

ISOAGONAL CONJUGACY WITHOUT ANGLES: GENERALIZATIONS AND PROPERTIES OF REMARKABLE CEVIAN POINTS

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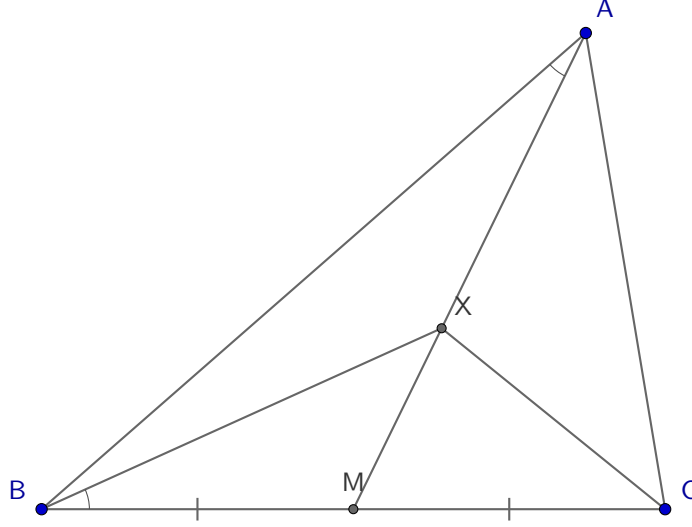
Abstract

This paper systematically investigates the geometric properties of two remarkable configurations involving unique points on the median and the symmedian of a triangle. To the best of our knowledge, several of the central characterizations and generalizations presented here are original and have not previously appeared in the literature. Based on empirical observations of recurring configurations in advanced geometry, we introduce a comprehensive framework that unifies the characteristic angle equalities, cyclicity, collinearity, and perpendicularity relations associated with these unique points. By drawing structural analogies with other isogonal conjugations, we extend these properties to arbitrary pairs of isogonal cevians, culminating in the formulation and proof of the *Lemma on Isogonal Conjugate Points*. Exploring special cases of this lemma, we identify the $\frac{AB+AC}{BC} = 2$ triangle and detail its significant properties. Recognizing that segment equalities are, in fact, a specific case of equal ratios, we subsequently establish the *Lost Angle Theorem*. Building on this, we demonstrate that isogonal conjugation can be redefined using only a single pair of equal angles. This naturally raises the question of whether such conjugation can be expressed without relying on angles entirely, which we address by formulating and proving the *Two Lost Angles Theorem*. To illustrate the practical value of the developed theory, we apply these foundational concepts to streamline the analysis of high-level olympiad geometry problems, offering elegant synthetic alternatives to the traditionally complex or algebraic official solutions from prestigious national competitions. Finally, we utilize this framework to synthesize several original geometry problems of varying difficulty. The theoretical toolkit established in this work contributes directly to competitive mathematics and can be effectively integrated into advanced elective courses, specialized seminars, or training curricula for gifted students.

Median's Unique Point

Definition

Let M be the midpoint of side BC in triangle ABC . Let X be the *median's unique point* if X lies on ray MA such that $\angle XBM = \angle MAB$.



This point has many helpful properties. Firstly, let us introduce some additional designations:

Ω — circle symmetric to $\omega(ABC)$ with respect to BC , H — orthocenter of $\triangle ABC$,

$$B_1 = BH \cap AC, \quad C_1 = CH \cap AB, D = \omega(ABC) \cap AM.$$

Property 1. $MX \cdot MA = MB^2 = MC^2 = MC \cdot MB$

Proof. $\angle XBM = \angle MAB \Rightarrow MB$ is tangent to $\omega(AXB) \Rightarrow MB^2 = MX \cdot MA$. Since $MB = MC$, we get $MB^2 = MC^2 = MB \cdot MC$. $\square$

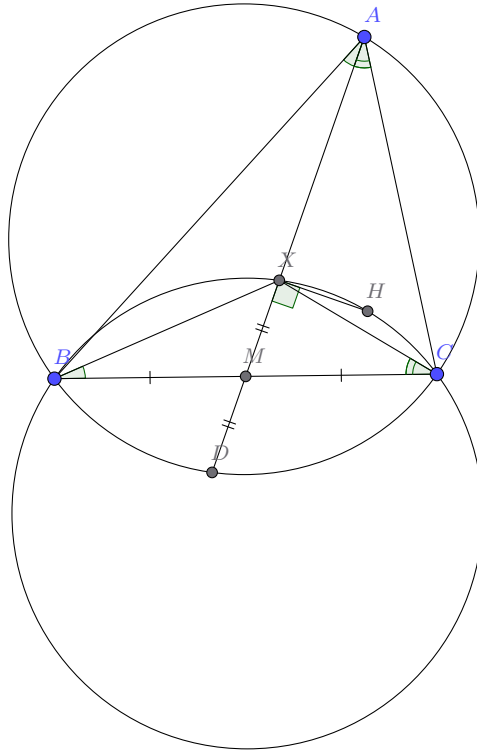
Property 2. $\angle XCM = \angle MAC$

Proof. Using Property 1: $MC^2 = MB \cdot MC \Rightarrow MC$ is tangent to $\omega(AXC)$. It implies that $\angle XCM = \angle MAC$ $\square$

Property 3. $X, H \in \Omega$

Proof. $\angle BXC = 180^\circ - (\angle XBM + \angle XCM) = 180^\circ - (\angle MAB + \angle MAC) = 180^\circ - \angle BAC$.

Also, $\angle BB_1A + \angle CC_1A = 180^\circ \Rightarrow \angle BHC = 180^\circ - \angle BAC$. Therefore $\angle BXC = \angle BHC = 180^\circ - \angle BAC$, which is equivalent to the required statement. $\square$



Property 4. $HX \perp MA$

Proof. Let $D = \Omega \cap AM$, $D \neq X$. Since $ABCD$ is a parallelogram (as D is symmetric to A with respect to M), we have $BD \parallel AC$, and $AC \perp BH$, so $BD \perp BH$.

By Property 3, $X, H \in \Omega$, hence $\angle HMD = 180^\circ - \angle HBD = 90^\circ$, which means $HX \perp AM$. $\square$

Property 5. $MX = MD$

Proof. Let us calculate the power of point M with respect to $\omega(ABC)$:

$$MB \cdot MC = MD \cdot MA \Rightarrow MD \cdot MA = MX \cdot MA \text{ (by Property 1)}$$

$$MD \cdot MA = MX \cdot MA \Rightarrow MX = MD$$

$\square$

Property 6. $\frac{XB}{XC} = \frac{AB}{AC}$

Proof. Using property 2 and the Law of Sines:

$$\frac{AB}{AC} = \frac{\sin \angle MAC}{\sin \angle MAB} = \frac{\sin \angle XCM}{\sin \angle XBM} = \frac{XB}{XC}$$

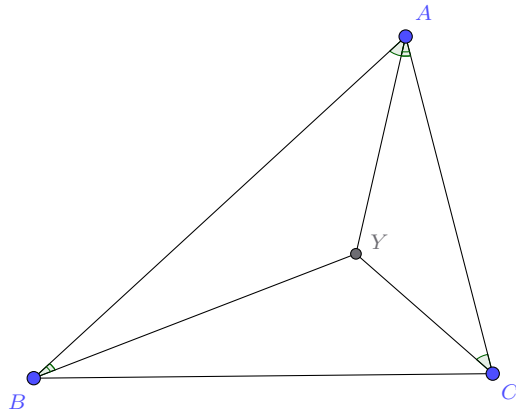
$\square$

Symmedian's Unique Point

Definition. From point A in triangle ABC , draw the median AM . The line symmetric to AM with respect to the angle bisector is called a *symmedian*.

Furthermore, let O be the centre of the circumcircle $\omega(ABC)$; S be the intersection point of the symmedian with $\omega(ABC)$; and T be the intersection point of the symmedian with BC .

Definition of a special point (Property 0): On the median, we observed some nice angle equalities for a certain special point. This naturally raises the question of what happens when we rearrange the equal angles. Let us take a point Y such that $\angle YAC = \angle YBA$ and $\angle YCA = \angle YAB$. This new point Y will be our special point.



