$ has rank 1. Moreover,

$$\operatorname{im} A \subseteq \ker A$$

because $A^2 = 0$. Both spaces have dimension 1, so

$$\operatorname{im} A = \ker A.$$

Similarly,

$$\operatorname{im} B = \ker B.$$

On the other hand, (12) and $BA = 0$ imply

$$\operatorname{im} B \subseteq \ker A, \quad \operatorname{im} A \subseteq \ker B.$$

Since all four spaces are one-dimensional, they are the same line; denote it by L . Thus both A and B vanish on L and map $\mathbb{R}^2$ into L .

The vector space of all linear maps

$$\mathbb{R}^2/L \longrightarrow L$$

is one-dimensional. It follows that A and B are linearly dependent. Hence $B = \lambda A$ for some $\lambda \in \mathbb{R}$. Taking $N = A$, $a = 1$, and $b = \lambda$ proves statement 2.

Conversely, suppose $A = aN$ and $B = bN$, where $N^2 = 0$. Then every quadratic product involving A and B is zero, and

$$\det(A - B) = \det((a - b)N) = 0.$$

Thus statement 1 holds. This proves the equivalence and also proves (11).

There is an accompanying polynomial version. If $p \in \mathbb{R}[x]$, then $A^2 = B^2 = 0$ gives

$$p(A) = p(0)I_2 + p'(0)A, \quad p(B) = p(0)I_2 + p'(0)B.$$

Therefore every pair satisfying the hypotheses obeys

$$p(A) + p(B) = 2p(0)I_2 + p'(0)(A + B). \quad (13)$$

Taking $p(x) = x^n$ in (13) recovers (11).

Remark on the dimension. The complete characterization above is special to 2×2 matrices. For example, let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \in M_3(\mathbb{R}).$$

Here the displayed block matrices have block sizes $2 + 1$. Since $J^2 = -I_2$, direct calculation gives

$$\det(\tilde{A} - \tilde{B}) = 0 \quad \text{and} \quad \tilde{A}^2 + \tilde{B}\tilde{A} + \tilde{B}^2 = \tilde{A}\tilde{B}.$$

Nevertheless, neither $\tilde{A}^2$ nor $\tilde{B}^2$ is zero. Thus one cannot extend the square-zero conclusion to arbitrary matrix sizes without adding further assumptions.

Also solved by Adam Deli, IIT Bombay, Mumbai, India; Bogdan Lataianu, Canada; Daniel Văcaru, National Economic College, Pitești, Romania; José Luis Díaz-Barrero, Barcelona, Spain; Mohammed Aassila, Strasbourg, France; Rovsen Pirgulyev, Sumqayıt, Azerbaijan; Seán M. Stewart, King Abdullah University of Science and Technology, Saudi Arabia; Tiberiu Popa, San Francisco, CA, USA; Adam John Frederickson, Utah Valley University, UT, USA.

U741. Let $\{a_n\}_{n \geq 1}$ be a sequence such that $|a_{n+1}| \leq c|a_n|$, $c > 0$. If

$$\sum_{n=1}^{\infty} |a_n(\cos(3nx_0) + 7\cos(nx_0) - 2\cos(2nx_0) - 2)| < +\infty.$$

for some $x_0 \in [0, 2\pi)$, prove that $\sum_{n=1}^{\infty} |a_n| < +\infty$.

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

Solution by Mohammed Aassila, Strasbourg, France

Define, for $t \in \mathbb{R}$, $f(t) := \cos(3t) + 7\cos(t) - 2\cos(2t) - 2$. Let $x = \cos t$. Using the identities $\cos(3t) = 4x^3 - 3x$ and $\cos(2t) = 2x^2 - 1$, we get

$$f(t) = (4x^3 - 3x) + 7x - 2(2x^2 - 1) - 2 = 4x^3 - 4x^2 + 4x = 4x(x^2 - x + 1).$$

Since $\frac{3}{4} \leq x^2 - x + 1 \leq 3$, we deduce that

$$3|\cos t| \leq |f(t)| \leq 12|\cos t| \quad \text{for all } t \in \mathbb{R}. \quad (1)$$

By the lower bound in (1), the hypothesis $\sum_{n=1}^{\infty} |a_n| |f(nx_0)| < \infty$ implies

$$\sum_{n=1}^{\infty} |a_n| |\cos(nx_0)| < \infty. \quad (2)$$

Thus it suffices to prove the following statement.

If $\{a_n\}_{n \geq 1}$ satisfies $|a_{n+1}| \leq c|a_n|$ for some fixed $c > 0$ and for all $n \geq 1$, and (2) holds, then

$$\sum_{n=1}^{\infty} |a_n| < \infty.$$

Lemma 1. *For every $n \geq 1$ and every integer $r \geq 0$,*

$$|a_{n+r}| \leq c^r |a_n|.$$

Consequently, for any two indices $n < m$,

$$|a_m| \leq c^{m-n} |a_n| \leq \max(1, c)^{m-n} |a_n|.$$

Proof. Immediate by induction on r , applying $|a_{k+1}| \leq c|a_k|$ repeatedly:

$$|a_{n+r}| \leq c|a_{n+r-1}| \leq c^2|a_{n+r-2}| \leq \cdots \leq c^r|a_n|.$$

The final inequality $c^r \leq \max(1, c)^r$ is trivial since $\max(1, c) \geq c$ and $r \geq 0$. □

Claim. *There exist an integer $N \geq 1$ and a constant $\delta > 0$, depending only on x_0 , such that for every integer $m \geq 0$,*

$$\max_{0 \leq j < N} |\cos((m+j)x_0)| \geq \delta. \quad (3)$$

Proof. Case 1: $\sin x_0 = 0$. Then $x_0 \in \{0, \pi\}$ and $|\cos(nx_0)| = 1$ for every n . Take $N = 1$, $\delta = 1$.

Case 2: $\sin x_0 \neq 0$. For any integer m and any $N \geq 1$, using the geometric sum formula,

$$\left| \sum_{j=0}^{N-1} \cos(2(m+j)x_0) \right| = \left| \operatorname{Re} \left(e^{2imx_0} \sum_{j=0}^{N-1} e^{2ijx_0} \right) \right| \leq \left| \frac{e^{2iNx_0} - 1}{e^{2ix_0} - 1} \right| = \frac{|\sin(Nx_0)|}{|\sin x_0|} \leq \frac{1}{|\sin x_0|}.$$

Using $\cos^2 \alpha = \frac{1}{2}(1 + \cos 2\alpha)$,

$$\sum_{j=0}^{N-1} \cos^2((m+j)x_0) = \frac{N}{2} + \frac{1}{2} \sum_{j=0}^{N-1} \cos(2(m+j)x_0) \geq \frac{N}{2} - \frac{1}{2|\sin x_0|}.$$

Choose $N = \left\lceil \frac{2}{|\sin x_0|} \right\rceil$. Then $\frac{1}{|\sin x_0|} \leq \frac{N}{2}$, so

$$\frac{N}{2} - \frac{1}{2|\sin x_0|} \geq \frac{N}{2} - \frac{N}{4} = \frac{N}{4}.$$

Hence

$$\sum_{j=0}^{N-1} \cos^2((m+j)x_0) \geq \frac{N}{4}.$$

Since this is an average bound over N terms, at least one index $j \in \{0, \dots, N-1\}$ satisfies

$$\cos^2((m+j)x_0) \geq \frac{1}{4}, \quad \text{i.e.} \quad |\cos((m+j)x_0)| \geq \frac{1}{2}.$$

This proves (3) with $\delta = 1/2$ (and N as chosen, which depends only on x_0 , not on m). $\square$

Fix N and δ as in the Claim above, and set $K := \max(1, c)^{2N-1}$, a finite constant depending only on c and N (hence, ultimately, only on c and x_0). Partition $\{1, 2, 3, \dots\}$ into consecutive blocks of length N :

$$B_k := \{kN + 1, kN + 2, \dots, (k+1)N\}, \quad k = 0, 1, 2, \dots$$

By the Claim applied with $m = kN + 1$ (which starts the block B_k), each block B_k contains an index $n_k \in B_k$ such that

$$|\cos(n_k x_0)| \geq \delta. \quad (4)$$

Now fix $k \geq 1$ and let $m \in B_k$, so $m \geq kN + 1$. Since $n_{k-1} \in B_{k-1} = \{(k-1)N + 1, \dots, kN\}$, we have

$$n_{k-1} \leq kN < m \leq (k+1)N,$$

so that

$$1 \leq m - n_{k-1} \leq (k+1)N - ((k-1)N + 1) = 2N - 1.$$

By Lemma 1 applied with $n = n_{k-1}$ and $r = m - n_{k-1} \in \{1, \dots, 2N - 1\}$,

$$|a_m| \leq \max(1, c)^{m-n_{k-1}} |a_{n_{k-1}}| \leq \max(1, c)^{2N-1} |a_{n_{k-1}}| = K |a_{n_{k-1}}|. \quad (5)$$

Summing (5) over the N elements of B_k , and using (4) for the index n_{k-1} :

$$\sum_{m \in B_k} |a_m| \leq NK |a_{n_{k-1}}| \leq \frac{NK}{\delta} |a_{n_{k-1}}| |\cos(n_{k-1} x_0)|. \quad (6)$$

Since the blocks $B_0, B_1, B_2, \dots$ are pairwise disjoint, the chosen indices $n_0, n_1, n_2, \dots$ (with $n_k \in B_k$) are pairwise distinct. Summing (6) over $k = 1, 2, 3, \dots$:

$$\sum_{k=1}^{\infty} \sum_{m \in B_k} |a_m| \leq \frac{NK}{\delta} \sum_{k=1}^{\infty} |a_{n_{k-1}}| |\cos(n_{k-1} x_0)| \leq \frac{NK}{\delta} \sum_{n=1}^{\infty} |a_n| |\cos(n x_0)| < \infty,$$

where the last inequality is exactly (2), and the middle inequality holds because the n_{k-1} are distinct positive integers, so the sum over k is a sub-series of the full series $\sum_{n \geq 1} |a_n| |\cos(nx_0)|$ (all terms of which are non-negative).

