

THREE USEFUL LEMMAS IN SOLVING CERTAIN INEQUALITIES OF 4 VARIABLES

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In this article we target a class of inequalities in four nonnegative variables with the constraint $3(a + b + c + d) = 2(ab + ac + ad + bc + bd + cd)$. The purpose of this article is to establish some lower and upper bounds for the product $abcd$ and the sum $abc + bcd + cda + dab$ so that we can reduce a symmetric inequality in four variables to an inequality in one variable.

Lemma 1. Let a, b, c, d be nonnegative real numbers such that

$$3(a + b + c + d) = 2(ab + ac + ad + bc + bd + cd) > 0.$$

If we set $s = a + b + c + d$ then we have the following inequalities

$$abcd \geq \frac{s^3(9 - 2s)}{64}, \quad abc + bcd + cda + dab \geq \frac{s^2(11 - 2s)}{12}.$$

Proof. First, note that according to Maclaurin's Inequality, we have

$$\frac{a + b + c + d}{4} \geq \sqrt{\frac{ab + bc + cd + da + ac + bd}{6}} \implies s \geq 4.$$

After homogenizing, the first inequality becomes (Vasile Cîrtoaje)

$$(a + b + c + d)^4 + 32abcd \geq 3(a + b + c + d)^2(ab + ac + ad + bc + bd + cd) \quad (1)$$

and the second can be written as follows

$$3(a + b + c + d)^3 + 18(abc + bcd + cda + dab) \geq 11(a + b + c + d)(ab + ac + ad + bc + bd + cd). \quad (2)$$

We can consider $a \geq b \geq c \geq d$ and $a + b + c = 1$. Denoting $t = abc$, by the AM-GM Inequality it follows that $t \leq \frac{1}{27}$. Obviously we also have $d \leq \frac{1}{3}$.

For (1) we need to show that

$$(d + 1)^4 + 32td \geq 3(d + 1)^2(ab + bc + ca + d).$$

By Schur's Inequality, i.e.

$$(a + b + c)^2 + \frac{9abc}{a + b + c} \geq 4(ab + bc + ca),$$

it follows that

$$ab + bc + ca \leq \frac{9t + 1}{4}. \quad (3)$$

It suffices to show that

$$(d + 1)^4 + 32td \geq 3(d + 1)^2 \left(\frac{9t + 1}{4} + d \right),$$

or

$$4d^4 + 4d^3 - 3d^2 - 2d + 1 - t(27d^2 - 74d + 27) \geq 0.$$

But $27t \leq 1$ so it remains to show that

$$27(4d^4 + 4d^3 - 3d^2 - 2d + 1) - 27d^2 + 74d - 27 \geq 0,$$

or

$$4d(3d - 1)^2(3d + 5) \geq 0,$$

clearly true.

For (2) we need to show that

$$3(d+1)^3 + 18(t + d(ab + bc + ca)) \geq 11(d+1)(d + ab + bc + ca),$$

or

$$3(d+1)^3 - 11d(d+1) + 18t - (11-7d)(ab + bc + ca) \geq 0.$$

It suffices to show that

$$3(d+1)^3 - 11d(d+1) + 18t - \frac{(11-7d)(9t+1)}{4} \geq 0,$$

or

$$12d^3 - 8d^2 - d + 1 - 9t(3-7d) \geq 0.$$

Since $3-7d > 0$ and $27t \leq 1$ it remains to show that

$$36d^3 - 24d^2 - 3d + 3 - 3 + 7d \geq 0,$$

or

$$4d(3d-1)^2 \geq 0,$$

clearly true. □

In both inequalities, the equality holds when $a = b = c = d = 1$ or when $a = b = c = \frac{3}{2}$, $d = 0$ and its cyclic permutations.

Lemma 2. Let a, b, c, d be nonnegative real numbers such that

$$3(a+b+c+d) = 2(ab+ac+ad+bc+bd+cd) > 0.$$

If we set $s = a+b+c+d$ then we have the following inequalities

$$abc + bcd + cda + dab \geq \frac{s^2(6-s)}{9}.$$

Proof. After homogenizing, the inequality becomes (Vasile Cîrtoaje)

$$2(a^2 + b^2 + c^2 + d^2)(a+b+c+d) + 9(abc + bcd + cda + dab) \geq (a+b+c+d)^3. \quad (3)$$

Without loss of generality, we may assume that $a \geq b \geq c \geq d$, then let $a = d+x$, $b = d+y$, $c = d+z$ where $x \geq y \geq z \geq 0$. After expanding and rearranging terms the inequality is equivalent to

$$4d^3 + 3(x+y+z)d^2 + 2(xy+yz+zx)d + x^3 + y^3 + z^3 + 3xyz - \sum_{cyc} xy(x+y) \geq 0,$$

clearly true by Schur's Inequality.