Property 1. $\omega(YAC)$ is tangent to AB at point A , $\omega(YAB)$ is tangent to AC at point A

Proof: this is a consequence of the equality of angles: $\angle YAC = \angle YBA \rightarrow \omega(ABY)$ is tangent to AC at point A . Similarly, $\angle YCA = \angle YAB \rightarrow \omega(YAC)$ is tangent to AB at point A

Property 2. Y lies on the A -symmedian of triangle ABC .

Proof. First, let us prove that the symmedian from point A divides BC in the ratio $\frac{BA^2}{AC^2}$.

Since the median and the symmedian are symmetric with respect to the angle bisector, we have $\angle MAB = \angle TAC$.

Let us prove the property of the median:

$$\frac{\sin \angle MAB}{\sin \angle MAC} \cdot \frac{AB}{AC} = \frac{BM}{MC} = 1.$$

By the sine rule:

$$\frac{\sin \angle BAM}{\sin \angle BMA} = \frac{BM}{AB}, \quad \frac{\sin \angle MAC}{\sin \angle AMC} = \frac{CM}{AC}.$$

Dividing the first equation by the second:

$$\frac{\sin \angle BAM}{\sin \angle BMA} \cdot \frac{\sin \angle AMC}{\sin \angle MAC} = \frac{BM}{CM} \cdot \frac{AC}{AB}.$$

Since $\sin \angle BMA = \sin \angle AMC$ (they sum to 180°) and $BM = CM$:

$$\frac{\sin \angle BAM}{\sin \angle MAC} \cdot \frac{AB}{AC} = 1.$$

Now, since $\angle MAB = \angle TAC$ and $\angle MAC = \angle TAB$:

$$\frac{\sin \angle BAM}{\sin \angle MAC} = \frac{\sin \angle TAC}{\sin \angle TAB} = \frac{AC}{AB}.$$

By the sine rule:

$$\frac{\sin \angle BAT}{\sin \angle ATB} = \frac{BT}{AB}; \quad \frac{\sin \angle TAC}{\sin \angle ATC} = \frac{CT}{AC}.$$

Dividing:

$$\frac{\sin \angle BAM}{\sin \angle BMA} \cdot \frac{\sin \angle AMC}{\sin \angle MAC} = \frac{BM}{CM} \cdot \frac{AC}{AB}.$$

Since $\sin \angle ATB = \sin \angle ATC$ (they sum to 180°):

$$\frac{\sin \angle TAC}{\sin \angle TAB} = \frac{CT}{BT} \cdot \frac{AB}{AC}.$$

As $\frac{\sin \angle TAC}{\sin \angle TAB} = \frac{AC}{AB}$, we get $\frac{CT}{BT} = \frac{AC^2}{BA^2}$, which is what we wanted to prove.

Returning to our point Y . Let AY intersect BC at T' . Since $\angle BYA = \angle AYC$, we have $\angle BYT' = \angle CYT'$. By the well-known property of the angle bisector, $\frac{BY}{YC} = \frac{BT'}{CT'}$. If we prove that $\frac{BT'}{CT'} = \frac{AB^2}{BC^2}$, then AY is a symmedian.

By the similarity of triangles AYB and CYA :

$$\frac{BY}{BA} = \frac{AY}{AC} \Rightarrow \frac{BA}{AC} = \frac{BY}{AY} = \frac{\sin \angle BAY}{\sin \angle ABY} = \frac{\sin \angle BAY}{\sin \angle YAC}.$$

By the sine rule:

$$\frac{\sin \angle BAT'}{\sin \angle AT'B} = \frac{BT'}{AB}, \quad \frac{\sin \angle T'AC}{\sin \angle AT'C} = \frac{CT'}{AC}.$$

Dividing:

$$\frac{\sin \angle T'AC}{\sin \angle T'AB} = \frac{CT'}{BT'} \cdot \frac{AB}{AC}.$$

But we already obtained:

$$\frac{BA}{CA} = \frac{\sin \angle BAY}{\sin \angle YAC} = \frac{\sin \angle T'AB}{\sin \angle T'AC} = \frac{AC}{BC} \cdot \frac{BT'}{CT'}.$$

Therefore $\frac{BT'}{CT'} = \frac{AB^2}{BC^2}$, which completes the proof. □

Property 3. Points B , O , Y , and C lie on the same circle.

Proof.

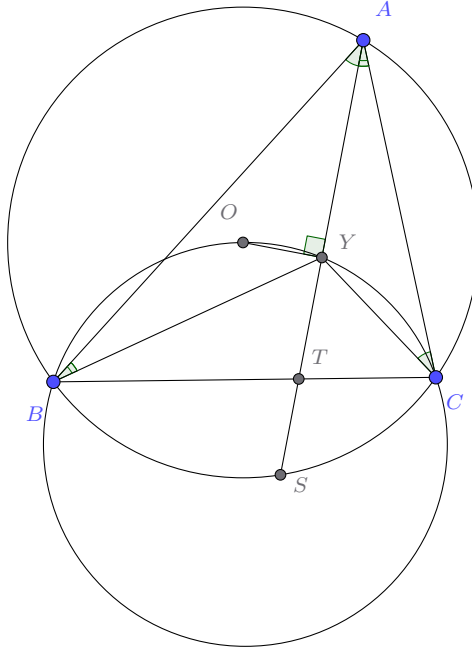
$$\begin{aligned}
 \angle BYC &= (180^\circ - \angle AYB) + (180^\circ - \angle AYC) \\
 &= (\angle YBA + \angle YAB) + (\angle YAC + \angle YCA) \\
 &= 2\angle BAC.
 \end{aligned}$$

At the same time, $\angle BOC = 2\angle BAC$, since $\angle BOC$ is a central angle in triangle ABC . From $\angle BOC = \angle BYC$, it follows that B, O, Y , and C lie on the same circle. $\square$

Property 4. $OY \perp AY$

Proof. Let $\angle BAC = \alpha$. Then $\angle AYB = 180^\circ - \alpha$, since $\angle ABY = \angle YAC$. Hence $\angle BYT = \alpha$, and $\angle BOC = 2\alpha$, so $\angle OCB = \frac{180^\circ - 2\alpha}{2} = 90^\circ - \alpha$.

By Property 3, $\angle BYO = 180^\circ - \angle OCB = 90^\circ + \alpha$. Since $\angle BYT = \alpha$, we get $\angle TYO = 90^\circ$, which is what we wanted to prove. $\square$



Property 4.5. Y is the midpoint of AS .

Proof. This follows from Property 4, since OY is perpendicular to AY . Therefore OY is the perpendicular bisector of AS . $\square$

Application of the Unique Point of the Symmedian in Olympiad Problems

Let us present several relevant Olympiad problems of various levels, where knowledge of the unique points and the ability to apply their properties can significantly simplify the solution.

Problem 1 (Kazakhstan's National Olympiad 2024, Grade 9, Problem 3)