Since $\bigcup_{k \geq 1} B_k = \{N+1, N+2, N+3, \dots\}$, this shows

$$\sum_{n=N+1}^{\infty} |a_n| < \infty.$$

Finally, the finitely many remaining terms $|a_1|, |a_2|, \dots, |a_N|$ contribute a finite sum, so

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^N |a_n| + \sum_{n=N+1}^{\infty} |a_n| < \infty.$$

Including the (irrelevant, finite) term $|a_0|$ if present, we conclude

$$\sum_{n=0}^{\infty} |a_n| < \infty,$$

as required.

Also solved by Theo Koupelis, Clark College, Washington, USA.

U742. Let $f \in C^3([0, 1])$ satisfy $f'(0) = f'(1) = 0$ and $f'''(x) > 0$ for all $x \in (0, 1)$. Prove that there exists exactly one point $c \in (0, 1)$ such that $f(c) - f(0) = cf'(c)$ and $f''(c) > 0$.

Proposed by Alessandro Ventullo, Milan, Italy

Solution 1 by Daniel Pascuas, Barcelona, Spain

Since $f' \in C^2([0, 1])$ satisfies that $f'(0) = f'(1) = 0$, Rolle's theorem shows that there is $x_0 \in (0, 1)$ such that $f''(x_0) = 0$. The hypothesis $f'''(x) > 0$ for all $x \in (0, 1)$ implies that f'' is increasing on $(0, 1)$. It follows that $f''(x) < f''(x_0) = 0$, for all $x \in (0, x_0)$, and $f''(x) > f''(x_0) = 0$, for all $x \in (x_0, 1)$. Therefore f' is decreasing on $[0, x_0]$ and increasing on $[x_0, 1]$ (recall that f' is continuous on $[0, 1]$ and differentiable on $(0, 1)$), so $f'(x) < f'(0) = 0$, for all $x \in [0, x_0)$, and $f'(x) < f'(1) = 0$, for all $x \in [x_0, 1)$, that is, $f'(x) < 0$, for all $x \in (0, 1)$.

Now consider the function $g(x) = f(x) - f(0) - xf'(x)$, $x \in [0, 1]$. We want to show that there is a unique point $c \in (0, 1)$ such that $g(c) = 0$ and $f''(c) > 0$, that is, there is a unique point $c \in (x_0, 1)$ such that $g(c) = 0$.

Since $f \in C^3([0, 1])$, it is clear that $g \in C^2([0, 1])$. Moreover, $g'(x) = -xf''(x)$, for all $x \in (0, 1)$, and therefore $g'(x) > 0$, for all $x \in (0, x_0)$, and $g'(x) < 0$, for all $x \in (x_0, 1)$. As a consequence, g is increasing on $[0, x_0]$ and decreasing on $[x_0, 1]$, so $0 = g(0) < g(x_0)$ and g is injective on $[x_0, 1]$. On the other hand, Lagrange's mean value theorem shows that there is a point $\xi \in (0, 1)$ such that

$$g(1) = f(1) - f(0) - f'(1) = f(1) - f(0) = f'(\xi),$$

and so $g(1) < 0$. By Bolzano's theorem, there is a point $c \in (x_0, 1)$ such that $g(c) = 0$. The injectivity of g on $(x_0, 1)$ implies that this point c is unique. And we are done.

Solution 2 by Mihály Bencze, Braşov, Romania

The particular interval $[0, 1]$, the value 0 of the endpoint derivatives, and even the existence of f''' are not essential. The following stronger result contains the original problem as a special case.

Generalized theorem. Let $\alpha < \beta$, and let $f \in C^2([\alpha, \beta])$. Suppose that

$$f'(\alpha) = f'(\beta)$$

and that f'' is strictly increasing on $[\alpha, \beta]$. Then there exists exactly one $c \in (\alpha, \beta)$ such that

$$f(c) - f(\alpha) = (c - \alpha)f'(c) \quad \text{and} \quad f''(c) > 0.$$

Proof. Set

$$m = f'(\alpha) = f'(\beta).$$

By the fundamental theorem of calculus,

$$\int_{\alpha}^{\beta} f''(x) dx = f'(\beta) - f'(\alpha) = 0.$$

Because f'' is continuous and strictly increasing, this identity implies

$$f''(\alpha) < 0 < f''(\beta).$$

Therefore there is a unique $a \in (\alpha, \beta)$ such that

$$f''(a) = 0,$$

with $f'' < 0$ on $[\alpha, a)$ and $f'' > 0$ on $(a, \beta]$.

We next show that

$$f'(x) < m \quad (\alpha < x < \beta).$$

Fix such an x . The mean value theorem, applied to f' on $[\alpha, x]$ and $[x, \beta]$, gives $\xi \in (\alpha, x)$ and $\eta \in (x, \beta)$ such that

$$\frac{f'(x) - m}{x - \alpha} = f''(\xi), \quad \frac{m - f'(x)}{\beta - x} = f''(\eta).$$

Since $\xi < \eta$ and f'' is strictly increasing,

$$\frac{f'(x) - m}{x - \alpha} < \frac{m - f'(x)}{\beta - x}.$$

Multiplying by the positive number $(x - \alpha)(\beta - x)$ and simplifying yields

$$(\beta - \alpha)(f'(x) - m) < 0,$$

which proves $f'(x) < m$.

Now define

$$\Psi(x) = f(x) - f(\alpha) - (x - \alpha)f'(x), \quad \alpha \leq x \leq \beta.$$

Then

$$\Psi'(x) = -(x - \alpha)f''(x).$$

Thus Ψ is strictly increasing on $[\alpha, a]$ and strictly decreasing on $[a, \beta]$. Since $\Psi(\alpha) = 0$, it follows that

$$\Psi(a) > 0.$$

At the other endpoint,

$$\begin{aligned} \Psi(\beta) &= f(\beta) - f(\alpha) - (\beta - \alpha)f'(\beta) \\ &= \int_{\alpha}^{\beta} (f'(x) - m) dx < 0, \end{aligned}$$

because $f'(x) < m$ throughout (α, β) .

The intermediate value theorem now gives a zero $c \in (a, \beta)$ of Ψ . Its uniqueness follows from the strict decrease of Ψ on $[a, \beta]$; moreover, $\Psi > 0$ on $(\alpha, a]$, so there are no earlier zeros. Finally, $c > a$ implies $f''(c) > 0$. Hence

$$\boxed{f(c) - f(\alpha) = (c - \alpha)f'(c), \quad f''(c) > 0}.$$

The original problem follows by taking $\alpha = 0$, $\beta = 1$, and $m = 0$. Its assumption $f'''(x) > 0$ guarantees precisely the strict increase of f'' used in the generalized theorem.

Also solved by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Corneliu Mănescu-Avram, Ploiești, Romania; José Luis Díaz-Barrero, Barcelona, Spain; Mohammed Aassila, Strasbourg, France; Rousen Pirgulyev, Sumqayit, Azerbaijan; Theo Koupelis, Clark College, Washington, USA; Tiberiu Popa, San Francisco, CA, USA; Teodor Cristian Petrescu, Romania.