The equality holds when $x = y = z$ which means $a = b = c$, $d = 0$ or when $x = y$, $z = 0$ which means $a = b$, $c = d = 0$. □

Starting from a result by Nguyen Tuan Anh for 3 variables, we found the following generalization for 4 variables:

Lemma 3. Let a, b, c, d be nonnegative real numbers such that

$$3(a+b+c+d) = 2(ab+ac+ad+bc+bd+cd) > 0.$$

If we set $s = a+b+c+d = \frac{4t^2}{2t-1}$ where $t \geq 1$, then we have the following inequalities

$$\frac{3t^4 - 2t^5}{2t-1} \leq abcd \leq \frac{4t^5 - 3t^4}{(2t-1)^4},$$

$$\frac{4(2t^3 - t^4)}{2t-1} \leq abc + bcd + cda + dab \leq \frac{4(3t^4 - 2t^3)}{(2t-1)^3}.$$

Proof. First, note that $t = \frac{s + \sqrt{s(s-4)}}{4} \geq 1$ and the necessity of this substitution will be seen below. We consider the polynomial $P : \mathbb{R} \mapsto \mathbb{R}$,

$$P(x) = (x-a)(x-b)(x-c)(x-d) = x^4 - sx^3 + \frac{3s}{2}x^2 - qx + r,$$

where we denote $q = abc + bcd + cda + dab$ and $r = abcd$.

1. First we will consider the case where $q > 0$ and $r > 0$, so a, b, c, d are positive numbers. By Rolle's theorem, the first derivative $P'(x)$ of P has three positive roots. We have

$$P'(x) = 4x^3 - 3sx^2 + 3sx - q$$

and second derivative of P that is $P''(x) = 3(4x^2 - 2sx + s)$ has the roots $\frac{s \pm \sqrt{s(s-4)}}{4}$ that is $\frac{t}{2t-1}$ and t . It follows that P' is strictly increasing on $\left[0, \frac{t}{2t-1}\right] \cup [t, \infty)$ and strictly decreasing on $\left[\frac{t}{2t-1}, t\right]$ and since it has three positive roots we deduce that $P'\left(\frac{t}{2t-1}\right) \geq 0$ and $P'(t) \leq 0$. Doing calculations, we get

$$4t^3 - 3st^2 + 3st \leq q \leq \frac{4t^3}{(2t-1)^3} - \frac{3st^2}{(2t-1)^2} + \frac{3st}{2t-1},$$

or

$$4t^3 - \frac{12t^4}{2t-1} + \frac{12t^3}{2t-1} \leq q \leq \frac{4t^3}{(2t-1)^3} - \frac{12t^4}{(2t-1)^3} + \frac{12t^3}{(2t-1)^2},$$

or

$$\frac{4(2t^3 - t^4)}{2t-1} \leq q \leq \frac{4(3t^4 - 2t^3)}{(2t-1)^3}.$$

On the left side the equality holds when $\left(\frac{3t-2t^2}{2t-1}, t, t, t\right)$ and on the right side the equality holds when $\left(\frac{t}{2t-1}, \frac{t}{2t-1}, \frac{t}{2t-1}, \frac{4t^2-3t}{2t-1}\right)$ (or any cyclic permutation).

Now consider the function $f : (0, \infty) \mapsto \mathbb{R}$ defined as

$$f(x) = \frac{P(x)}{x} = x^3 - sx^2 + \frac{3s}{2}x - q + \frac{r}{x}.$$

Since $f(a) = f(b) = f(c) = f(d) = 0$, by Rolle's theorem there is three positive points at which the first derivative $f'(x)$ of the function f is zero. We have

$$f'(x) = 3x^2 - 2sx + \frac{3s}{2} - \frac{r}{x^2} = \frac{6x^4 - 4sx^3 + 3sx^2 - 2r}{x^2} = \frac{Q(x)}{x^2}.$$

We deduce that the polynomial Q , of degree 4, has three positive roots and since $\lim_{x \rightarrow 0} Q(x) = -2r < 0$, $\lim_{x \rightarrow \infty} Q(x) = \infty$ it follow that the fourth root of Q , defined on $\mathbb{R}$, is negative.

