

Junior Problems

J745. Let

$$a_k = k + \frac{1}{k} - 2027, \quad k = 1, 2, 3, \dots$$

There is an integer n such that

$$\frac{n}{2a_4} + \frac{n}{4a_{16}} + \frac{n}{5a_{25}} = \frac{1}{29} + \frac{1}{31} + \frac{1}{55} + \frac{1}{65} + \frac{1}{67} + \frac{1}{83}.$$

Find n .

Proposed by Adrian Andreescu, Dallas, USA

J746. Point E is on the side AB of rectangle $ABCD$, and point F is on the segment EB such that

$$AE : EB = EF : AF = 1 : 3.$$

Prove that if $AB = 2BC$, then $\angle DFA = 2\angle ADE$.

Proposed by A.N. Afanasyev, North-Eastern Federal University, Yakutsk, Russia

J747. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$ab(3a + 4) + bc(3b + 4) + ca(3c + 4) \leq 21.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J748. Let a, b, c be positive real numbers. Prove that

$$\frac{a + b + c}{(27a^3 + 8)(27b^3 + 8)(27c^3 + 8)} \leq \frac{1}{729}.$$

Proposed by An Zhenping, Xianyang Normal University, China

J749. Let ABC be a triangle such that $AC > AB$. A circle Γ passes through B and C , meets the ray AB beyond B again at Y , and meets the segment AC again at X . Suppose that Γ is chosen so that $BY^2 + CX^2$ is minimal. Through X draw the line parallel to BC , meeting AB at T . Let M be the midpoint of YC , and let $TM \cap BC = Z$.

Determine the value of $\frac{CZ}{ZB}$.

Proposed by Mihaela Berindeanu, Bucharest, Romania

J750. Prove that if n is a positive integer such that $n + 6$ is divisible by 36, then the sum of all positive divisors of n is also divisible by 36.

Proposed by Alessandro Ventullo, Milan, Italy

Senior Problems

S745. Find all solutions (x, y, z, u, v, w) in positive integers to the system

$$\begin{cases} x^2 + y + 2 = u^2 \\ y^2 + z + 2 = v^2 \\ z^2 + x + 2 = w^2. \end{cases}$$

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

S746. Let $a \geq 1 \geq b \geq c \geq 0$ such that $a + b + c = 3$. Prove that

$$ab + bc + ca \leq \sqrt{4 + 5abc}.$$

Proposed by Marius Stănean, Zalău, Romania

S747. Find all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$f(y^2 + xf(y)) = \sqrt{xy}f(\sqrt{xy}) + y^2$$

for all positive real numbers x and y .

Proposed by Murad Safarli, The Physics, Mathematics, and Informatics Lyceum, Azerbaijan

S748. Let $p_1, \dots, p_k$ be distinct odd primes. Prove that there exist infinitely many integers n such that each p_i divides $n^2 - 1$ but p_i^2 does not divide $n^2 - 1$.

Proposed by Alessandro Ventullo, Milan, Italy

S749. Find all positive real constants k such that

$$9(a^2 + k)(b^2 + k)(c^2 + k) \leq (1 + k)^3(a + b + c)^2$$

for any nonnegative real numbers a, b, c with at most one of them larger than 1 and $ab + bc + ca = 3$.

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania

S750. Let ABC be an acute triangle with orthocenter H . Let AF , BE , and CK be the altitudes of $\triangle ABC$, where $F \in BC$, $E \in CA$, $K \in AB$. Prove that

$$\frac{BC \cdot AC \cdot HE \cdot HF}{EC \cdot FC} + \frac{AB \cdot BC \cdot HF \cdot HK}{FB \cdot KB} + \frac{AB \cdot AC \cdot HK \cdot HE}{KA \cdot EA} \leq \frac{3AB \cdot AC \cdot BC}{AB + AC + BC}.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

Undergraduate Problems

U745. Let (u_n) be a sequence defined by

$$\begin{aligned} u_1 &= 1 \\ u_2 &= 2 + 4 \\ u_3 &= 3 + 5 + 7 \\ u_4 &= 6 + 8 + 10 + 12 \\ &\dots \quad \dots \quad \dots \end{aligned}$$

Evaluate $\lim_{n \rightarrow \infty} \frac{u_n}{n^3}$.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U746. Let $(x_n)_{n \in \mathbb{N}}$ be the sequence defined by

$$x_1 \in [0, 2], \quad x_{n+1} = \frac{5x_n + 3}{x_n + 1} \quad \text{for all } n \geq 1.$$

Let $p \in \mathbb{N}$ be fixed. Evaluate

$$\lim_{n \rightarrow \infty} n^p (2 + \sqrt{7} - x_n).$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

U747. Let $L = \lim_{\lambda \rightarrow +\infty} \frac{\lambda^{x^n}}{\int_a^b \lambda^{t^n} dt}$ and $n > 0$.