U743. Let $n \geq 3$ and $1 \leq k \leq n-1$. If $a_1, a_2, \dots, a_n$ are positive real numbers such that

$$a_1 \geq a_2 \geq \dots \geq a_k \geq 1 \geq a_{k+1} \geq \dots \geq a_n$$

and $a_1 a_2 \dots a_n = 1$, prove that

$$\frac{1}{ka_1 + 1} + \frac{1}{ka_2 + 1} + \dots + \frac{1}{ka_n + 1} \geq \frac{n}{k+1}.$$

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania

Solution by the author

Let $z = (a_1 a_2 \dots a_k)^{1/k}$. Since the function

$$f(u) = \frac{1}{ke^u + 1}$$

is convex for $u \geq 0$, by Jensen's inequality

$$\frac{1}{ka_1 + 1} + \frac{1}{ka_2 + 1} + \dots + \frac{1}{ka_k + 1} \geq \frac{k}{kz + 1}.$$

So, it suffices to show that

$$\frac{k}{kz + 1} + \frac{1}{ka_{k+1} + 1} + \dots + \frac{1}{ka_n + 1} \geq \frac{n}{k+1}.$$

Let $b_{k+1}, \dots, b_n$ be defined by

$$a_i b_i^k = 1, \quad i = k+1, \dots, n.$$

Since

$$z^k = a_1 a_2 \dots a_k = \frac{1}{a_{k+1} a_{k+2} \dots a_n} \geq \frac{1}{a_n} = b_n^k,$$

we have

$$1 \leq b_{k+1} \leq b_{k+2} \leq \dots \leq b_n \leq z.$$

By Lemma below, we have

$$\frac{1}{ka_i + 1} \geq 1 - \frac{k}{kb_i + 1}, \quad i = k+1, \dots, n.$$

By adding these inequalities, we get

$$\frac{1}{ka_{k+1} + 1} + \dots + \frac{1}{ka_n + 1} \geq n - k - k \left(\frac{1}{kb_{k+1} + 1} + \dots + \frac{1}{kb_n + 1} \right).$$

So, it suffices to show that

$$\frac{k}{kz + 1} + n - k - k \left(\frac{1}{kb_{k+1} + 1} + \dots + \frac{1}{kb_n + 1} \right) \geq \frac{n}{k+1},$$

which is equivalent to

$$\frac{1}{kz + 1} + \frac{n - k - 1}{k + 1} \geq \frac{1}{kb_{k+1} + 1} + \dots + \frac{1}{kb_n + 1}.$$

Let $z = e^x$ and $b_i = e^{c_i}$ for $i = k+1, k+2, \dots, n$. From

$$a_i b_i^k = 1 \quad (i = k+1, \dots, n)$$

and

$$z^k = a_1 a_2 \dots a_k = \frac{1}{a_{k+1} a_{k+2} \dots a_n},$$

we obtain

$$(b_{k+1}b_{k+2}\cdots b_n)^k = z^k,$$

and hence

$$b_{k+1}b_{k+2}\cdots b_n = z.$$

Therefore

$$x = c_{k+1} + c_{k+2} + \cdots + c_n.$$

Moreover,

$$x \geq c_n \geq c_{n-1} \geq \cdots \geq c_{k+1} \geq 0.$$

Thus the inequality becomes

$$f(x) + (n - k - 1)f(0) \geq f(c_{k+1}) + f(c_{k+2}) + \cdots + f(c_n).$$

Since f is convex on $[0, \infty)$ and

$$(x, 0, \dots, 0)$$

majorizes

$$(c_n, c_{n-1}, \dots, c_{k+1}),$$

the desired inequality follows from Karamata's inequality.

Thus, the proof is finished.

If $k \geq 2$, equality holds if and only if $a_1 = a_2 = \cdots = a_n = 1$.

If $k = 1$, equality holds precisely for $(a_1, a_2, \dots, a_n) = (t, 1, \dots, 1, \frac{1}{t})$, where $t \geq 1$.

Lemma. Let $k \geq 1$ be an integer numbers. If a and b are positive numbers such that $ab^k = 1$, then

$$\frac{1}{ka+1} + \frac{k}{kb+1} \geq 1.$$

Proof. For $k = 1$, the inequality is an identity, and for $k \geq 2$ we have

$$\frac{1}{ka+1} + \frac{k}{kb+1} - 1 = \frac{b^k}{b^k+k} + \frac{k}{kb+1} - 1 = \frac{k(b^k - kb + k - 1)}{(b^k+k)(kb+1)}.$$

Furthermore,

$$b^k - kb + k - 1 = (b-1)[(b^{k-1}-1) + (b^{k-2}-1) + \cdots + (b-1)],$$

and hence

$$\frac{1}{ka+1} + \frac{k}{kb+1} - 1 = \frac{k(b-1)[(b^{k-1}-1) + (b^{k-2}-1) + \cdots + (b-1)]}{(b^k+k)(kb+1)} \geq 0.$$

Remark. Note that k is the least positive value of m such that

$$\frac{1}{ma_1+1} + \frac{1}{ma_2+1} + \cdots + \frac{1}{ma_n+1} \geq \frac{n}{m+1}$$

for any $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1 \geq a_{k+1} \geq \cdots \geq a_n > 0$ with $a_1 a_2 \cdots a_n = 1$.

For $a_1 = a_2 = \cdots = a_k := a$ and $a_{k+1} = a_{k+2} = \cdots = a_{n-1} = 1$, the constraint becomes $a^k a_n = 1$ with $a \geq 1$, and the inequality can be written as follows:

$$\begin{aligned} \frac{k}{ma+1} + \frac{1}{ma_n+1} &\geq \frac{k+1}{m+1}, \\ \frac{k}{ma+1} + \frac{a^k}{a^k+m} &\geq \frac{k+1}{m+1}. \end{aligned}$$

For $a \rightarrow \infty$, we get $1 \geq \frac{k+1}{m+1}$, hence $m \geq k$.

Also solved by Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Theo Koupelis, Clark College, Washington, USA; Tiberiu Popa, San Francisco, CA, USA.

U744. Let $x > 0$. Evaluate

$$\frac{1 \cdot \binom{2n}{0}}{x} - \frac{2^2 \cdot \binom{2n}{1}}{x+1} + \frac{3^2 \cdot \binom{2n}{2}}{x+2} - \frac{4^2 \cdot \binom{2n}{3}}{x+3} + \dots + \frac{(2n+1)^2 \cdot \binom{2n}{2n}}{x+2n}.$$

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

Solution 1 by Murad Safarli, Physics Mathematics and Informatics Lyceum, Baku, Azerbaijan and Turan Musayev, Shaki City Lyceum with Chemistry and Biology Specialisation, Shaki, Azerbaijan

Put $m = 2n \geq 2$. Polynomial division gives

$$\frac{(j+1)^2}{x+j} = j+2-x + \frac{(x-1)^2}{x+j}.$$

The binomial theorem implies

$$\sum_{j=0}^m (-1)^j \binom{m}{j} = 0, \quad \sum_{j=0}^m (-1)^j \binom{m}{j} j = -m \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} = 0.$$

It follows that

$$S = (x-1)^2 \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{x+j}.$$

We now use the partial-fraction identity

$$\frac{m!}{x(x+1) \cdots (x+m)} = \sum_{j=0}^m \frac{(-1)^j \binom{m}{j}}{x+j}.$$

Indeed, the coefficient of $1/(x+j)$ on the left is

$$\frac{m!}{\prod_{\substack{0 \leq \ell \leq m \\ \ell \neq j}} (\ell-j)} = \frac{m!}{(-1)^j j! (m-j)!} = (-1)^j \binom{m}{j}.$$

Therefore

$$S = \frac{(2n)!(x-1)^2}{x(x+1) \cdots (x+2n)}.$$

If $n = 0$ is allowed, the original sum is simply $1/x$; the displayed formula is for $n \geq 1$.

Solution 2 by Tiberiu Popa, San Francisco, CA, USA

Set

$$f_n(x) = \sum_{k=0}^{2n} \frac{(-1)^k (k+1)^2 \binom{2n}{k}}{x+k} = \frac{p_n(x)}{q_n(x)}, \quad q_n(x) = \prod_{i=0}^{2n} (x+i),$$

where

$$p_n(x) = \sum_{k=0}^{2n} (-1)^k (k+1)^2 \binom{2n}{k} \prod_{\substack{0 \leq i \leq 2n \\ i \neq k}} (x+i).$$

Clearly, $\deg p_n \leq 2n$. For $j = 0, 1, \dots, 2n$, substituting $x = -j$ annihilates every term in the last sum except the one with $k = j$. Hence

$$\begin{aligned} p_n(-j) &= (-1)^j (j+1)^2 \binom{2n}{j} \prod_{\substack{0 \leq i \leq 2n \\ i \neq j}} (i-j) \\ &= (-1)^j (j+1)^2 \binom{2n}{j} ((-1)^j j! (2n-j)!) \\ &= (j+1)^2 (2n)! = (2n)! (-j-1)^2. \end{aligned}$$

Thus the polynomials $p_n(x)$ and $(2n)!(x-1)^2$, both of degree at most $2n$, agree at the $2n+1$ distinct points $0, -1, \dots, -2n$. Therefore

$$\boxed{\sum_{k=0}^{2n} \frac{(-1)^k (k+1)^2 \binom{2n}{k}}{x+k} = \frac{(2n)!(x-1)^2}{\prod_{i=0}^{2n} (x+i)}}.$$

Generalization. The same argument gives, for every polynomial P with $\deg P \leq 2n$,

$$\sum_{k=0}^{2n} \frac{(-1)^k \binom{2n}{k} P(k)}{x+k} = \frac{(2n)!P(-x)}{\prod_{i=0}^{2n} (x+i)}.$$

For an arbitrary function $f(k)$, the same formula holds after replacing P on the right by the unique polynomial of degree at most $2n$ which agrees with f at $0, 1, \dots, 2n$.

Also solved by Adam Deli, Devis Alvarado, UNAH and UPNFM, Tegucigalpa, Honduras; Kousik Sett, India; Mihály Bencze, Braşov, Romania; Mohammed Aassila, Strasbourg, France; Nicuşor Zlota, “Traian Vuia” Technical College, Focşani, Romania; Seán M. Stewart, King Abdullah University of Science and Technology, Saudi Arabia; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Etisha Sharma and Toyesh Prakash Sharma, Agra College, Agra, India.