To study the monotonicity of the function Q we need to evaluate first derivative on $(0, \infty)$. We have

$$Q'(x) = 24x^3 - 12sx^2 + 6sx = 6x(4x^2 - 2sx + s) = 24x \left(x - \frac{t}{2t-1}\right)(x - t).$$

It follows that Q is strictly increasing on $\left[0, \frac{t}{2t-1}\right] \cup [t, \infty)$ and strictly decreasing on $\left[\frac{t}{2t-1}, t\right]$. Since it has three positive roots we deduce that $Q\left(\frac{t}{2t-1}\right) \geq 0$ and $Q(t) \leq 0$. Doing calculations, we get

$$6t^4 - 4st^3 + 3st^2 \leq 2r \leq \frac{6t^4}{(2t-1)^4} - \frac{4st^3}{(2t-1)^3} + \frac{3st^2}{(2t-1)^2},$$

or

$$\frac{3t^4 - 2t^5}{2t - 1} \leq r \leq \frac{4t^5 - 3t^4}{(2t - 1)^4}.$$

On the left side the equality holds when $\left(\frac{3t-2t^2}{2t-1}, t, t, t\right)$ and on the right side the equality holds when $\left(\frac{t}{2t-1}, \frac{t}{2t-1}, \frac{t}{2t-1}, \frac{4t^2-3t}{2t-1}\right)$ (or any cyclic permutation).

2. If $q = 0$ which means that two of the numbers a, b, c, d are zero, let's say $c = d = 0$. We have

$$3(a + b) = 2ab \leq \frac{(a + b)^2}{2} \implies a + b \geq 6$$

therefore

$$\frac{4t^2}{2t - 1} \geq 6 \implies t \geq \frac{3 + \sqrt{3}}{2} > 2,$$

so

$$\frac{4(2t^3 - t^4)}{2t - 1} < 0 = q < \frac{4(3t^4 - 2t^3)}{(2t - 1)^3}.$$

Also, if $r = 0$ we have two possibilities:

i) $a, b, c > 0, d = 0$ so

$$3(a + b + c) = 2(ab + bc + ca) \leq \frac{2(a + b + c)^2}{3} \implies a + b + c \geq \frac{9}{2}$$

therefore

$$\frac{4t^2}{2t - 1} \geq \frac{9}{2} \implies t \geq \frac{3}{2},$$

so

$$\frac{3t^4 - 2t^5}{2t - 1} \leq 0 = r < \frac{4t^5 - 3t^4}{(2t - 1)^4}.$$

ii) $a, b > 0, c = d = 0$ so

$$3(a + b) = 2ab \implies t \geq \frac{3 + \sqrt{3}}{2} > 2,$$

so

$$\frac{3t^4 - 2t^5}{2t - 1} < 0 = r < \frac{4t^5 - 3t^4}{(2t - 1)^4}.$$

□

Remarks.

1) With the substitution $s = \frac{4t^2}{2t-1}$, $t \geq 1$, as in Lemma 3, the conclusion of Lemma 1 can be written as follows

$$abcd \geq \frac{t^6(4t - 3)(3 - 2t)}{(2t - 1)^4} \quad (4)$$

and

$$abc + bcd + cda + dab \geq \frac{4t^4(22t - 11 - 8t^2)}{3(2t - 1)^3}. \quad (5)$$

In (4) the expression on the right side is an acceptable lower bound only for $1 \leq t \leq \frac{3}{2}$ and in (5) for $22t - 11 - 8t^2 \geq 0 \implies 1 \leq t \leq \frac{11+\sqrt{33}}{8} \approx 2.09307$.

Also, the conclusion of Lemma 2 in the same terms is written as

$$abc + bcd + cda + dab \geq \frac{32t^4(6t - 3 - 2t^2)}{9(2t - 1)^3}. \quad (6)$$

which is an acceptable lower bound only for $6t - 3 - 2t^2 \geq 0 \implies 1 \leq t \leq \frac{3+\sqrt{3}}{2} \approx 2.36603$.

