Contents

1 Preface

The main purpose of this book is to provide an introduction to central topics in elementary Algebra from a problem-solving point of view. While working with students who were preparing for various mathematics competitions or exams, I observed that fundamental algebraic techniques were not part of their mathematical repertoire. Since algebraic skills are not only critical to Algebra itself but also to numerous other mathematical fields, a lack of such knowledge can drastically hinder a student's performance. Taking the above observations into account, I put together this introductory book using both simple and challenging examples which shed light upon essential algebraic strategies and techniques, as well as their application in diverse meaningful problems. This work is the first volume in a series of such books.

Regarding the structure of the book, the featured topics are elementary and classical, including factorizations, algebraic identities, inequalities, algebraic equations and systems of equations. More advanced concepts such as complex numbers, exponents and logarithms, as well as other topics are generally avoided. Nevertheless, some problems are constructed using properties of complex numbers which challenge and expose the reader to a broader spectrum of mathematics. Each chapter focuses on specific methods or strategies and provides an ample collection of accompanying problems that graduate in difficulty and complexity. In order to assist the reader with verifying mastery of the theoretical component, I included 105 problems in the last sections of the book, of which 52 are introductory and 53 advanced. All problems come together with solutions, many employing several approaches and providing the motivation behind the solutions offered.

Enjoy the problems!

2 Completing the square and quadratic equations

While the identity
$$(a+b)^2 = a^2 + 2ab + b^2$$
is easy to check (simply expand the left hand-side, by writing it as $(a+b)(a+b)$), things are a little bit more difficult in real life. Indeed, most of the time we have to do exactly the opposite: we are given a quadratic expression and we want to express it as a sum of squares (or some linear combination of squares). The idea is quite simple: we fix one variable, say x of that expression, such that the expression becomes a quadratic polynomial in x. Say our expression is ax^2+bx+c for some real numbers a, b, c (which may themselves be complicated expressions depending on other real numbers!). Then we complete the square by writing
$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) =$$
$$a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right) = a\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a},$$
where $\Delta = b^2 - 4ac$ is the discriminant of the expression ax^2+bx+c. This eliminates the variable x, by including all of its appearances in the term $a\left(x + \frac{b}{2a}\right)^2$. Now, $-\frac{\Delta}{4a}$ may (or may not...) itself be some quadratic expression in different variables, so we can apply the same reasoning to write it as a sum of squares.

In particular, the previous discussion applies to the quadratic equation
$$ax^2 + bx + c = 0,$$
where a, b, c are given real numbers, with a nonzero (if $a = 0$, then we obtain a linear equation). The previous paragraph shows that the equation can be written as
$$\left(x + \frac{b}{2a}\right)^2 = \frac{\Delta}{4a^2}.$$
If the equation has real solutions, then the left hand-side must be nonnegative (as is the square of any real number). Hence so must be the right hand-side, which means that $\Delta \geq 0$. In this case, we can solve the previous equation by taking square roots, and we end up with the solutions
$$x_1 = \frac{-b + \sqrt{\Delta}}{2a}, \quad x_2 = \frac{-b - \sqrt{\Delta}}{2a},$$
which are equal if and only if $\Delta = 0$. Hence we can summarize our discussion in:

Theorem 2.1. *Let a, b, c be real numbers with $a \neq 0$ and let*

$$\Delta = b^2 - 4ac.$$

Then the quadratic equation

$$ax^2 + bx + c = 0$$

has either:

- *no real solution, if $\Delta < 0$.*
- *exactly one real solution, if $\Delta = 0$.*
- *two real solutions if $\Delta > 0$.*