Olympiad problems

O739. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence defined recursively as follows: $a_1 = 1$ and $a_{n+1} = a_n + \left\lfloor \sqrt{a_n} + \frac{1}{2} \right\rfloor$. Find $n \in \mathbb{N}$ for which both solutions of the equation $x^2 + 2x - a_n = 0$ are integers.

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution 1 by Mohammed Aassila, Strasbourg, France

The roots of the quadratic equation are $-1 \pm \sqrt{1 + a_n}$. Hence both roots are integers if and only if $1 + a_n = m^2$ for some integer $m \geq 1$, that is,

$$a_n = m^2 - 1.$$

It therefore suffices to determine all terms of the sequence of the form $m^2 - 1$.

For each integer $k \geq 1$, define

$$I_k = \{a \in \mathbb{N} : k^2 - k + 1 \leq a \leq k^2 + k\}.$$

It is immediate that

$$\left\lfloor \sqrt{a} + \frac{1}{2} \right\rfloor = k \quad \text{if and only if} \quad a \in I_k.$$

We prove by induction on k that, for every integer $k \geq 1$

$$a_{2k-1} = k^2 - k + 1, \quad a_{2k} = k^2 + 1.$$

Base case. For $k = 1$, $I_1 = \{1, 2\}$. Since $a_1 = 1$, we obtain $a_2 = a_1 + 1 = 2$, so the assertion holds.

Inductive step. Assume that $a_{2k-1} = k^2 - k + 1$, $a_{2k} = k^2 + 1$ for some integer $k \geq 1$. Since $k^2 + 1 \leq k^2 + k$, we have

$$a_{2k} = k^2 + 1 \in I_k.$$

Therefore,

$$a_{2k+1} = a_{2k} + k = k^2 + k + 1 = (k+1)^2 - (k+1) + 1,$$

which is precisely the first element of I_{k+1} . Hence $a_{2k+1} \in I_{k+1}$, and the next increment equals $k+1$. Thus

$$a_{2k+2} = a_{2k+1} + (k+1) = k^2 + 2k + 2 = (k+1)^2 + 1.$$

This completes the induction.

Case 1. Even indices. Let $n = 2k$. Then $a_n + 1 = k^2 + 2$. Suppose that $k^2 + 2 = m^2$. Then $m^2 - k^2 = 2$, so $(m-k)(m+k) = 2$. The only positive factorization of 2 is $1 \cdot 2$, which yields $m - k = 1$, $m + k = 2$, and hence $m = \frac{3}{2}$, a contradiction. Therefore no even index satisfies the required condition.

Case 2. Odd indices. Let $n = 2k - 1$. Then $a_n + 1 = k^2 - k + 2$.

For $k = 1$, we obtain 2, which is not a perfect square.

For $k = 2$, $k^2 - k + 2 = 4 = 2^2$, so $n = 3$ is a solution.

Now assume that $k \geq 3$. Since $k^2 - k + 2 - (k-1)^2 = k+1 > 0$, and $k^2 - (k^2 - k + 2) = k-2 > 0$, it follows that $(k-1)^2 < k^2 - k + 2 < k^2$. Hence $k^2 - k + 2$ lies strictly between two consecutive perfect squares and therefore cannot itself be a perfect square.

Consequently, the only natural number n satisfying the required condition is $n = 3$.

Solution 2 by Theo Koupelis, Clark College, Washington, USA

From the given conditions we get that the sequence $(a_n)_{n \in \mathbb{N}}$ is composed of positive and increasing integers. The first few terms of the sequence are $a_2 = 2$, $a_3 = 3$, $a_4 = 5$, $a_5 = 7$, $a_6 = 10$, $a_7 = 13$, and $a_8 = 17$.

There seems to be a pattern between the odd and even terms of the sequence; namely, $a_{2m} = m^2 + 1$, and $a_{2m-1} = m^2 - m + 1$ for integers $m \geq 1$. We show this by induction. We have

$$\begin{aligned} a_{2m} &= a_{2m-1} + \left\lfloor \sqrt{a_{2m-1}} + \frac{1}{2} \right\rfloor = m^2 - m + 1 + \left\lfloor \sqrt{m^2 - m + 1} + \frac{1}{2} \right\rfloor \\ &= m^2 - m + 1 + m = m^2 + 1, \end{aligned}$$

because $m - \frac{1}{2} < \sqrt{m^2 - m + 1} \leq m$ for any $m \geq 1$. Also,

$$\begin{aligned} a_{2m+1} &= a_{2m} + \left\lfloor \sqrt{a_{2m}} + \frac{1}{2} \right\rfloor = m^2 + 1 + \left\lfloor \sqrt{m^2 + 1} + \frac{1}{2} \right\rfloor \\ &= m^2 + m + 1 = (m+1)^2 - (m+1) + 1, \end{aligned}$$

because $m < \sqrt{m^2 + 1} < m + \frac{1}{2}$ for any $m \geq 1$.

The solutions of the equation $x^2 + 2x - a_n = 0$ are $x_{1,2} = -1 \pm \sqrt{a_n + 1}$; for both solutions to be integers, we must have $a_n = k^2 - 1$, where $k \geq 2$ is a positive integer, for some positive integer n . Thus, if $n = 2m$, we are looking for a value m such that

$$k^2 - 1 = m^2 + 1 \iff (k-m)(k+m) = 2 \iff k-m = 1 \text{ and } k+m = 2$$

which is not possible on the integers. If $n = 2m - 1$, we are looking for a value m such that

$$\begin{aligned} k^2 - 1 &= m^2 - m + 1 \iff 4k^2 - 7 = 4m^2 - 4m + 1 = (2m-1)^2 \\ &\iff (2k-2m+1)(2k+2m-1) = 7 \\ &\iff 2k-2m+1 = 1 \text{ and } 2k+2m-1 = 7. \end{aligned}$$

Therefore, $(k, m) = (2, 2)$ is the only solution. Thus, only for $n = 3$, and the term $a_3 = 3$, we get that both solutions of the equation $x^2 + 2x - 3 = 0 \iff (x-1)(x+3) = 0$ are integers.

Also solved by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Corneliu Mănescu-Avram, Ploiești, Romania; José Luis Díaz-Barrero, Barcelona, Spain; Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Rousen Pirgulyev, Sumqayit, Azerbaijan; The Ring Lords, Baden-Württemberg, Germany; Anderson Torres, Sao Paulo, Brazil; Saarang Agarwal, AMMOC; Teodor Cristian Petrescu, Romania.

O740. Find all pairs of positive integers (m, n) satisfying

$$3^m - 1 = 2^n \cdot m.$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution 1 by the author

Take 2-adic valuations of both sides:

$$v_2(3^m - 1) = v_2(2^n \cdot m) = n + v_2(m).$$

By the Lifting the Exponent Lemma

$$v_2(3^m - 1) = \begin{cases} 1 & \text{if } m \text{ is odd,} \\ 2 + v_2(m) & \text{if } m \text{ is even.} \end{cases}$$

(i) If m is odd, then $1 = n + v_2(m) = n$, so $n = 1$. The equation becomes

$$3^m - 1 = 2m.$$

Clearly, $m = 1$ is a solution. For $m \geq 3$, by Bernoulli's Inequality we have

$$3^m - 1 = (1 + 2)^m - 1 > 1 + 2m - 1 = 2m,$$

so no further solutions. Thus $(m, n) = (1, 1)$ is the only solution with m odd.

(ii) If m is even, then

$$2 + v_2(m) = v_2(3^m - 1) = n + v_2(m),$$

so $n = 2$. Plugging back, the equation becomes

$$3^m - 1 = 4m.$$

Clearly, $m = 2$ is a solution. For $m \geq 4$, by Bernoulli's Inequality we have

$$3^m - 1 = (3^{\frac{m}{2}} - 1)(3^{\frac{m}{2}} + 1) > \left(1 + 2 \cdot \frac{m}{2} - 1\right) \cdot 4 = 4m,$$

so no further solutions.

Combining both cases, the only solutions are $(m, n) \in \{(1, 1), (2, 2)\}$.

Solution 2 by Mihály Bencze, Braşov, Romania

The same method completely solves the equation when the base 3 is replaced by an arbitrary integer greater than 1.

Theorem 1. *Let $a > 1$, m , and n be positive integers. Then*

$$a^m - 1 = 2^n m$$

if and only if

$$\boxed{(a, m, n) = (2^n + 1, 1, n) \quad (n \geq 1)}$$

or

$$\boxed{(a, m, n) = (3, 2, 2)}.$$

Proof. If a is even, then $a^m - 1$ is odd, whereas $2^n m$ is even. Therefore a must be odd.

For odd a , the same factorization argument used in the lemma gives the more general valuation formula

$$v_2(a^m - 1) = \begin{cases} v_2(a - 1), & m \text{ odd}, \\ v_2(a^2 - 1) + v_2(m) - 1, & m \text{ even}. \end{cases} \quad (1)$$

Indeed, when m is odd, the quotient

$$\frac{a^m - 1}{a - 1} = 1 + a + \cdots + a^{m-1}$$

is a sum of an odd number of odd terms and is therefore odd. When $m = 2^s q$ with $s \geq 1$ and q odd, the quotient $(a^m - 1)/(a^2 - 1)$ is again odd. Starting with $a^2 - 1$, each subsequent factorization

$$a^{2^j} - 1 = (a^{2^{j-1}} - 1)(a^{2^{j-1}} + 1), \quad j \geq 2,$$

adds exactly one to the valuation, because $a^{2^{j-1}} + 1 \equiv 2 \pmod{8}$. This proves (1).