2) For $1 \leq t \leq \frac{3}{2}$ the lower bounds in Lemma 3 for $abcd$ and $abc + bcd + cda + dab$ are stronger than those in Lemma 1. Indeed, we have

$$\frac{3t^4 - 2t^5}{2t - 1} - \frac{t^6(4t - 3)(3 - 2t)}{(2t - 1)^4} = \frac{t^4(t - 1)^2(3 - 2t)(4t - 1)}{(2t - 1)^4} \geq 0$$

and

$$\frac{4(2t^3 - t^4)}{2t - 1} - \frac{4t^4(22t - 11 - 8t^2)}{3(2t - 1)^3} = \frac{8t^3(t - 1)^2(3 - 2t)}{3(2t - 1)^3} \geq 0.$$

Also, for $1 \leq t \leq \frac{3}{2}$ the lower bound for $abc + bcd + cda + dab$ in Lemma 3 are stronger than those in Lemma 2. We have

$$\frac{4(2t^3 - t^4)}{2t - 1} - \frac{32t^4(6t - 3 - 2t^2)}{9(2t - 1)^3} = \frac{4t^3(3 - 2t)(10t^2 - 15t + 6)}{9(2t - 1)^3} \geq 0.$$

Note that for $t \geq \frac{3}{2}$ the lower bound for $abc + bcd + cda + dab$ in Lemma 2 are stronger than those in Lemma 1. We have

$$\frac{32t^4(6t - 3 - 2t^2)}{9(2t - 1)^3} - \frac{4t^4(22t - 11 - 8t^2)}{3(2t - 1)^3} = \frac{4t^4(4t - 3)(2t - 3)}{9(2t - 1)^3} \geq 0.$$

Example 1.(Marius Stănean) Let a, b, c, d be nonnegative real numbers such that

$$3(a + b + c + d) = 2(ab + ac + ad + bc + bd + cd) > 0.$$

Prove that

$$8(a^3 + b^3 + c^3 + d^3) + \frac{196abcd}{a + b + c + d} \geq 81.$$

Solution. We have

$$a^3 + b^3 + c^3 + d^3 = (a + b + c + d)^3 - 3(a + b + c + d) \sum_{sym} ab + 3 \sum_{cyc} abc,$$

therefore our inequality becomes

$$8(a + b + c + d)^3 - 24 \sum_{cyc} a \sum_{sym} ab + 24 \sum_{cyc} abc + \frac{196abcd}{a + b + c + d} \geq 81. \quad (7)$$

Setting $s = a + b + c + d$, according to Lemma 1 we have

$$abcd \geq \frac{s^3(9 - 2s)}{64},$$

$$abc + bcd + cda + dab \geq \frac{s^2(11 - 2s)}{12}.$$

We have two cases:

1. If $4 \leq s \leq \frac{9}{2}$, then returning to inequality (7) and using these results, we have to prove the following inequality

$$8s^3 - 36s^2 + 2s^2(11 - 2s) + \frac{196s^2(9 - 2s)}{64} - 81 \geq 0.$$

After factoring, this becomes

$$(s - 4)(9 - 2s)(17s + 36) \geq 0,$$

clearly true.

2. If $s > \frac{9}{2}$, then by Cauchy-Schwarz Inequality, we have

$$\begin{aligned}(a^3 + b^3 + c^3 + d^3)(a + b + c + d) &\geq (a^2 + b^2 + c^2 + d^2)^2 \\ &= ((a + b + c + d)^2 - 2(ab + ac + ad + bc + bd + cd))^2 \\ &= (a + b + c + d)^2(a + b + c + d - 3)^2\end{aligned}$$

which means

$$a^3 + b^3 + c^3 + d^3 \geq s(s - 3)^2 > \frac{9}{2} \left(\frac{9}{2} - 3 \right)^2 = \frac{81}{8},$$

hence

$$8(a^3 + b^3 + c^3 + d^3) + \frac{196abcd}{a + b + c + d} > 81$$

The equality holds when $a = b = c = d = 1$ or $a = b = c = \frac{3}{2}$, $d = 0$ and its cyclic permutations. $\square$

Remark. The inequality is stronger than the inequality O701 from Mathematical Reflections 3, 2025: Let a, b, c, d be nonnegative real numbers such that

$$3(a + b + c + d) = 2(ab + ac + ad + bc + bd + cd) > 0.$$

Prove that

$$8(a^3 + b^3 + c^3 + d^3) + 49abcd \geq 81.$$

Example 2. Let a, b, c, d be nonnegative real numbers. Prove that

$$\frac{a^2 + b^2 + c^2 + d^2}{ab + ac + ad + bc + bd + cd} + \frac{12abcd}{(ab + ac + bc + ad + bd + cd)^2} \geq 1.$$

Solution. The inequality can be rewritten as

$$\frac{(a + b + c + d)^2}{ab + ac + ad + bc + bd + cd} + \frac{12abcd}{(ab + ac + bc + ad + bd + cd)^2} \geq 3.$$