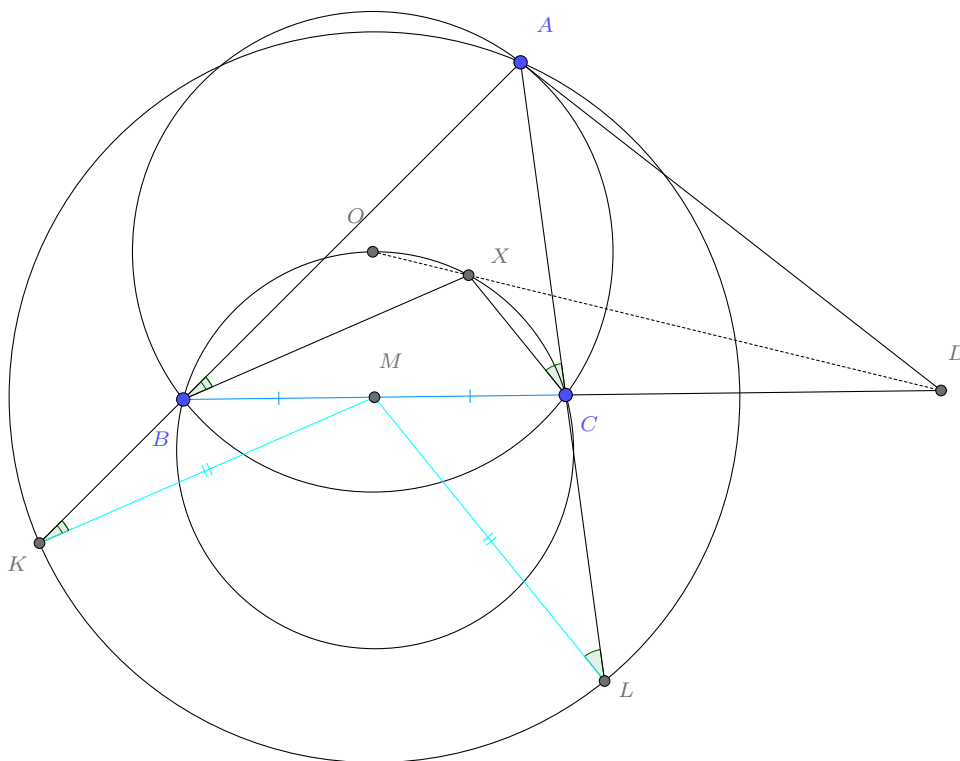
In a circle ω with centre O , an acute triangle ABC is inscribed ($AB \neq AC$). Point M is the midpoint of side BC . The tangent to ω at point A intersects the extension of side BC at point D . The circle with centre M and radius MA intersects the extensions of sides AB and AC at points K and L , respectively. Let X be the point such that $BX \parallel KM$ and $CX \parallel LM$. Prove that points X , D , and O lie on the same straight line.

Solution that uses properties of unique points:

First, note a simple angle equality coming from parallelism and the isosceles triangles:

$$\angle XBA = \angle MKA = \angle MAB, \quad \angle XCA = \angle MLA = \angle MAC.$$

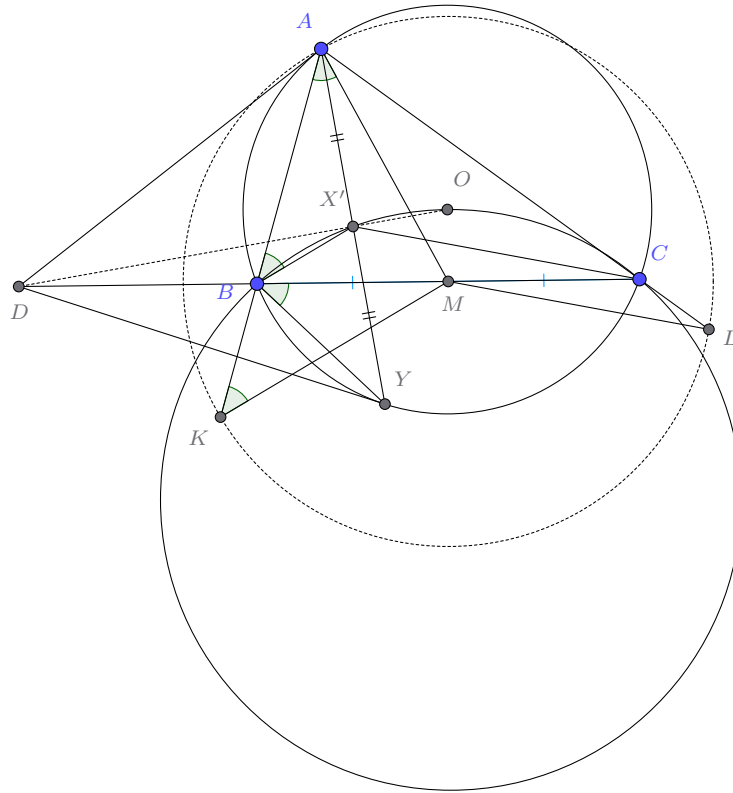
From this we immediately see that the point X is the Symmedian's Unique point. This observation greatly simplifies the problem, eliminating the need to consider points M , K , and L . From Property 4, OX is the perpendicular bisector of AS , where $S = AX \cap \omega$. At this point, it becomes clear that, by the symmedian theorem, the problem is solved.



Official Solution:

Draw the second tangent DY to ω . Let X' be the midpoint of AY . From the symmedian theorem, it is known that line AX' is the symmedian from vertex A in triangle ABC , and line BC contains the symmedian from vertex B in triangle ABY . Then $\angle ABX' = \angle CBY = \angle CAY = \angle MAB = \angle MKA$.

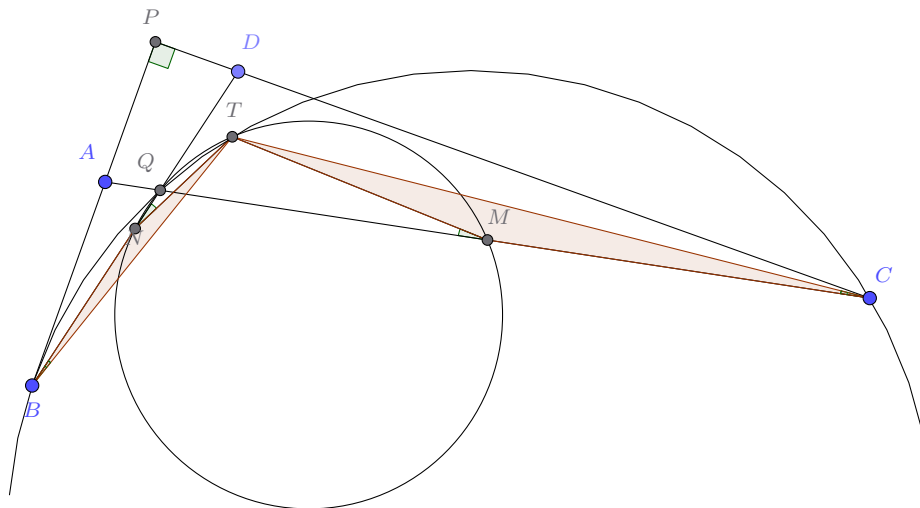
Therefore $BX' \parallel KM$. Similarly, $CX' \parallel LM$, which means $X = X'$. Thus, points O , X , and D lie on the perpendicular bisector of segment AY .



Problem 2 (Kazakhstan's National Olympiad 2021, Grade 9, Problem 3)

The extensions of sides AB and CD of a convex quadrilateral $ABCD$ intersect at point P , and the diagonals AC and BD intersect at point Q . Points M and N are the midpoints of diagonals AC and BD , respectively. The circumcircles of triangles BCQ and MNQ intersect at point T ($T \neq Q$). Prove that if $\angle APD = 90^\circ$, then the line PT bisects the segment MN .

Solution that uses properties of unique points:



Since T lies on the circumcircle of $\triangle MNQ$, we have:

$$T \in \omega(MNQ) \implies \angle TNQ = \angle TMQ$$

Similarly, since T lies on the circumcircle of $\triangle BCQ$, we have:

$$T \in \omega(BCQ) \implies \angle TBN = \angle TCM \implies \triangle TBN \sim \triangle TCM \quad (1)$$

From the similarity (1), it follows that:

$$\frac{TN}{TM} = \frac{BN}{CM} = \frac{PN}{PM}$$

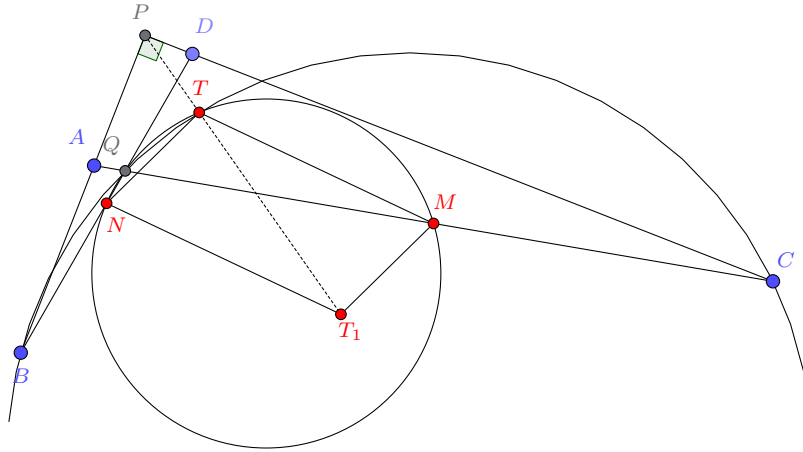
(The last equality holds because $BN = PN$ and $CM = PM$, since $\angle P$ is a right angle in the respective right-angled triangles).