Suppose first that m is odd. Set $t = v_2(a - 1)$ and write

$$a - 1 = 2^t u,$$

where u is odd. Comparing the valuations of the two sides by (1) gives $n = t$. After factoring and cancelling 2^t , the original equation becomes

$$u(1 + a + a^2 + \cdots + a^{m-1}) = m. \quad (2)$$

If $m > 1$, the sum in parentheses consists of m positive terms and is strictly greater than m , since $a > 1$. This contradicts (2). Thus $m = 1$, and (2) then gives $u = 1$. Consequently

$$a - 1 = 2^n,$$

so this case gives precisely

$$(a, m, n) = (2^n + 1, 1, n), \quad n \geq 1.$$

Now suppose that m is even, and write $m = 2r$. Put

$$w = v_2(a^2 - 1), \quad a^2 - 1 = 2^w u,$$

where u is odd. Formula (1) and the original equation give

$$w + v_2(m) - 1 = n + v_2(m),$$

so $n = w - 1$. Factoring $a^{2r} - 1$ and cancelling 2^w now gives

$$u(1 + a^2 + a^4 + \cdots + a^{2(r-1)}) = r. \quad (3)$$

If $r > 1$, the sum in parentheses is strictly greater than r , which contradicts (3). Hence $r = 1$, so $m = 2$, and (3) gives $u = 1$. It follows that

$$a^2 - 1 = (a - 1)(a + 1)$$

is a power of 2. Thus both $a - 1$ and $a + 1$ are powers of 2. They differ by 2, and the only two positive powers of 2 with this difference are 2 and 4. Therefore $a = 3$. Finally, $a^2 - 1 = 8$ gives $w = 3$, and hence $n = w - 1 = 2$. This produces the single even- m solution

$$(a, m, n) = (3, 2, 2).$$

Conversely, direct substitution shows that every listed triple satisfies the equation. This completes the classification. $\square$

Also solved by Amanda Goodwin, SUNY Brockport, NY, USA; Arkan Manva, IIT Bombay, Mumbai, India; Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam; Jamal Dine Chergui, École Normal Supérieure, Tétouan, Morocco; José Luis Díaz-Barrero, Barcelona, Spain; Kousik Sett, India; Mohammed Aassila, Strasbourg, France; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Anderson Torres, Sao Paulo, Brazil; Saarang Agarwal, AMMOC; Shilong Zhuang, California, USA; Teodor Cristian Petrescu, Romania; Yeonwoo Amelia Tae, Northfield Mount Hermon School, Gill, Massachusetts, USA.

O741. Let $a_1, \dots, a_n$ be pairwise relatively prime nonzero integers, and let c be an integer. For every $1 \leq i \leq n$, let b_i be the product of all the a_j with $j \in \{1, \dots, n\} \setminus \{i\}$. Prove that

$$f(x_1, \dots, x_n) := \left\lfloor \frac{x_1}{a_1} \right\rfloor + \dots + \left\lfloor \frac{x_n}{a_n} \right\rfloor$$

is constant for all the n -tuples $(x_1, \dots, x_n)$ of integers satisfying the equation

$$b_1x_1 + \dots + b_nx_n = c.$$

Proposed by Titu Andreescu, USA and Marian Tetiva, Romania

Solution by the authors

Note first that $b_ia_i = a_1 \cdots a_n$ and $\gcd(b_i, a_i) = 1$ for all $i \in \{1, \dots, n\}$ by the very definition of the numbers b_i , and due to the fact that any two among $a_1, \dots, a_n$ are relatively prime. Let $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ be two arbitrary solutions of the equation, so we have

$$b_1x_1 + \dots + b_nx_n = c \quad \text{and} \quad b_1y_1 + \dots + b_ny_n = c,$$

hence

$$b_1x_1 + \dots + b_nx_n = b_1y_1 + \dots + b_ny_n.$$

Because each b_j is divisible by a_i for $i \neq j$ the above equality implies

$$b_ix_i \equiv b_iy_i \pmod{a_i}, \quad \forall i \in \{1, \dots, n\},$$

and since $\gcd(b_i, a_i) = 1$ this yields

$$x_i \equiv y_i \pmod{a_i}, \quad \forall i \in \{1, \dots, n\}.$$

Thus, for all i there exists some $t_i \in \mathbb{Z}$ such that

$$y_i = x_i + a_it_i.$$

Replacing in the above equality, we get

$$b_1x_1 + \dots + b_nx_n = b_1(x_1 + a_1t_1) + \dots + b_n(x_n + a_nt_n),$$

that is

$$b_1a_1t_1 + \dots + b_n a_n t_n = 0 \Leftrightarrow a_1 \cdots a_n (t_1 + \dots + t_n) = 0 \Leftrightarrow t_1 + \dots + t_n = 0,$$

since $b_ia_i = a_1 \cdots a_n \neq 0$ for all i . Thus we have

$$\begin{aligned} f(y_1, \dots, y_n) &= \sum_{i=1}^n \left\lfloor \frac{y_i}{a_i} \right\rfloor = \sum_{i=1}^n \left\lfloor \frac{x_i}{a_i} + t_i \right\rfloor = \\ &= \sum_{i=1}^n \left(\left\lfloor \frac{x_i}{a_i} \right\rfloor + t_i \right) = \sum_{i=1}^n \left\lfloor \frac{x_i}{a_i} \right\rfloor + \sum_{i=1}^n t_i = f(x_1, \dots, x_n), \end{aligned}$$

as the sum of all the t_i is 0. Since this equality holds for any n -tuples $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ of integers satisfying $b_1x_1 + \dots + b_nx_n = c$ and $b_1y_1 + \dots + b_ny_n = c$ the required conclusion follows.

Also solved by Corneliu Mănescu-Avram, Ploiești, Romania; Mohammed Aassila, Strasbourg, France; Ayaan Alam, IIT Delhi.

O742. Let ABC be a non-right triangle with $AB \neq AC$ and let H be its orthocenter, I be its incenter and M be the midpoint of BC . Let D be the reflection of I in the line BC and let E be the reflection of A in H . Knowing that AI is parallel to DE , prove that the points H, I, M are collinear.

Proposed by Todor Zaharinov, Sofia, Bulgaria

Solution 1 by the author

Let $BC = a, CA = b, AB = c$ are the side lengths of the triangle ABC . Denote with R the circumradius and with r - the inradius of the triangle ABC and $s = \frac{a+b+c}{2}$. Let S be the area of the $\triangle ABC$.

$$S = rs = \frac{abc}{4R} = \sqrt{s(s-a)(s-b)(s-c)}$$

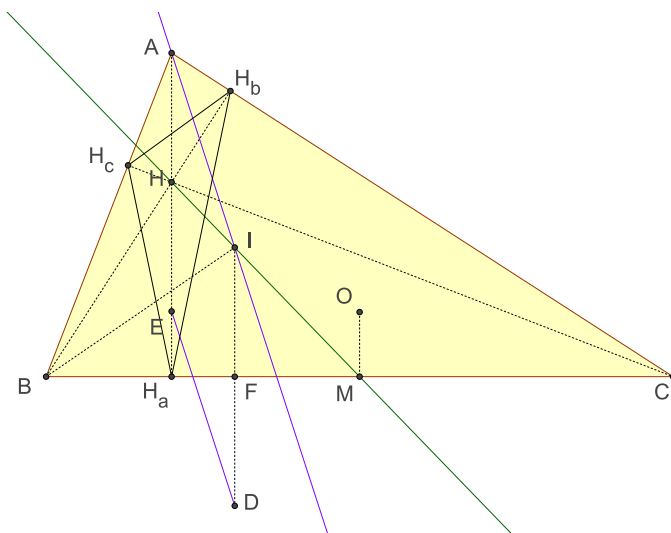


Figure 2:

Let $F = ID \cap BC$ so $IF = FD = r$; $ID = 2r$. Let O be the circumcenter of $\triangle ABC$, $2OM = AH$. Let H_a, H_b, H_c be feet of the perpendiculars at H to BC, CA, AB .

Without loss of generality we may assume that $b > c$; $\angle B > \angle C$ and M is between C and the points H_a, F . Since $ID \perp BC$ and $AE \perp BC$, that is $ID \parallel AE$, and $AI \parallel DE$, hence $AEDI$ is a parallelogram and $AE = ID$. But $AE = 2AH = 4OM = 4R \cos A$ and $ID = 2r$, so $r = 2R \cos A$.

The points H, F are the midpoints of AE, ID , so $AHFI$ is a parallelogram and $HF \parallel AI$. Point I is an interior of triangle ABC , therefore point H is also interior. Then $\triangle ABC$ is acute-angled.