By normalizing to $3(a + b + c + d) = 2(ab + ac + ad + bc + bc + cd) > 0$, we need to show that

$$\frac{2}{3}(a + b + c + d) + \frac{16abcd}{3(a + b + c + d)^2} \geq 3.$$

Setting $a + b + c + d = \frac{4t^2}{2t-1}$, $t \geq 1$, the inequality becomes

$$\frac{8t^2}{2t-1} + \frac{abcd(2t-1)^2}{t^4} \geq 9,$$

or

$$abcd \geq \frac{(4t-3)(3-2t)t^4}{(2t-1)^3}.$$

Now, if $t > \frac{3}{2}$ the inequality is obviously true since the right side of the inequality is a negative number. So it remains to study the case $t \leq \frac{3}{2}$. According to Lemma 3, we have

$$abcd \geq \frac{3t^4 - 2t^5}{2t-1},$$

so it remains to prove that

$$\frac{3t^4 - 2t^5}{2t - 1} \geq \frac{(4t - 3)(3 - 2t)t^4}{(2t - 1)^3}$$

that is

$$\frac{4t^4(t - 1)^2(3 - 2t)}{(2t - 1)^3} \geq 0,$$

clearly true.

The equality holds when $t = 1$ which means $a = b = c = d = 1$ or when $t = \frac{3}{2}$ which means $a = b = c = \frac{3}{2}$, $d = 0$ and its cyclic permutations. $\square$

Example 3. Let a, b, c, d be positive real numbers such that $a + b + c + d = 2$. Prove that

$$a^2 + b^2 + c^2 + d^2 \leq \frac{4}{9\sqrt{3abcd}}.$$

Solution. After homogenizing, the inequality becomes

$$36\sqrt{3abcd}(a^2 + b^2 + c^2 + d^2) \leq (a + b + c + d)^4.$$

By normalizing to $3(a + b + c + d) = 2(ab + ac + ad + bc + bc + cd) > 0$, we need to show that

$$36\sqrt{3abcd}(a + b + c + d - 3) \leq (a + b + c + d)^3.$$

Setting $a + b + c + d = \frac{4t^2}{2t-1}$, $t \geq 1$, the inequality becomes

$$9\sqrt{3abcd} \leq \frac{16t^6}{(4t^2 - 6t + 3)(2t - 1)^2},$$

or

$$abcd \leq \frac{256t^{12}}{243(4t^2 - 6t + 3)^2(2t - 1)^4}.$$

According to Lemma 3, we have

$$abcd \leq \frac{4t^5 - 3t^4}{(2t - 1)^4},$$

so it remains to prove that

$$\frac{4t^5 - 3t^4}{(2t - 1)^4} \leq \frac{256t^{12}}{243(4t^2 - 6t + 3)^2(2t - 1)^4},$$

or

$$256t^8 \geq 243(4t - 3)(4t^2 - 6t + 3)^2,$$

that is

$$(3 - 2t)^2(64t^6 + 192t^5 + 432t^4 - 3024t^3 + 4536t^2 - 2916t + 729) \geq 0,$$

which is true for $t \geq 1$.

The equality holds when $t = \frac{3}{2}$ which means $a = b = c = \frac{3}{4}$, $d = \frac{9}{4}$. In the original inequality, the equality holds when $a = b = c = \frac{1}{3}$, $d = 1$ and its cyclic permutations. $\square$

Example 4. Let a, b, c, d nonnegative real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that

$$a^3 + b^3 + c^3 + d^3 + abc + bcd + cda + dab \leq 8.$$

Solution. We have

$$a^3 + b^3 + c^3 + d^3 = (a + b + c + d)^3 - 3(a + b + c + d) \sum_{sym} ab + 3 \sum_{cyc} abc,$$

therefore our inequality becomes

$$(a + b + c + d)^3 - 3(a + b + c + d) \sum_{sym} ab + 4 \sum_{cyc} abc \leq 8,$$

or after homogenizing

$$(a + b + c + d)^3 - 3(a + b + c + d) \sum_{sym} ab + 4 \sum_{cyc} abc \leq (a^2 + b^2 + c^2 + d^2) \sqrt{a^2 + b^2 + c^2 + d^2}.$$

If $\sum_{sym} ab = 0$, which means that three of the numbers a, b, c, d are equal to 0 in the above inequality we have the equality. Otherwise, by normalizing to $3(a + b + c + d) = 2(ab + ac + ad + bc + bc + cd) > 0$, we need to show that

$$(a + b + c + d)^3 - \frac{9(a + b + c + d)^2}{2} + 4 \sum_{cyc} abc \leq [(a + b + c + d)(a + b + c + d - 3)]^{3/2}.$$