The problem requires us to prove that PT passes through the midpoint of MN in $\triangle MPN$. The previously obtained equality implies that, T must be a unique point of the P -median of $\triangle MPN$. This can be verified in a straightforward manner using a basic angle chase and Property 3:

$$\begin{aligned} 180^\circ - \angle NPM &= 180^\circ - (90^\circ - \angle BPN - \angle CPM) \\ &= 90^\circ + \angle PBN + \angle PCM \\ &= \angle QDC + \angle PCM = \angle NQM = \angle NTM. \end{aligned}$$

Since on the circle symmetric to $\omega(NTM)$ with respect to NM , there is only one point that satisfies $\frac{TN}{TM} = \frac{BN}{CM}$ and lies on the same side of NM as P (which is a well-known fact), this point must be the unique point, which completes the proof.

Official Solution:



Notice that $\angle TMQ = \angle TNQ$ and $\angle TCQ = \angle TBQ$. Therefore, triangles TMC and TNB are similar, which implies:

$$\frac{CM}{TM} = \frac{BN}{TN}$$

Let us construct a point T_1 in the plane such that TMT_1N is a parallelogram. Then the following equalities hold:

$$\frac{PM}{T_1N} = \frac{CM}{TM} = \frac{BN}{TN} = \frac{PN}{T_1M} \implies PM \cdot T_1M = PN \cdot T_1N \quad (*)$$

On the other hand, notice that:

$$\begin{aligned}\angle MT_1N &= \angle MTN = \angle MQN = \angle PCA + \angle PBD + \angle APD \\ &= 90^\circ + (\angle CPM + \angle BPN) \\ &= 90^\circ + 90^\circ - \angle MPN\end{aligned}$$

This implies:

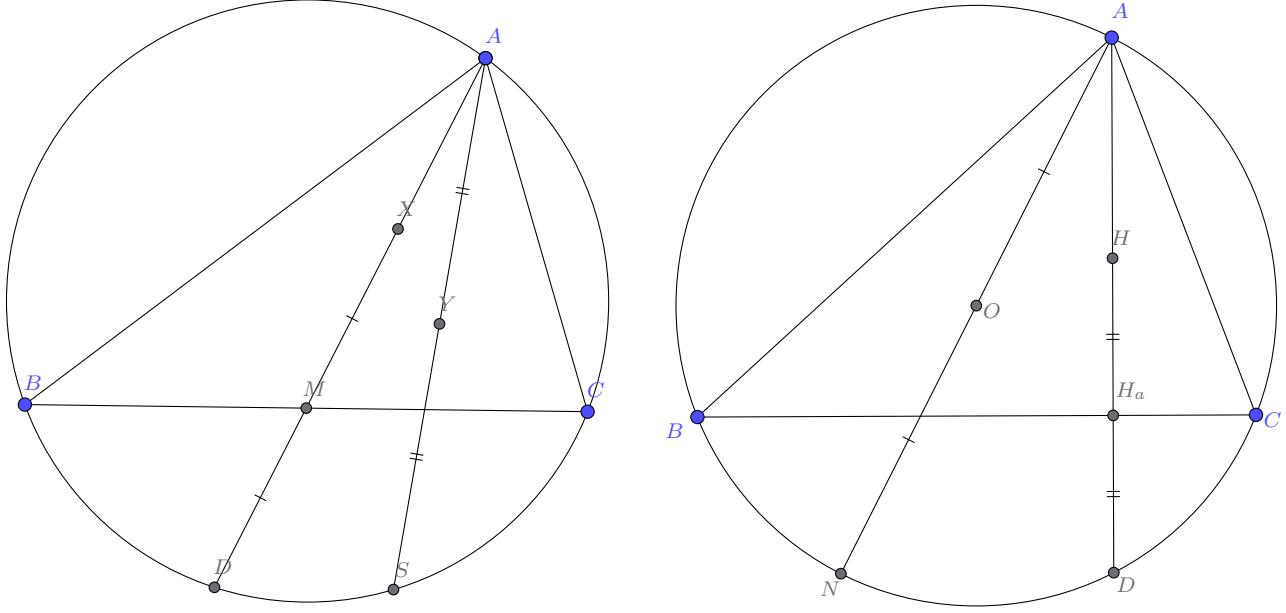
$$\angle MT_1N + \angle MPN = 180^\circ \implies \angle PNT_1 + \angle PMT_1 = 180^\circ \implies \sin \angle PMT_1 = \sin \angle PNT_1 \quad (**)$$

From (*) and (**), we obtain that $S_{\triangle PMT_1} = S_{\triangle PNT_1}$, which implies that PT_1 passes through the midpoint of MN . By the properties of a parallelogram, the line TT_1 also passes through the midpoint of MN . It follows that the line PT bisects the segment MN .

Lemma on Isogonal Conjugate Points

The following lemma, which we believe to be original, generalizes the segment equalities observed for the median's and symmedian's unique points to arbitrary pairs of isogonal cevians.

In triangle ABC , let X be the Median's Unique point, Y be the Symmedian's Unique point, M be the midpoint of segment BC , D be the intersection of AM with the circumcircle $\omega(ABC)$, and S be the intersection of the symmedian from A with $\omega(ABC)$. Then the following holds: $XM = MD$ and $AY = YS$. In other words, Y is the midpoint of AS , and X is the reflection of D with respect to M .



The same property holds for two other isogonal conjugate points: the circumcentre O and the orthocenter H of triangle ABC .

In triangle ABC , let $N = AO \cap \omega(ABC)$, $H_a = AH \cap BC$, and $D = AH \cap \omega(ABC)$. Then: $AO = ON$ and $HH_a = H_aD$. In other words, O is the midpoint of AN , and H is the reflection of H_a with respect to D .

Thus, the following lemma is formulated:

Lemma. In triangle ABC , from point A two isogonal rays l_1 and l_2 are drawn. Let $l_1 \cap \omega(ABC) = D$, $l_2 \cap \omega(ABC) = E$, and $l_1 \cap BC = S$. Then, if Y is the midpoint of AE and X is the reflection of D with respect to S , points X and Y are isogonal conjugates.

Proof 1 (sine rule approach):

From the sine rule property for the median:

$$\frac{\sin \angle YAC}{\sin \angle AEC} = \frac{\sin \angle ACY}{\sin \angle YCE}.$$

Triangle ABC with $\frac{AB + AC}{BC} = 2$

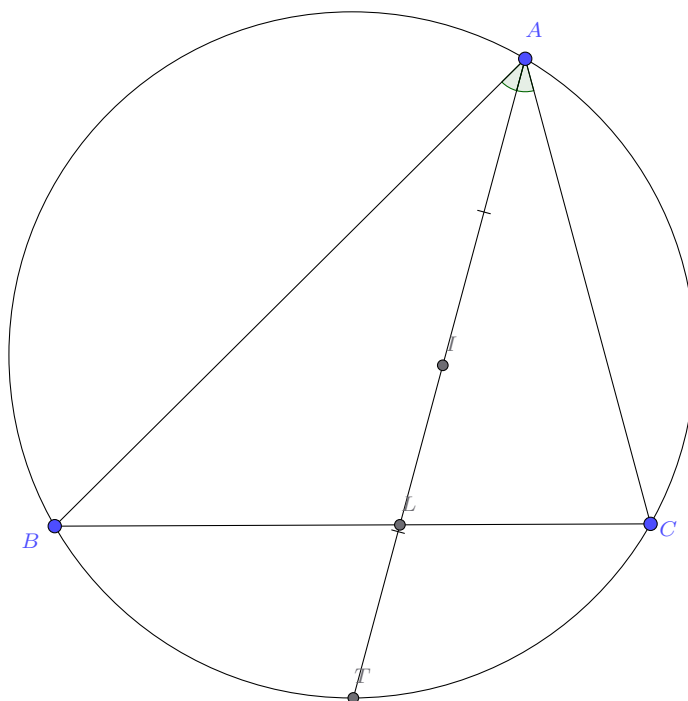
The lemma allows us to explore some special cases. For example, the incenter I of triangle ABC is its own isogonal conjugate. Let us consider several important corollaries of the Lemma.