Since $AI^2 = \frac{r^2}{\sin^2 \frac{A}{2}} = \frac{bc(s-a)}{s}$,

$$\begin{aligned} AI^2 \cdot BI^2 \cdot CI^2 &= \frac{r^6}{\sin^2 \frac{A}{2} \cdot \sin^2 \frac{B}{2} \cdot \sin^2 \frac{C}{2}} = \\ &= \frac{a^2 b^2 c^2 (s-a)(s-b)(s-c)}{s^3} = 16R^2 r^4 \\ \text{or } r &= 4R \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2} \end{aligned}$$

Now

$$\begin{aligned} \frac{r}{R} &= 4 \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2} = 2 \sin \left(90^\circ - \frac{B+C}{2} \right) \left(\cos \frac{B-C}{2} - \cos \frac{B+C}{2} \right) = \\ &= 2 \cos \frac{B+C}{2} \cdot \cos \frac{B-C}{2} - 2 \cos \frac{B+C}{2} \cdot \cos \frac{B+C}{2} = \\ &= \cos A + \cos B + \cos C - 1 = 2 \cos A \end{aligned}$$

So

$$\cos B + \cos C - \cos A - 1 = 0$$

From the Law of Cosines that is:

$$\frac{a^2 - b^2 + c^2}{2ac} + \frac{a^2 + b^2 - c^2}{2ab} - \frac{-a^2 + b^2 + c^2}{2bc} - 1 = 0$$

So, the condition $AI \parallel DE$ is equivalent to

$$a^3 + a^2b - ab^2 - b^3 + a^2c - 2abc + b^2c - ac^2 + bc^2 - c^3 = 0 \quad (1)$$

$$\begin{aligned} H_aM &= H_aC - MC = b \cos C - \frac{a}{2} = \frac{a^2 + b^2 - c^2}{2a} - \frac{a}{2} = \frac{b^2 - c^2}{2a} \\ FM &= FC - MC = \frac{a + b - c}{2} - \frac{a}{2} = \frac{b - c}{2} \\ \frac{H_aM}{FM} &= \frac{b + c}{a} \end{aligned}$$

It is well known that $\triangle H_aH_bC \sim \triangle ABC$, $\angle H_aH_bC = \angle B$. The angle $\angle HH_bC = \angle HH_aC = 90^\circ$ and HH_bCH_a is a cyclic quadrilateral. So $\angle HCH_a = \angle HH_bH_a = 90^\circ - \angle H_aH_bC = 90^\circ - \angle B$.

$$HH_a = CH \cdot \sin \angle HCH_a = 2R \cdot \cos C \cdot \sin(90^\circ - B) = 2R \cdot \cos C \cdot \cos B$$

$$IF = r = 2R \cos A$$

$$\frac{HH_a}{IF} = \frac{\cos B \cdot \cos C}{\cos A} = \frac{(a^2 - b^2 + c^2)(a^2 + b^2 - c^2)}{2a^2(-a^2 + b^2 + c^2)}$$

The points H, I, M are collinear, if and only if $\triangle MHH_a \sim \triangle MIF$ that is

$$\frac{HH_a}{IF} = \frac{H_aM}{FM} \iff \frac{(a^2 - b^2 + c^2)(a^2 + b^2 - c^2)}{2a^2(-a^2 + b^2 + c^2)} = \frac{b + c}{a}$$

Now

$$\begin{aligned} &\frac{(a^2 - b^2 + c^2)(a^2 + b^2 - c^2)}{2a^2(-a^2 + b^2 + c^2)} - \frac{b + c}{a} = \\ &= \frac{(a^2 - b^2 + c^2)(a^2 + b^2 - c^2) - 2a(b + c)(-a^2 + b^2 + c^2)}{2a^2(-a^2 + b^2 + c^2)} = \\ &= \frac{2s(a^3 + a^2b - ab^2 - b^3 + a^2c - 2abc + b^2c - ac^2 + bc^2 - c^3)}{2a^2(-a^2 + b^2 + c^2)} \stackrel{(1)}{=} 0 \end{aligned}$$

and the points H, I, M are collinear.

Solution 2 by Kousik Sett, India

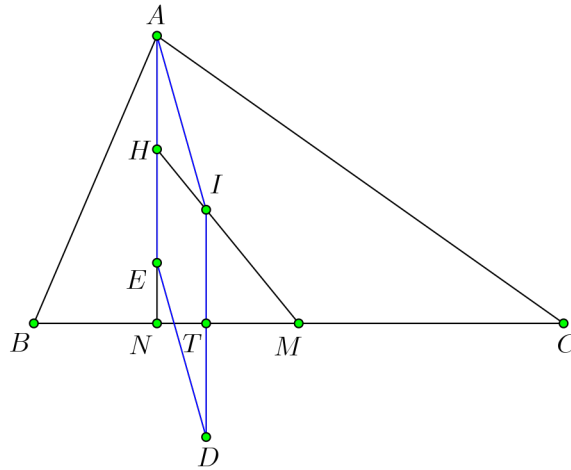
Let N be the foot of the altitude from A to BC , and let T be the point where the incircle touches BC . Let $BC = a$, $CA = b$, $AB = c$ where $b \neq c$.

Since D is the reflection of I in BC , we have $ID \perp BC$. Since E is the reflection of A in H , and A, H lie on the altitude from A , we infer that $AE \perp BC$. Hence $ID \parallel AE$. It is given that $AI \parallel DE$. Therefore, quadrilateral $AIDE$ is a parallelogram. Consequently, $AE = ID$.

Now H is the midpoint of AE , while $ID = 2r$, where r is the inradius of triangle ABC . Hence $AE = 2AH = 2r$, that is, $AH = r$.

Let s be the semi-perimeter of triangle ABC . Since T is the touch-point of the incircle, we have $BT = s - b$, $CT = s - c$. Since M is the midpoint of BC , we get

$$MT = \left| BT - \frac{a}{2} \right| = \frac{|c - b|}{2}.$$



Also,

$$BN = AB \cos B = \frac{a^2 + c^2 - b^2}{2a}.$$

Therefore,

$$MN = \left| BN - \frac{a}{2} \right| = \frac{|c^2 - b^2|}{2a} = \frac{|c - b|(b + c)}{2a}.$$

Let $AN = h$. Since $[ABC] = rs = \frac{1}{2}ah$, we have $h = \frac{2rs}{a}$. Since $2s = a + b + c$, we have

$$h - r = \frac{2rs}{a} - r = \frac{(2s - a)r}{a} = \frac{(b + c)r}{a}.$$

Since $AH = r$, we obtain

$$NH = AN - AH = h - r = \frac{(b+c)r}{a}.$$

Since Both IT and HN are perpendicular to BC , with $IT = r$, we have

$$\frac{MT}{IT} = \frac{|c - b|}{2r}.$$

On the other hand, we get

$$\frac{MN}{NH} = \frac{\frac{|c-b|(b+c)}{2a}}{\frac{(b+c)r}{a}} = \frac{|c-b|}{2r}.$$

Thus,

$$\frac{MT}{IT} = \frac{MN}{NH},$$

and since $\angle MTI = \angle MNH = 90^\circ$, the right triangles MTI and MNH are similar.

Moreover, M , T , N lie on the same side of M . Therefore, the corresponding angles at M are equal, that is, $\angle TMI = \angle NMH$. But MT and MN are the same ray from M . Hence the rays MI and MH coincide. Therefore, H , I , and M are collinear.

Also solved by Andrea Fanchini, Cantù, Italy; Arkan Manva, IIT Bombay, Mumbai, India; The Ring Lords, Baden-Württemberg, Germany; Theo Koupelis, Clark College, Washington, USA; Tiberiu Popa, San Francisco, CA, USA.

O743. Let $x_1, x_2, \dots, x_n$ be positive real numbers such that $x_1 + x_2 + \dots + x_n = n$, $n \geq 3$. Prove that

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} - n \geq \frac{27(n-1)^2}{(2n^2 + n - 1)^2} (x_1^3 + x_2^3 + \dots + x_n^3 - n).$$

Proposed by Marius Stănean, Zalău, Romania

Solution 1 by the author

Setting

$$\sum_{1 \leq i < j \leq n} x_i x_j = \frac{n(n-1)}{2} (1 - t^2), \quad t \in [0, 1),$$

then for any $i \in \{1, 2, \dots, n\}$ we have that

$$1 - (n-1)t \leq x_i \leq 1 + (n-1)t.$$

Indeed, applying the Cauchy-Schwarz Inequality, we have

$$n + n(n-1)t^2 = \sum_{i=1}^n x_i^2 \geq x_1^2 + \frac{(x_2 + x_3 + \dots + x_n)^2}{n-1} = x_1^2 + \frac{(n-x_1)^2}{n-1},$$

or, equivalently,

$$x_1^2 - 2x_1 + 1 - (n-1)^2 t^2 \leq 0,$$

which implies

$$1 - (n-1)t \leq x_1 \leq 1 + (n-1)t.$$

Analogously, we get

$$1 - (n-1)t \leq x_2, x_3, \dots, x_n \leq 1 + (n-1)t.$$

By the Cauchy-Schwarz Inequality,

$$\begin{aligned} \frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} &= \frac{n}{1 + (n-1)t} + \sum_{i=1}^n \left(\frac{1}{x_i} - \frac{1}{1 + (n-1)t} \right) \\ &= \frac{n}{1 + (n-1)t} + \sum_{i=1}^n \frac{1 + (n-1)t - x_i}{x_i [1 + (n-1)t]} \\ &\geq \frac{n}{1 + (n-1)t} + \frac{\left(\sum_{i=1}^n [1 + (n-1)t - x_i] \right)^2}{\sum_{i=1}^n [1 + (n-1)t - x_i] x_i [1 + (n-1)t]} \\ &= \frac{n}{1 + (n-1)t} + \frac{n^2 (n-1)^2 t^2}{n(n-1)t(1-t)[1 + (n-1)t]} \\ &= \frac{n[1 + (n-2)t]}{(1-t)[1 + (n-1)t]}. \end{aligned}$$