Setting $a + b + c + d = \frac{4t^2}{2t-1}$, $t \geq 1$, the inequality becomes

$$\sum_{cyc} abc \leq \frac{2t^3(4t^2 - 6t + 3)^{3/2} - 16t^6 + 36t^5 - 18t^4}{(2t - 1)^3}.$$

According to Lemma 3, we have

$$abc + bcd + cda + dab \leq \frac{4(3t^4 - 2t^3)}{(2t - 1)^3},$$

so it remains to prove that

$$2(3t - 2) \leq (4t^2 - 6t + 3)^{3/2} - 8t^3 + 18t^2 - 9t,$$

or

$$(4t^2 - 6t + 3)^{3/2} \geq (2t - 1)(4t^2 - 7t + 4)$$

or, after squaring both sides and a few calculations,

$$(t - 1)^2(12t^2 - 20t + 11) \geq 0$$

clearly true.

The equality holds when $t = 1$ which means $a = b = c = d = 1$. □

Example 5. Let a, b, c, d nonnegative real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that

$$2(a^3 + b^3 + c^3 + d^3) + abcd \geq 9.$$

Solution. We can rewrite the inequality as

$$2 \left[(a + b + c + d)^3 - 3(a + b + c + d) \sum_{sym} ab + 3 \sum_{cyc} abc \right] + abcd \geq 9,$$

or after homogenizing

$$16\sqrt{a^2 + b^2 + c^2 + d^2} \left[(a + b + c + d)^3 - 3(a + b + c + d) \sum_{sym} ab + 3 \sum_{cyc} abc \right] + 16abcd$$

$$\geq 9(a^2 + b^2 + c^2 + d^2)^2.$$

If $\sum_{sym} ab = 0$, which means that three of the numbers a, b, c, d are equal to 0 the inequality is true. Otherwise, by normalizing to $3(a + b + c + d) = 2(ab + ac + ad + bc + bc + cd) > 0$, we need to show that

$$16\sqrt{(a+b+c+d)(a+b+c+d-3)} \left[(a + b + c + d)^3 - \frac{9}{2}(a + b + c + d)^2 + 3 \sum_{cyc} abc \right] + 16abcd$$

$$\geq 9(a + b + c + d)^2(a + b + c + d - 3)^2.$$

Setting $a + b + c + d = \frac{4t^2}{2t-1}$, $t \geq 1$, the inequality becomes

$$\frac{32t\sqrt{4t^2 - 6t + 3}}{2t - 1} \left(\frac{8t^4(4t - 3)(2t - 3)}{(2t - 1)^3} + 3 \sum_{cyc} abc \right) + 16abcd \geq \frac{144t^4(4t^2 - 6t + 3)^2}{(2t - 1)^4}. \quad (8)$$

We have three cases:

1. If $t \geq \frac{3+\sqrt{3}}{2}$ we consider in (8) that $abcd \geq 0$ and $\sum_{cyc} abc \geq 0$ so it suffices to prove that

$$\frac{32t\sqrt{4t^2 - 6t + 3}}{2t - 1} \left(\frac{8t^4(4t - 3)(2t - 3)}{(2t - 1)^3} \right) \geq \frac{144t^4(4t^2 - 6t + 3)^2}{(2t - 1)^4}.$$

By simplifying and squaring, it becomes

$$256t^2(4t - 3)^2(2t - 3)^2 \geq 81(4t^2 - 6t + 3)^3,$$

or

$$(20t^2 - 54t + 27)(20t^2 - 6t - 9)(28t^2 - 42t + 9) \geq 0$$

which is true for $t \geq \frac{27+3\sqrt{21}}{20} \approx 2.03739$.

2. If $\frac{3}{2} \leq t < \frac{3+\sqrt{3}}{2}$ we consider in (8) that $abcd \geq 0$ and $\sum_{cyc} abc \geq \frac{32t^4(6t-3-2t^2)}{9(2t-1)^3}$, according to (6). We will prove that

$$\frac{32t\sqrt{4t^2 - 6t + 3}}{2t - 1} \left(\frac{8t^4(4t - 3)(2t - 3)}{(2t - 1)^3} + \frac{32t^4(6t - 3 - 2t^2)}{3(2t - 1)^3} \right) \geq \frac{144t^4(4t^2 - 6t + 3)^2}{(2t - 1)^4}. \quad (9)$$