Corollary 1. In triangle ABC , let AL be the angle bisector. Let $T = AL \cap \omega(ABC)$, X be the midpoint of AT , and let $Y \neq T$ be a point on the bisector such that $YL = LT$. Then X and Y are isogonal conjugates.

Corollary 2. In triangle ABC , let I be the incenter. Then $AI = IT$ if and only if $IL = LT$.

From Corollary 2, the following two facts follow:

Fact 1. $\frac{AI}{IL} = 2$ if and only if $AI = IT$.



Proof. If $AI = IT$, then by Corollary 2 we have $IL = LT$, which implies $\frac{AI}{IL} = 2$.

Conversely, suppose $\frac{AI}{IL} = 2$. Let $BL = a$, $LC = b$. Then, by the angle bisector property, $\frac{AB}{BL} = \frac{AI}{IL} = \frac{AC}{LC}$, hence $AB = 2a$, $AC = 2b$.

Using the formula for the length of the angle bisector:

$$AL^2 = AB \cdot AC - BL \cdot LC = 4ab - ab = 3ab,$$

so $AL = \sqrt{3ab}$. Also, $AL \cdot LT = BL \cdot LC$, hence $LT = \frac{ab}{\sqrt{3ab}} = \sqrt{\frac{ab}{3}}$.

Since $\frac{AI}{IL} = 2$ and $AL = \sqrt{3ab}$, we get $IL = \frac{AL}{3} = \frac{\sqrt{3ab}}{3} = \sqrt{\frac{ab}{3}} = LT$. Thus $IL = LT$, and by Corollary 2 it follows that $AI = IT$. $\square$

Fact 2. $\frac{AL}{LT} = 3$ if and only if $\frac{AI}{IL} = 2$.

Proof. If $\frac{AI}{IL} = 2$, then by Fact 1 we have $AI = IT$ and $IL = LT$. Hence $AL = 3LT$, which is what we wanted to prove.

Conversely, suppose $\frac{AL}{LT} = 3$. From cyclicity, $BL \cdot LC = AL \cdot LT = \frac{AL^2}{3} = \frac{AB \cdot AC - BL \cdot LC}{3}$. It follows that:

$$AB \cdot AC = 4BL \cdot LC \implies \frac{AB \cdot AC}{BL \cdot LC} = 4.$$

By the angle bisector property, $\frac{AB}{BL} = \frac{AI}{IL} = \frac{AC}{LC}$. Hence $\left(\frac{AI}{IL}\right)^2 = 4 \implies \frac{AI}{IL} = 2$. $\square$

Corollary 3. It is actually an addition to Corollary 2: in triangle ABC , $\frac{AB + AC}{BC} = 2$ is true if and only if $AI = IT$ and $IL = LT$.

Proof. If the conditions of Corollary 2 are satisfied, then $\frac{AI}{IL} = 2$. Let $BL = a$, $LC = b$. By the angle bisector property $AB = 2a$, $AC = 2b$. Therefore:

$$AB + AC = 2a + 2b = 2(a + b) \implies \frac{AB + AC}{BC} = 2.$$

Conversely, suppose $\frac{AB + AC}{BC} = 2$. Let $BI \cap AC = B_1$ and $CI \cap AB = C_1$. Then, by Van Aubel's theorem and the angle bisector property:

$$\frac{AI}{IL} = \frac{AB_1}{B_1C} + \frac{AC_1}{C_1B} = \frac{AB}{BC} + \frac{AC}{BC} = \frac{AB + AC}{BC} = 2.$$

Thus, Fact 1 holds, and therefore $AI = IT$, and $IL = LT$, from Corollary 2. $\square$

In Issue 5, 2024 of Math Reflections, Mihaela Berindeanu (Bucharest, Romania) proposed an elegant problem, published under the code J678.

Problem. Let ABC be a triangle with circumcenter O , incenter I , and such that $2AC = AB + BC$. Let M be the midpoint of AB and S the midpoint of OB . Prove that $SI = SM$.

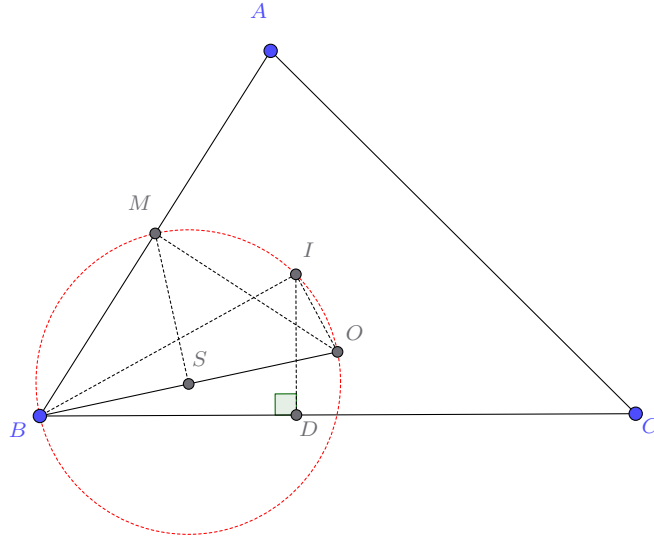
Official Solution by Kousik Sett, India:

Since $2AC = AB + BC$, we have $2b = c + a$. So $3b = 2s$ where s is the semiperimeter of triangle ABC . Let D be the projection of I on BC . Therefore, $ID = r$ which is the inradius of triangle ABC . Also, it is known that $BD = s - b$. Let R be the circumradius and Δ be the area of triangle ABC .

Therefore,

$$\begin{aligned}
 BI^2 &= r^2 + (s - b)^2 = \frac{\Delta^2}{s^2} = \frac{s(s - a)(s - b)(s - c)}{s^2} + (s - b)^2 \\
 &= (s - b) \left[\frac{(s - a)(s - c)}{s} + (s - b) \right] \\
 &= (s - b) \left[s - (a + c) + \frac{ac}{s} + (s - b) \right] \\
 &= (s - b) \cdot \frac{ac}{s} \\
 &= (2s - 2b) \cdot \frac{ac}{2s} = (3b - 2b) \cdot \frac{ac}{3b} = \frac{ac}{3}.
 \end{aligned}$$

Since M is the midpoint of AB and O is the circumcenter of triangle ABC , we infer that MO is perpendicular to AB , i.e., $\angle BMO = 90^\circ$. So there exists a circle passing through B , O , and M and OB is the diameter of the circle. Since S is the midpoint of OB , the center of the circle is S . We will prove I lies on this circle. So it is enough to prove $\angle BIO = 90^\circ$.



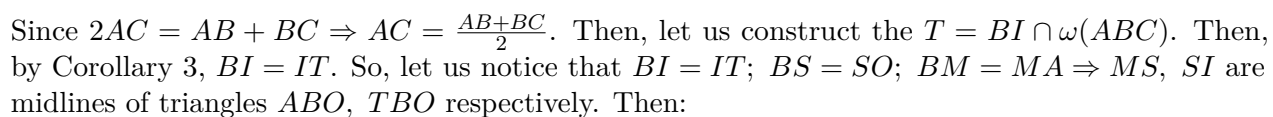
Since $BO = R$ and from the well-known fact $IO = \sqrt{R^2 - 2Rr}$, we have

$$BO^2 - IO^2 = R^2 - (R^2 - 2Rr) = 2Rr = 2 \cdot \frac{abc}{4\Delta} \cdot \frac{\Delta}{s} = \frac{abc}{2s} = \frac{abc}{3b} = \frac{ac}{3} = BI^2.$$

So $BO^2 = BI^2 + IO^2$ and hence $\angle BIO = 90^\circ$. Therefore, I lies on the circle. Thus $SI = SM$.