Also, by the Cauchy-Schwarz Inequality, we have

$$\begin{aligned} \sum_{i=1}^n x_i^2 [-x_i + 1 + (n-1)t] &\sum_{i=1}^n [-x_i + 1 + (n-1)t] \\ &\geq \left(\sum_{i=1}^n x_i [-x_i + 1 + (n-1)t] \right)^2, \end{aligned}$$

or

$$n[1 + (n-1)t][1 + (n-1)t^2] - \sum_{i=1}^n x_i^3 \geq n(n-1)t(1-t)^2,$$

or

$$\sum_{i=1}^n x_i^3 \leq n[1 + 3(n-1)t^2 + (n-1)(n-2)t^3].$$

Therefore, it suffices to prove that

$$\frac{n(n-1)t^2}{(1-t)[1 + (n-1)t]} \geq \frac{27(n-1)^2 \cdot n[3(n-1)t^2 + (n-1)(n-2)t^3]}{(2n^2 + n - 1)^2},$$

or after clearing denominators and factoring

$$\frac{n(n-1)t^2[3(n-1)t - 2(n-2)]^2[3(n-1)(n-2)t + n^2 + 5n - 5]}{(n+1)^2(2n-1)^2(1-t)[1 + (n-1)t]} \geq 0$$

which is clearly true.

The equality holds when $t = 0$ which means $x_1 = x_2 = \dots = x_n = 1$ or $t = \frac{2(n-2)}{3(n-1)}$ which means $x_1 = \frac{2n-1}{3}$, $x_2 = x_3 = \dots = x_n = \frac{n+1}{3(n-1)}$ and its cyclic permutations.

Solution 2 by Theo Koupelis, Clark College, Washington, USA

Let $a = \frac{27(n-1)^2}{(2n^2 + n - 1)^2} = \frac{27(n-1)^2}{(n+1)^2(2n-1)^2} > 0$, and $f(x) = \frac{1}{x} - ax^3$, where $x > 0$. Then $f'(x) = -\frac{1}{x^2} - 3ax^2$, and $f''(x) = \frac{2}{x^3} - 6ax$. Thus, $f''(x) = 0 \iff x = \frac{1}{(3a)^{1/4}}$. Therefore, $f(x)$ has one inflection point; by the $(n-1)$ -equal value principle, the expression $\sum_{i=1}^n f(x_i)$ achieves a maximal or minimal value when $n-1$ of the x_i variables are equal. Clearly, $\lim_{x \rightarrow 0^+} f(x) = +\infty$. Let $x_1 = x$ and $x_2 = x_3 = \dots = x_n = \frac{n-x}{n-1}$, where $0 < x < n$. Thus,

$$\begin{aligned} g(x) &:= \sum_{i=1}^n f(x_i) - n(1-a) = \frac{1-ax^4}{x} + \frac{(n-1)^4 - a(n-x)^4}{(n-1)^2(n-x)} - n(1-a) \\ &= \frac{n(x-1)^2(2n-1-3x)^2[(n+1)^2 + 3x(n-2)]}{x(n-x)(n+1)^2(2n-1)^2} \geq 0. \end{aligned}$$

Therefore, the desired inequality holds, with equality when $x_1 = x_2 = \dots = x_n = 1$, or $x_1 = \frac{2n-1}{3}$ and $x_2 = x_3 = \dots = x_n = \frac{n+1}{3(n-1)}$ and cyclic permutations.

Solution 3 by Kevin Xiong, of Canyon Crest Academy, California, USA

Set

$$U = \sum_{i=1}^n \frac{1}{x_i} - n, \quad V = \sum_{i=1}^n x_i^3 - n.$$

If $x_1 = \dots = x_n = 1$, then $U = V = 0$ and there is nothing to prove. Hence assume that the variables are not all equal to 1. Since their arithmetic mean is 1, strict AM-HM gives $U > 0$.

Let $M = \max\{x_1, \dots, x_n\}$, and, after relabeling, suppose that $x_1 = M$. Because the average of the x_i is 1 and the variables are not all equal, $M > 1$; moreover $M < n$ since $x_2, \dots, x_n > 0$. Put

$$b = \frac{n-M}{n-1}.$$

Then

$$0 < b < 1, \quad M + (n-1)b = n,$$

and b is the arithmetic mean of $x_2, \dots, x_n$.

For the comparison tuple $(M, b, \dots, b)$, define

$$U_0 = \frac{1}{M} + \frac{n-1}{b} - n, \quad V_0 = M^3 + (n-1)b^3 - n.$$

Using $M + (n-1)b = n$, we obtain

$$U_0 = \frac{(M-1)(M-b)}{Mb}, \quad V_0 = (M-1)(M-b)(M+b+1).$$

Thus, with

$$L = Mb(M+b+1),$$

we have

$$V_0 = LU_0.$$

We now compare $(x_1, \dots, x_n)$ with $(M, b, \dots, b)$. Since

$$\sum_{i=2}^n (x_i - b) = 0,$$

the identities

$$\frac{1}{x_i} - \frac{1}{b} = \frac{(x_i - b)^2}{b^2 x_i} - \frac{x_i - b}{b^2}$$

and

$$x_i^3 - b^3 = (x_i - b)^2(x_i + 2b) + 3b^2(x_i - b)$$

give

$$U - U_0 = \sum_{i=2}^n \frac{(x_i - b)^2}{b^2 x_i}$$

and

$$V - V_0 = \sum_{i=2}^n (x_i - b)^2(x_i + 2b).$$

For every $i \geq 2$, since $x_i \leq M$,

$$b^2 x_i(x_i + 2b) \leq b^2 M(M + 2b).$$

Also,

$$\begin{aligned} L - b^2 M(M + 2b) &= Mb(M + b + 1 - b(M + 2b)) \\ &= Mb(1 - b)(M + 2b + 1) > 0. \end{aligned}$$

Consequently,

$$(x_i - b)^2(x_i + 2b) \leq L \frac{(x_i - b)^2}{b^2 x_i},$$

and therefore

$$V - V_0 \leq L(U - U_0).$$

Together with $V_0 = LU_0$, this yields

$$V \leq LU.$$

Since $U > 0$,

$$\frac{V}{U} \leq L = Mb(M + b + 1) = \frac{M(n - M)((n - 2)M + 2n - 1)}{(n - 1)^2}. \quad (1)$$

It remains to bound the last numerator. Consider the three positive quantities

$$(n+1)M, \quad (2n-1)(n-M), \quad (n-2)M+2n-1.$$

Their sum is

$$(n+1)M + (2n-1)(n-M) + (n-2)M + 2n-1 = (2n-1)(n+1).$$

Hence AM–GM gives

$$\begin{aligned} & (n+1)(2n-1)M(n-M)((n-2)M+2n-1) \\ & \leq \left(\frac{(2n-1)(n+1)}{3} \right)^3. \end{aligned}$$

Therefore

$$M(n-M)((n-2)M+2n-1) \leq \frac{(2n-1)^2(n+1)^2}{27}.$$

Combining this with (1), and using

$$(2n-1)(n+1) = 2n^2 + n - 1,$$

we obtain

$$\frac{V}{U} \leq \frac{(2n^2 + n - 1)^2}{27(n-1)^2}.$$

Equivalently,

$$\boxed{\sum_{i=1}^n \frac{1}{x_i} - n \geq \frac{27(n-1)^2}{(2n^2 + n - 1)^2} \left(\sum_{i=1}^n x_i^3 - n \right)}.$$

Equality. The constant is sharp. Besides $x_1 = \cdots = x_n = 1$, there is a nonconstant equality case. Indeed, because

$$b^2 M(M+2b) < L,$$

equality in the comparison $V \leq LU$ can hold only if

$$x_2 = \cdots = x_n = b.$$

Equality in the final AM–GM step then requires

$$(n+1)M = (2n-1)(n-M) = (n-2)M + 2n-1,$$

which gives

$$M = \frac{2n-1}{3}, \quad b = \frac{n+1}{3(n-1)}.$$

Thus, up to permutation, equality holds at

$$\boxed{\left(\frac{2n-1}{3}, \frac{n+1}{3(n-1)}, \dots, \frac{n+1}{3(n-1)} \right)}.$$

Hence the coefficient

$$\frac{27(n-1)^2}{(2n^2 + n - 1)^2}$$

cannot be increased.

Also solved by Mihály Bencze, Braşov, Romania; Nicuşor Zlota, “Traian Vuia” Technical College, Focşani, Romania; Tiberiu Popa, San Francisco, CA, USA.