By simplifying and squaring, it becomes

$$256t^2(16t^2 - 30t + 15)^2 \geq 729(4t^2 - 6t + 3)^3.$$

But this inequality is true because

$$16t^2 - 30t + 15 \geq 2(4t^2 - 6t + 3) \iff (4t - 3)(2t - 3) \geq 0$$

and

$$2t^2(16t^2 - 30t + 15) \geq 3(4t^2 - 6t + 3)^2$$

$$\iff 2(4t - 3)(2t - 3) \left(t - \frac{3 - \sqrt{3}}{2} \right) \left(t - \frac{3 + \sqrt{3}}{2} \right) \leq 0$$

so

$$\frac{t^2(16t^2 - 30t + 15)^2}{(4t^2 - 6t + 3)^3} \geq \frac{2t^2(16t^2 - 30t + 15)}{(4t^2 - 6t + 3)^2} \geq 3 > \frac{729}{256}.$$

3. If $1 \leq t < \frac{3}{2}$, according to Lemma 3 (see remark 2) we have

$$abcd \geq \frac{3t^4 - 2t^5}{2t - 1}$$

and

$$\sum_{cyc} abc \geq \frac{4(2t^3 - t^4)}{2t - 1}.$$

so to prove (8) it suffices to show that

$$\begin{aligned} & \frac{32t\sqrt{4t^2 - 6t + 3}}{2t - 1} \left(\frac{8t^4(4t - 3)(2t - 3)}{(2t - 1)^3} + \frac{12t^3(2 - t)}{2t - 1} \right) + \frac{16t^4(3 - 2t)}{2t - 1} \\ & \geq \frac{144t^4(4t^2 - 6t + 3)^2}{(2t - 1)^4}, \end{aligned}$$

or after clearing denominators and simplifying

$$2\sqrt{4t^2 - 6t + 3}(4t^3 - 9t + 6) \geq 40t^4 - 120t^3 + 147t^2 - 86t + 21.$$

After squaring both sides and some calculations, denoting $x = t - 1 \in [0, \frac{1}{2})$, we get

$$x^2(24 + 88x + 311x^2 + 208x^3 - 240x^4 - 1536x^5 - 1344x^6) \geq 0$$

which is true because

$$\begin{aligned} & 24 + 88x + 311x^2 + 208x^3 - 240x^4 - 1536x^5 - 1344x^6 \\ & > 24 + 88x + 311x^2 + 208x^3 - \frac{1}{2^3}(240x + 1536x^2 + 1344x^3) \\ & = 24 + 88x + 311x^2 + 208x^3 - (30x + 192x^2 + 168x^3) > 0. \end{aligned}$$

The equality holds when $t = 1$ which means $a = b = c = d$. □

Example 6. Let a, b, c, d nonnegative real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that

$$a^3 + b^3 + c^3 + d^3 + \frac{abc + bcd + cda + dab}{4} \geq 5.$$

Solution. We can rewrite the inequality as

$$(a + b + c + d)^3 - 3(a + b + c + d) \sum_{sym} ab + 3 \sum_{cyc} abc + \frac{abc + bcd + cda + dab}{4} \geq 5,$$

or after homogenizing

$$8(a + b + c + d)^3 - 24(a + b + c + d) \sum_{sym} ab + 26 \sum_{cyc} abc \geq 5(a^2 + b^2 + c^2 + d^2) \sqrt{a^2 + b^2 + c^2 + d^2}.$$

If $\sum_{sym} ab = 0$, which means that three of the numbers a, b, c, d are equal to 0 the inequality is true. Otherwise, by normalizing to $3(a + b + c + d) = 2(ab + ac + ad + bc + bd + cd) > 0$, we need to show that

$$4(a + b + c + d)^2 [2(a + b + c + d) - 9] + 26 \sum_{cyc} abc \geq 5(a + b + c + d)^{3/2} (a + b + c + d - 3)^{3/2}.$$

Setting $a + b + c + d = \frac{4t^2}{2t-1}$, $t \geq 1$, the inequality becomes

$$\frac{32t^4(4t-3)(2t-3)}{(2t-1)^3} + 13 \sum_{cyc} abc \geq \frac{20t^3(4t^2-6t+3)\sqrt{4t^2-6t+3}}{(2t-1)^3}. \quad (10)$$

We have three cases:

1. If $t \geq \frac{3+\sqrt{3}}{2}$ it suffices to prove that

$$\frac{32t^4(4t-3)(2t-3)}{(2t-1)^3} \geq \frac{20t^3(4t^2-6t+3)\sqrt{4t^2-6t+3}}{(2t-1)^3},$$

or after clearing denominators and simplifying

$$8t(4t-3)(2t-3) \geq 5(4t^2-6t+3)\sqrt{4t^2-6t+3},$$

true because it is easy to see that

$$2t(2t-3) \geq \frac{2}{3}(4t^2-6t+3)$$

and

$$4(4t-3) \geq \frac{3}{2} \cdot 5\sqrt{4t^2-6t+3}.$$

2. If $\frac{3}{2} \leq t < \frac{3+\sqrt{3}}{2}$, according to Lemma 2 (see remark 1) it suffices to prove that

$$\frac{32t^4(4t-3)(2t-3)}{(2t-1)^3} + \frac{13 \cdot 32t^4(6t-3-2t^2)}{9(2t-1)^3} \geq \frac{20t^3(4t^2-6t+3)\sqrt{4t^2-6t+3}}{(2t-1)^3},$$

or after clearing denominators and simplifying

$$16t(23t^2-42t+21) \geq 45(4t^2-6t+3)\sqrt{4t^2-6t+3}$$

After squaring both sides and some calculations, denoting $x = t - \frac{3}{2} \geq 0$, we get

$$5824x^6 + 141024x^5 + 393600x^4 + 442296x^3 + 232128x^2 + 48222x + 81 \geq 0$$

clearly true.

3. If $1 \leq t < \frac{3}{2}$, according to Lemma 3 (see remark 2) it suffices to prove that

$$\frac{32t^4(4t-3)(2t-3)}{(2t-1)^3} + \frac{52(2t^3-t^4)}{2t-1} \geq \frac{20t^3(4t^2-6t+3)\sqrt{4t^2-6t+3}}{(2t-1)^3},$$

or after clearing denominators and simplifying

$$12t^3 + 12t^2 - 45t + 26 \geq 5(4t^2-6t+3)\sqrt{4t^2-6t+3}.$$

After squaring both sides and some calculations, denoting $x = t - 1 \in [0, \frac{1}{2})$, we get

$$x^2(105 + 160x + 264x^2 - 1248x^3 - 1456x^4) \geq 0,$$

or

$$x^2 [4 + (1-2x)(728x^3 + 988x^2 + 362x + 101)] \geq 0$$

which is true.

The equality holds when $t = 1$ which means $a = b = c = d$. □

Proposed problems:

1. Let a, b, c, d be nonnegative real numbers such that $a^2 + b^2 + c^2 + d^2 = 3$. Prove that

$$3 + abcd \geq a + b + c + d.$$

2.(Sladjan Stanković) Let a, b, c, d be nonnegative numbers such that $a + b + c + d = 4$. Prove that

$$a^3 + b^3 + c^3 + d^3 + \frac{14}{3}(ab + ac + ad + bc + bd + cd) \geq 32.$$

3.(Vasile Cîrtoaje, 2005) Let a, b, c, d be nonnegative numbers such that $a + b + c + d = 4$. Prove that

$$5(a^2 + b^2 + c^2 + d^2) \geq a^3 + b^3 + c^3 + d^3 + 16.$$

4. Let a, b, c, d be nonnegative numbers such that $a + b + c + d = 4$. Prove that

$$27(abc + bcd + cda + dab) \leq 44abcd + 64.$$

5.(Marius Stănean, O379 Mathematical Reflections 4, 2016) Let a, b, c, d be nonnegative real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that

$$\frac{2}{3}(ab + bc + cd + da + ac + bd) \leq (3 - \sqrt{3})abcd + 1 + \sqrt{3}.$$

6.(Marius Stănean, S401 Mathematical Reflections 1, 2017) Let a, b, c, d be nonnegative real numbers such that $ab + ac + ad + bc + bd + cd = 6$. Prove that

$$a + b + c + d + (3\sqrt{2} - 4)abcd \geq 3\sqrt{2}.$$

7.(Marius Stănean) Let a, b, c, d be nonnegative real numbers such that

$$3(a + b + c + d) = 2(ab + ac + ad + bc + bd + cd) > 0.$$

Prove that

$$a + b + c + d + \frac{1}{a + b + c + d} + \frac{15}{4} \geq 8abcd.$$

8.(Marius Stănean) Let a, b, c, d be nonnegative real numbers such that

$$3(a + b + c + d) = 2(ab + ac + ad + bc + bd + cd) > 0.$$

Prove that

$$a^2 + b^2 + c^2 + d^2 + \frac{11abcd}{a + b + c + d} \geq \frac{27}{4}.$$