The official solution, while comprehensive, relies on rather heavy algebraic manipulations that somewhat obscure the underlying behavior of the system. The purpose of this paper is to present a fresh perspective. Instead of performing exhaustive expansions, we focus on the intermediate outputs and relations naturally generated by the given conditions. By analyzing these outputs directly, the required conclusion follows almost immediately, bypassing the tedious algebra.

Solution by using our properties:



But then, $AO = OT$, since it is radius of circumcircle ABC . Then

Remark: Corollary 3 and the subsequent application to Problem J678 appear to be new observations, though the individual ingredients (Van Aubel’s theorem and properties of the incenter) are classical.

Theorem on Isogonal Conjugate Points

Lemma (Sine Lemma for a Ceva Line). In triangle ABC , a cevian AS is drawn. Then the following equality holds:

$$\frac{BS}{SC} = \frac{AB}{AC} \cdot \frac{\sin \angle BAS}{\sin \angle CAS}.$$

Proof. From the sine rule: $\frac{BS}{\sin \angle BAS} = \frac{AB}{\sin \angle ASB}$ and $\frac{CS}{\sin \angle CAS} = \frac{AC}{\sin \angle ASC}$. Dividing:

$$\frac{BS}{CS} \cdot \frac{\sin \angle CAS}{\sin \angle BAS} = \frac{AB}{AC} \cdot \frac{\sin \angle ASB}{\sin \angle ASC}.$$

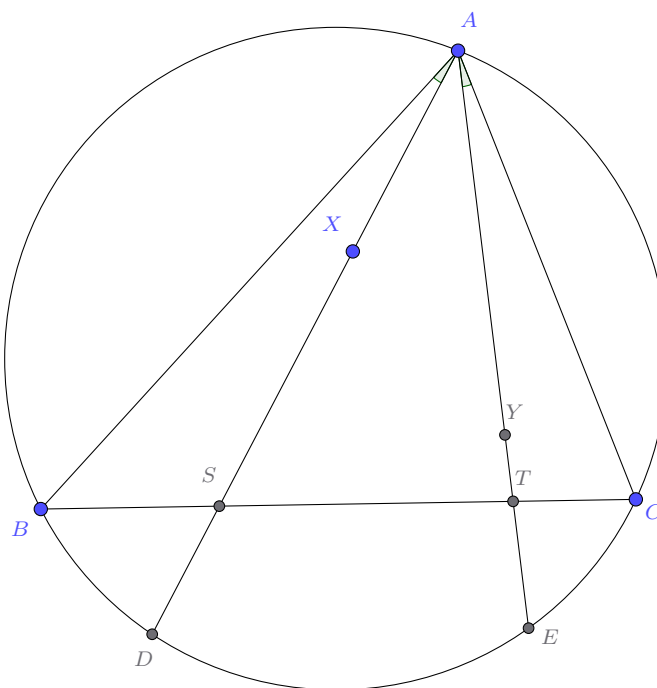
Since $\angle ASB = 180^\circ - \angle ASC$, we have $\frac{\sin \angle ASB}{\sin \angle ASC} = 1$. Therefore:

$$\frac{BS}{SC} = \frac{AB}{AC} \cdot \frac{\sin \angle BAS}{\sin \angle CAS}. \quad \square$$

□

In our lemma, we found that any two points satisfying the lemma's condition are isogonal conjugates. But what if we have equal ratios instead of just segment equalities? We address this question and discovered a theorem that is valid not only for isogonal conjugate points with certain properties, but also covers *all* isogonal conjugate points of a triangle.

Theorem (Lost Angle Theorem). In triangle ABC , two isogonal rays l_1, l_2 are drawn from point A . Let $l_1 \cap \omega(ABC) = D$; $l_2 \cap \omega(ABC) = E$, and $l_1 \cap BC = S$. Suppose Y is a point on segment AE such that $\frac{YE}{AY} = \frac{x}{y}$, and X is a point on line l_1 such that $\frac{DS}{SX} = \frac{x}{y}$, with S lying between X and D . Then X and Y are isogonal conjugates.



Proof. From the Sine Lemma for a Ceva Line:

$$\frac{YE}{AY} = \frac{CE}{AC} \cdot \frac{\sin \angle YCE}{\sin \angle ACY} = \frac{x}{y} = \frac{DS}{SX} = \frac{DC}{CX} \cdot \frac{\sin \angle DCS}{\sin \angle XCS}.$$

Now, from the sine theorem:

$$\begin{aligned} \frac{x}{y} &= \frac{CE}{AC} \cdot \frac{\sin \angle YCE}{\sin \angle ACY} = \frac{\sin \angle EAC}{\sin \angle AEC} \cdot \frac{\sin \angle YCE}{\sin \angle ACY}, \\ \frac{x}{y} &= \frac{DC}{CX} \cdot \frac{\sin \angle SCD}{\sin \angle XCS} = \frac{\sin \angle DXC}{\sin \angle CDX} \cdot \frac{\sin \angle DCS}{\sin \angle XCS}. \end{aligned}$$

Therefore:

$$\frac{\sin \angle DXC}{\sin \angle CDX} \cdot \frac{\sin \angle SCD}{\sin \angle XCS} = \frac{\sin \angle EAC}{\sin \angle AEC} \cdot \frac{\sin \angle YCE}{\sin \angle ACY}.$$

We also have the angle equalities: $\angle EAC = \angle BAD = \angle SCD$ and $\angle XDC = \angle ADC = \angle AEC$ (from cyclicity and isogonality). Thus:

$$\frac{\sin \angle DXC}{\sin \angle XCS} = \frac{\sin \angle YCE}{\sin \angle ACY}.$$

Let $\angle BAX = \angle CAY = \alpha$, $\angle XAY = \beta$, $\angle ACB = \gamma$. Then:

$$\angle DXC + \angle XCS = 180^\circ - \angle XSC = \alpha + \beta + \gamma = \angle ACB + \angle BAE = \angle ACE = \angle ACY + \angle YCE.$$

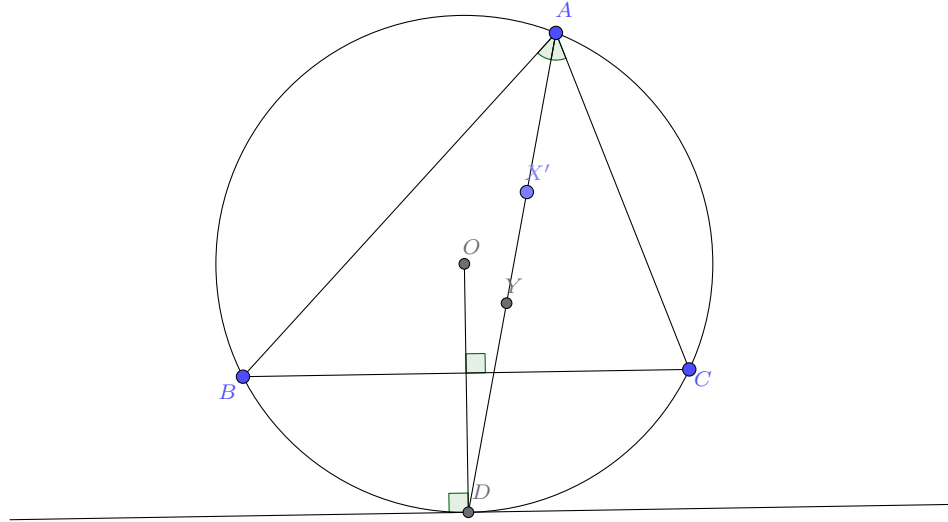
By the lemma: $\angle XCS = \angle ACY = \angle XCB$. Similarly, $\angle ABY = \angle XBC$. From this, the isogonal conjugacy of X and Y follows. $\square$

We have characterized isogonal conjugation by replacing the standard three angle equalities with a single angle equality and a ratio condition. This naturally raises a question: can we redefine isogonal conjugation without using any angle equalities at all? As we shall see, this is indeed possible. This theorem, to our knowledge, provides the first characterization of isogonal conjugacy purely in terms of segment ratios, without reference to any angle equalities.