O744. Let a, b, c, d be positive real numbers such that $a \geq b \geq 1 \geq c \geq d$ and $abcd = 1$. Prove that

$$\frac{1}{a^2 + a + 1} + \frac{1}{b^2 + b + 1} + \frac{1}{c^2 + c + 1} + \frac{1}{d^2 + d + 1} \geq \frac{4}{3}.$$

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania

Solution 1 by the author

Let $x = \sqrt{ab} \geq 1$, $y = cd \leq 1$ and $z = c + d$. From $(c - 1)(d - 1) \geq 0$, we get

$$y + 1 \geq z.$$

We will first show that

$$\frac{1}{a^2 + a + 1} + \frac{1}{b^2 + b + 1} \geq \frac{2}{x^2 + x + 1}. \quad (*)$$

By the Cauchy-Schwarz inequality, we have

$$\frac{1}{a^2 + a + 1} + \frac{1}{b^2 + b + 1} \geq \frac{(b + a)^2}{b^2(a^2 + a + 1) + a^2(b^2 + b + 1)} = \frac{(a + b)^2}{2x^4 + (a + b)x^2 + (a + b)^2 - 2x^2}.$$

So, it suffice to show that

$$(a + b)^2(x^2 + x + 1) \geq 4x^4 + 2(a + b)x^2 + 2(a + b)^2 - 4x^2,$$

that is

$$\begin{aligned} (x^2 - 1)[(a + b)^2 - 2x^2] + (a + b)x(a + b - 2x) &\geq 0, \\ (x^2 - 1)(a - b)^2 + (a + b)x(\sqrt{a} - \sqrt{b})^2 &\geq 0. \end{aligned}$$

By (*), we only need to prove that

$$\frac{2}{x^2 + x + 1} + \frac{1}{c^2 + c + 1} + \frac{1}{d^2 + d + 1} \geq \frac{4}{3} \quad (**)$$

for $x \geq 1 \geq c \geq d > 0$ such that $x^2cd = 1$.

Case 1: $\frac{1}{3} \leq y \leq 1$. Since

$$\frac{2}{x^2 + x + 1} = \frac{4}{2x^2 + 2x + 2} \geq \frac{4}{2x^2 + (x^2 + 1) + 2} = \frac{4}{3(x^2 + 1)} = \frac{4y}{3(y + 1)} = \frac{4}{3} - \frac{4}{3(y + 1)},$$

it suffices to show that

$$\frac{1}{c^2 + c + 1} + \frac{1}{d^2 + d + 1} \geq \frac{4}{3(y + 1)}.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{1}{c^2 + c + 1} + \frac{1}{d^2 + d + 1} \geq \frac{(d + c)^2}{d^2(c^2 + c + 1) + c^2(d^2 + d + 1)} = \frac{z^2}{2y^2 + yz + z^2 - 2y}.$$

So, we only need to show that

$$\frac{z^2}{2y^2 + yz + z^2 - 2y} \geq \frac{4}{3(y + 1)},$$

which is equivalent to

$$(3y - 1)z^2 - 4yz - 8y^2 + 8y \geq 0.$$

Indeed, since $z \leq y + 1$, we have

$$(3y - 1)z^2 - 4yz - 8y^2 + 8y \geq (3y - 1)z^2 - 4y(y + 1) - 8y^2 + 8y = (3y - 1)(z^2 - 4y) = (3y - 1)(c - d)^2 \geq 0.$$

Case 2: $y < \frac{1}{3}$. We will first show that

$$\frac{1}{c^2 + c + 1} + \frac{1}{d^2 + d + 1} \geq \frac{1}{y^2 + y + 1} + \frac{1}{3}. \quad (***)$$

Write this inequality as follows:

$$\frac{z^2 + z - 2y + 2}{z^2 + (y + 1)z + y^2 - y + 1} \geq \frac{y^2 + y + 4}{3(y^2 + y + 1)},$$

$$\begin{aligned} -(1 - 2y - 2y^2)z^2 - (1 + 2y - y^2 + y^3)z + (1 + y)(2 + y - 5y^2 - y^3) &\geq 0, \\ (1 + y - z)[2 + y - 5y^2 - y^3 + (1 - 2y - 2y^2)z] &\geq 0. \end{aligned}$$

It is true since $1 + y - z = (1 - c)(1 - d) \geq 0$, $2 + y - 5y^2 - y^3 > 2 - 9y^2 - y^3 > 1 - y^3 > 0$ and $1 - 2y - 2y^2 > 1 - 2/3 - 2/9 > 0$. By (**) and (***), it suffices to prove that

$$\frac{2}{x^2 + x + 1} + \frac{1}{y^2 + y + 1} \geq 1$$

for $x^2y = 1$. Indeed,

$$\begin{aligned} \frac{2}{x^2 + x + 1} + \frac{1}{y^2 + y + 1} - 1 &= \frac{2}{x^2 + x + 1} + \frac{x^4}{x^4 + x^2 + 1} - 1 = \frac{2}{x^2 + x + 1} - \frac{x^2 + 1}{x^4 + x^2 + 1} \\ &= \frac{2}{x^2 + x + 1} - \frac{x^2 + 1}{(x^2 + x + 1)(x^2 - x + 1)} = \frac{(x - 1)^2}{(x^2 + x + 1)(x^2 - x + 1)} \geq 0. \end{aligned}$$

The proof is completed. The equality occurs for $a = b = c = d = 1$.

Remark. Since $a^2 + a \geq 2a^{3/2}$, $b^2 + b \geq 2b^{3/2}$ and $c^2 + c \geq 2c^{3/2}$, we get the following result (Vasile Cîrtoaje, Crux Mathematicorum, 5, 2025):

- If $a \geq b \geq 1 \geq c \geq d > 0$ such that $abcd = 1$, then

$$\frac{1}{2a + 1} + \frac{1}{2b + 1} + \frac{1}{2c + 1} + \frac{1}{2d + 1} \geq \frac{4}{3}.$$

Solution 2 by Theo Koupelis, Clark College, Washington, USA

Let $f(a, b, c, d) = \sum_c \frac{1}{a^2 + a + 1}$. Keeping c, d fixed, we get $b = \frac{1}{cd} \cdot \frac{1}{a}$; taking the derivative with respect to a we get $b' = -\frac{1}{cda^2} = -\frac{b}{a}$, and thus

$$f'_a = -\frac{2a + 1}{(a^2 + a + 1)^2} + \frac{b(2b + 1)}{a(b^2 + b + 1)^2} > 0,$$

because

$$\begin{aligned} b(2b + 1)(a^2 + a + 1)^2 &> a(2a + 1)(b^2 + b + 1)^2 \iff (a - b)[2a^2b^2(a + b) + 4(a^2b^2 - 1) \\ &\quad + ab(a^2 + ab + b^2) + (ab - 1)(2a + 2b - 1) + 2] \geq 0, \end{aligned}$$

which is obvious. Thus, it suffices to show that $f(b, b, c, d) \geq 4/3$ under the condition $b^2 = \frac{1}{cd}$. Let $p = c + d$ and $r^2 = cd$, where $0 < r \leq 1$, $0 < p \leq 2$, and $p \geq 2r$. Substituting into $f(b, b, c, d)$ we get the equivalent expression

$$\frac{2r^2}{r^2 + r + 1} + \frac{p^2 - 2r^2 + p + 2}{r^4 + r^2p + p^2 - r^2 + p + 1} \geq \frac{4}{3}.$$

Clearing denominators we get

$$(5r^2 - r - 1)p^2 + (2r^4 - 4r^3 + r^2 - r - 1)p + 2(r^6 - 2r^5 - 6r^4 - r^3 + 3r^2 + r + 1) \geq 0.$$

But $2r^4 - 4r^3 + r^2 - r - 1 = 2r^3(r-2) + r(r-1) - 1 \leq -1$, and from $0 \leq (1-c)(1-d) = 1 + cd - (c+d) = 1 + r^2 - p$ we get $p \leq 1 + r^2$. Thus, it suffices to show that

$$(5r^2 - r - 1)p^2 + (2r^4 - 4r^3 + r^2 - r - 1)(1 + r^2) + 2(r^6 - 2r^5 - 6r^4 - r^3 + 3r^2 + r + 1) \geq 0,$$

or, after simplifying,

$$(5r^2 - r - 1)p^2 + 4r^6 - 8r^5 - 9r^4 - 7r^3 + 6r^2 + r + 1 \geq 0.$$

When $5r^2 - r - 1 \geq 0$, or $\frac{1 + \sqrt{21}}{10} \leq r \leq 1$, it suffices to show that

$$4r^2(5r^2 - r - 1) + 4r^6 - 8r^5 - 9r^4 - 7r^3 + 6r^2 + r + 1 \geq 0 \iff (r-1)^2(4r^4 + 7r^2 + 3r + 1) \geq 0,$$

which is obvious. When $5r^2 - r - 1 \leq 0$, or $0 < r \leq \frac{1 + \sqrt{21}}{10}$, it suffices to show that

$$(1 + r^2)^2(5r^2 - r - 1) + 4r^6 - 8r^5 - 9r^4 - 7r^3 + 6r^2 + r + 1 \geq 0 \iff 9r^2(r-1)^2(r^2 + r + 1) \geq 0,$$

which is obvious. Equality occurs when $r = 1$, and thus when $a = b = c = d = 1$.

Also solved by Nicușor Zlota, "Traian Vuia" Technical College, Focșani, Romania; Tiberiu Popa, San Francisco, CA, USA; Shilong Zhuang, California, USA; Teodor Cristian Petrescu, Romania.