(a) Show that if $0 \leq a \leq x < b$ and $n > 0$, then $L = 0$.

(b) Show also that if $0 \leq a < x = b$, then $L = +\infty$.

Proposed by Yagub Aliyev, ADA University, Baku, Azerbaijan

U748. Prove that 6 is the least positive value of the power k such that

$$\left(\frac{a^2 + b^2 + c^2 + d^2}{4} \right)^k \geq \frac{a^4 + b^4 + c^4 + d^4}{4}$$

for any nonnegative real numbers a, b, c, d with $a \geq b \geq c \geq d$ and $ab + bc + cd + da = 4$.

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania

U749. Let n and m be nonnegative integers such that $m \geq 1$, $m > n$. Evaluate

$$\int_0^1 \frac{x^n dx}{(x^2 + x + 1)(1 + x^m)}.$$

Proposed by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Italy

U750. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be an even, strictly convex, and real-analytic function. Assume moreover that $K := \varphi'' \in L^1(\mathbb{R})$ and that its Fourier transform $\widehat{K}(\xi)$ is nowhere zero on $\mathbb{R}$. We say that a function $f : [0, 1] \rightarrow [0, 1]$ is *awesome* if it is continuous, strictly monotone and

$$\int_0^1 \varphi(f(x) - t) dx = \int_0^1 \varphi(x - t) dx$$

for all $t \in [0, 1]$. Find all awesome functions.

Proposed by Alessandro Ventullo, Milan, Italy

Olympiad Problems

O745. Find all prime numbers p and q satisfying

$$p^3 - 4p^2 - 50p = q^9 - 36q.$$

Proposed by George-Florin Șerban, National Pedagogical College “D.P. Perpessicius”, Brăila, Romania

O746. Let a, b, c, d be real numbers such that $a \geq 1 \geq b \geq c \geq d \geq 0$ and $a + b + c + d = 4$. Prove that

$$a^3 + b^3 + c^3 + d^3 + 6(ab + bc + cd + da) \geq 28.$$

Proposed by Marius Stănean, Zalău, Romania

O747. Let $(F_n)_{n \geq 1}$ be the Fibonacci sequence, defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. Let $p > 5$ be a prime number. Prove that there exists $\varepsilon \in \{-1, 1\}$ such that

$$\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} (-1)^{i+j} \binom{i+j}{i}^2 \equiv \varepsilon (2F_{p+\varepsilon} - 1) \pmod{p^2}.$$

Proposed by Titu Andreescu, Dallas, USA and Gabriel Dospinescu, Lyon, France

O748. Let a, b, c be nonnegative real numbers such that at most one of them is less than 1 and $ab + bc + ca = 3$. Prove that

$$\frac{7}{a^3 + b^3 + c^3} + 6 \leq \frac{25}{a + b + c}.$$

Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania

O749. A coloring of the numbers $1, 2, \dots, n$ is called special if the equation $ab = cd$ has no solutions with $a, b, c, d \in \{1, 2, \dots, n\}$ distinct and of the same color. Prove that there exists a constant $c > 0$ such that for all integers $n > 1$, any special coloring of $1, 2, \dots, n$ requires at least $c \frac{n^{1/3}}{\sqrt{\log n}}$ different colors.

Proposed by Titu Andreescu, Dallas, USA and Gabriel Dospinescu, Lyon, France

O750. Let $n \geq 3$ be an integer divisible by 3. Around a circle there are n boxes labeled $0, 1, \dots, n-1$ (indices taken modulo n). In box i there are a_i chips ($a_i \in \mathbb{N}$).

A *move* consists of choosing an index t and adding one chip to each of the three consecutive boxes $t, t+1, t+2$.

Prove that one can reach a configuration in which all boxes contain the same number of chips if and only if

$$\sum_{i=0}^{n-1} a_i \omega^i = 0,$$

where $\omega = e^{2\pi i/3}$ is a primitive cube root of unity.

Proposed by Alessandro Ventullo, Milan, Italy