Theorem (Two Lost Angles Theorem). Given a triangle ABC , let X and Y be two points in the plane such that A , X , and Y are not collinear. Let lines AX and AY intersect BC at points S and T , respectively, and intersect the circumcircle $\omega(ABC)$ at points D and E , respectively. Suppose the following ratio equalities hold:

$$\frac{AX}{XD} = \frac{YT}{TE}, \quad \frac{XS}{SD} = \frac{AY}{YE}.$$

Then points X and Y are isogonal conjugates.



Now note: if we draw a line l through D' parallel to BC , this line will be tangent to the circumcircle at D' , because $OB = OC$ and $D'B = D'C$, meaning OD' is the perpendicular bisector of BC , so $OD' \perp BC$. In this case $l \perp OD'$, which means l is tangent to the circumcircle. But $DD' \parallel BC$ and D lies on the circumcircle, so $D = D'$. Therefore X also lies on this bisector, which means X and Y are isogonal conjugates by the Lost Angle Theorem. Hence $X = X'$ in all cases. $\square$

So, we have managed to rewrite the isogonal conjugacy of two points in a triangle in terms of a ratio of segments. The lemma, theorems, facts, and corollaries can serve as a useful tool not only for solving problems but also for creating them. After all, ratios can be used as hidden constructions of two isogonal conjugate points.

Original Problems on Finding Hidden Constructions

Below, we introduce a series of original problems that arose naturally from the developed framework.

Problem 1. Given a triangle ABC . The tangent to the circumcircle of ABC at point A meets line BC at point D . Let T be the foot of the tangent from D to the circle passing through points A and C and tangent to line BC . Prove that TC is parallel to the median from point A in triangle ABC .

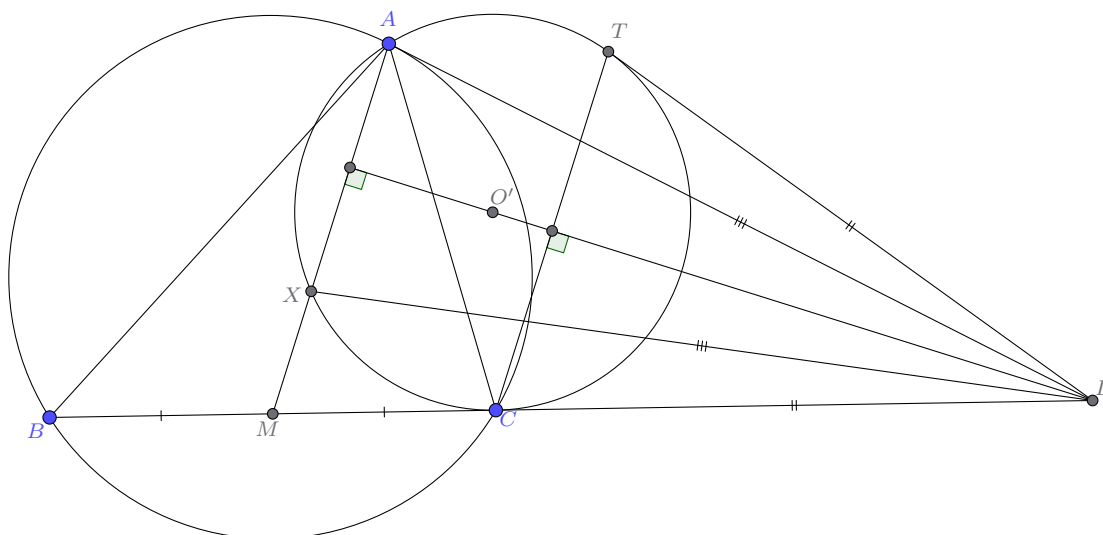
Solution:

Let the median intersect $\omega(ATC)$ at point X . By definition, X is the Median's Unique point of triangle ABC .

Next, note that D is the centre of the A -Apollonius circle of triangle ABC . We will now prove that X lies on the A -Apollonius circle. This follows from the relation:

$$\frac{BX}{XC} = \frac{\sin \angle XCB}{\sin \angle XBC} = \frac{\sin \angle BAX}{\sin \angle XAC} \quad (\text{from the properties of the Median's Unique point}).$$

Also, $\frac{\sin \angle BAX}{\sin \angle XAC} = \frac{BA}{AC}$ from the sine property of the median. Therefore $DA = DX$.



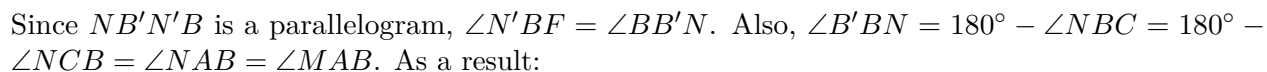
We also have $DT = DC$, since DT and DC are tangents to $\omega(ATC)$. Hence, if O' is the centre of $\omega(ATC)$, then $DO' \perp AX$ and $DO' \perp TC$, because AD is the perpendicular bisector of both AX and TC .

It follows that $AX \parallel TC$, meaning the median is parallel to TC . □

Problem 2 (Kazhymurat Aknazar Olympiad 2024, 9th grade, Problem 4). The external bisector of angle $\angle BAC$ in triangle ABC intersects its circumcircle and line BC at points N and F , respectively. Let B' be the reflection of B across F , and let M be the midpoint of AN . Prove that the circumcircle of triangle BMN is tangent to line $B'N$.

Solution:

Let N' be the image of point N under central symmetry with respect to point F . By the lemma, points N' and M are isogonal conjugates with respect to triangle ABC . Therefore $\angle N'BF = \angle MBA$.

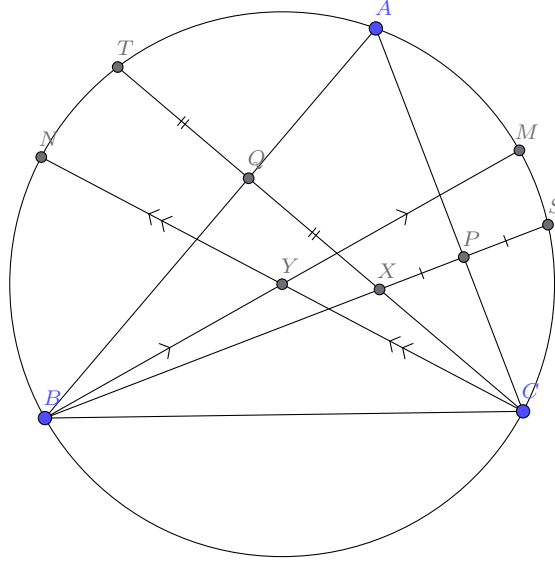


which means $\triangle BB'N \sim \triangle ABM$. Therefore:

This is exactly the angular criterion for tangency of a line and a circle, so the circumcircle of BMN is tangent to line $B'N$. $\square$

Solution:

Let us construct point Y , the isogonal conjugate of X . From point Y , draw the rays BY and CY , and let them intersect the circumcircle $\omega(ABC)$ at points M and N , respectively.



Then, according to the lemma, $BY = YM$ and $CY = YN$. Hence, quadrilateral $BNMC$ is a cyclic parallelogram, and therefore it must be a rectangle. It follows that Y is the circumcenter, and consequently, its isogonal conjugate is the orthocenter. $\square$

Problem 4. Given triangle ABC . From point A , two isogonal rays l_1 and l_2 are drawn. On these rays, points X and Y are chosen respectively. It turns out that quadrilateral $BXYC$ is inscribed in a circle ω .

The lines l_1 and l_2 intersect ω again at points P and Q , respectively; they also intersect side BC at points M and N , respectively, and the circumcircle $\omega(ABC)$ at points S and T , respectively.

It is given that $MY \parallel PT$.

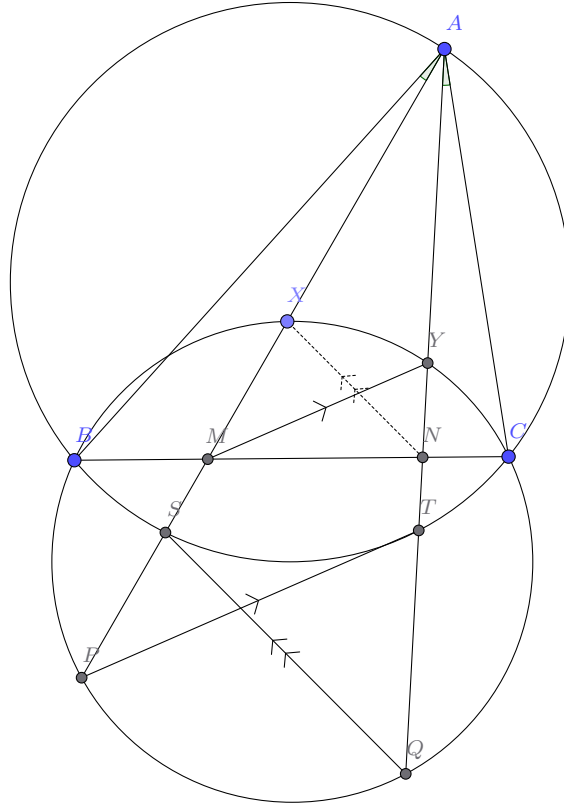
- (a) Prove that $NX \parallel QS$.
- (b) Prove that the incenter I of triangle ABC lies on ω .

Solution:

(a) From the parallelism, we obtain the equality $\frac{AM}{MP} = \frac{AY}{YT}$.

Since M lies on the radical axis of the circles $\omega(ABC)$ and ω , we have $MX \cdot MP = MS \cdot MA$, hence $\frac{AM}{MP} = \frac{XM}{MS}$.

Therefore $\frac{XM}{MS} = \frac{AY}{YT}$. Now, applying the theorem, we conclude that points X and Y are isogonal conjugates. Hence, analogously, we have $\frac{AX}{XS} = \frac{YN}{NT}$.



Since N lies on the radical axis of the circles $\omega(ABC)$ and ω , we get $NY \cdot NQ = NT \cdot NA$, so $\frac{AX}{XS} = \frac{YN}{NT} = \frac{AN}{NQ}$.

This implies $NX \parallel QS$.

(b) Since we have proved that X and Y are isogonal conjugates, and $BXYC$ is cyclic, let us prove the lemma: for any pair of isogonal conjugate points inside a triangle,

$$\angle BXC + \angle BYC = 180^\circ + \angle BAC.$$

Indeed:

$$\begin{aligned} \angle BXC &= 180^\circ - \angle XBC - \angle XCB, \\ \angle BYC &= 180^\circ - \angle YBC - \angle YCB = 180^\circ - \angle ABX - \angle ACX. \end{aligned}$$

Thus:

$$\angle BXC + \angle BYC = 360^\circ - \angle XBC - \angle XCB - \angle ABX - \angle ACX = 360^\circ - \angle ABC - \angle ACB = 180^\circ + \angle BAC.$$

But since $BXYC$ is cyclic:

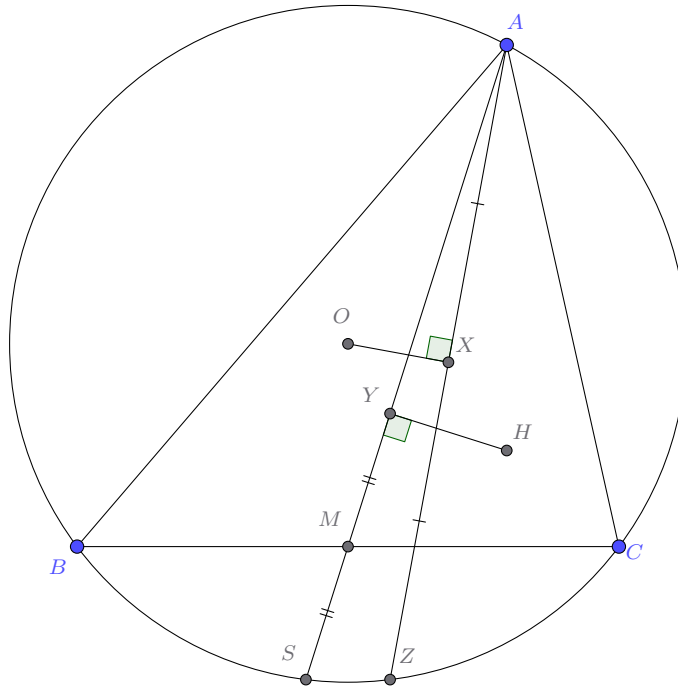
$$\angle BXC = \angle BYC = 90^\circ + \frac{\angle BAC}{2}.$$

Since $\angle BIC = 90^\circ + \frac{\angle BAC}{2}$, the result follows. $\square$

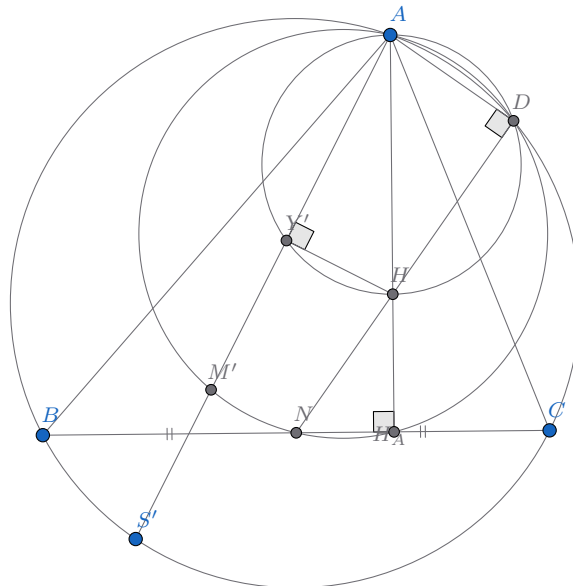
Problem 5. In a triangle ABC , let O be the circumcenter and H be the orthocenter. Let $X \neq O$ and $Y \neq H$ be isogonal conjugates with respect to $\triangle ABC$. If $OX \perp AX$ and $HY \perp AY$, prove that AY is a median of $\triangle ABC$.

Solution:

Let $AX \cap \omega(ABC) = Z$, $AY \cap \omega(ABC) = S$, $AY \cap BC = M$. Then, since $OX \perp AX \Rightarrow AX = XZ$. Using the lemma, we obtain that $YM = MS$. Now we can completely get rid of X and O .



Then, let us focus on new problem: $YM = MS$, $HY \perp AH$. We need to prove that AY is a median of $\triangle ABC$.



Let N be a midpoint of BC , and $H_A = AH \cap BC$. Let Y' be a point, such that $AY' \perp HY'$. Also, let $M' = \omega(ANH_A) \cap AY'$; $S' = AY' \cap \omega(ABC)$; $D = NH \cap \omega(ABC)$. Then, $\angle HDA = 90$ (a property of orthocenter), therefore $ADHY'$ is a cyclic quadrilateral, ADH_ANM' is a cyclic pentagon.

Now, let us use Forgotten Coaxiality Lemma [5], for $\omega(ABC), \omega(AM'N), \omega(AY'H)$:

$$\frac{\text{Power of point } S' \text{ with respect to } \omega(AM'N)}{\text{Power of point } S' \text{ with respect to } \omega(AY'H)} = \frac{S'M' \cdot S'A}{S'Y' \cdot S'A} = \frac{S'M'}{S'Y'} = \text{const}$$

Then, we can calculate it only for one case, and others will be same: since Y' is a point, such that $AY' \perp HY'$, let us take Y' as a Median's Unique point, then:

$$\frac{S'M'}{S'Y'} = \frac{1}{2}$$

So $S'M' = M'Y'$. It is true for any such that $AY' \perp HY'$. So actually, it means that if we take a point W , such that $AW \perp HW$, then, if $U = AW \cap \omega(ABC)$, and K is a midpoint of UW , then $K \in \omega(ANH_A)$. But then, Y is a point, such that $AY \perp HY$; $M = AY \cap BC \Rightarrow YM = MS$. So using the property we proved, $M \in \omega(ANH_A)$, but also $M \in BC$, so

$$M = N \text{ or } H_A$$

If $M = H_A$, then $Y = H$, which is not true. Then $M = N$, so AY is a median. □

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