Note that the previous discussion also gives a nice way of solving quadratic inequalities or proving inequalities involving quadratic expressions: since

$$ax^2 + bx + c = a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{-\Delta}{4a^2}\right),$$

we see that the expression $ax^2 + bx + c$ has constant sign (equal to that of a) when $\Delta \leq 0$. On the other hand, if $\Delta > 0$ and if $x_1 \leq x_2$ are the real solutions of the equation $ax^2 + bx + c = 0$, then the inequality $ax^2 + bx + c \leq 0$ is equivalent to $a(x - x_1)(x - x_2) \leq 0$. If $a > 0$, this is in turn equivalent to $x \in [x_1, x_2]$, while if $a < 0$, this is equivalent to $x \notin (x_1, x_2)$. To summarize, suppose for the sake of simplicity that $a > 0$. Then

- if $\Delta = b^2 - 4ac < 0$, then $ax^2 + bx + c > 0$ for all real numbers x.
- If $\Delta = 0$, then $ax^2 + bx + c \geq 0$ for all real numbers x, with equality if and only if $x = -\frac{b}{2a}$.
- If $\Delta > 0$, then the equation $ax^2 + bx + c = 0$ has two real roots, say $x_1 < x_2$, and we have $ax^2 + bx + c < 0$ if and only if $x \in (x_1, x_2)$.

A consequence of this discussion is the following important fact (which is a very special case of a general theorem in real analysis):

Theorem 2.2. *Let $f(x) = ax^2 + bx + c$ be a quadratic polynomial and let $u \leq v$ be real numbers such that $f(u)f(v) < 0$. Then the equation $f(x) = 0$ has at least one solution in (u, v).*

Proof. Since f changes sign between u and v, its discriminant $\Delta = b^2 - 4ac$ must be positive and so the equation $f(x) = 0$ has two distinct solutions $x_1 < x_2$. If none of them belongs to (u, v), then the previous discussion shows that either $f(u), f(v) > 0$ or $f(u), f(v) < 0$ (according to the sign of a). But this contradicts the hypothesis that $f(u)$ and $f(v)$ have different signs. □

Actually the previous theorem holds for any polynomial function (and more generally for continuous functions), but the proof is beyond the scope of this introductory book. Another very important result concerning quadratic equations is the following:

Theorem 2.3. *(Vieta's relations for quadratic equations) Let a, b, c be real numbers, with $a \neq 0$ and let x_1, x_2 be the roots of the equation $ax^2+bx+c=0$. Then*

$$x_1 + x_2 = -\frac{b}{a} \quad \textit{and} \quad x_1x_2 = \frac{c}{a}.$$

Proof. Since $ax^2 + bx + c = 0$ has roots x_1, x_2, we must have an equality of polynomials

$$ax^2 + bx + c = a(x - x_1)(x - x_2) = ax^2 - a(x_1 + x_2)x + ax_1x_2.$$

Identifying coefficients yields

$$x_1 + x_2 = -\frac{b}{a} \quad \text{and} \quad x_1x_2 = \frac{c}{a},$$

which is exactly what we wanted to prove. □

We remark that the previous theorem holds for complex roots, and roots with multiplicity, with the same proof.

It is now time for practice: we will see how the above theoretical facts really apply in practice.

Example 2.1. Solve the equation

$$\frac{(2x-1)^2}{2} + \frac{(3x-1)^2}{3} + \frac{(6x-1)^2}{6} = 1.$$

Solution. Expanding each term and collecting terms according to the successive powers of x yields the following equivalent equations

$$2x^2 - 2x + \frac{1}{2} + 3x^2 - 2x + \frac{1}{3} + 6x^2 - 2x + \frac{1}{6} = 1,$$

$$11x^2 - 6x = 0 \quad \text{or} \quad x(11x - 6) = 0.$$

Hence the solutions are $x = 0$ and $x = \frac{6}{11}$.

Example 2.2. Find the greatest integer n for which the equation

$$\frac{1}{x-1} - \frac{1}{nx} + \frac{1}{x+1} = 0$$

has real solutions.

Solution. Write the equation successively as

$$\frac{1}{x-1} + \frac{1}{x+1} = \frac{1}{nx},$$

then $\frac{2x}{x^2-1} = \frac{1}{nx}$, $2nx^2 = x^2 - 1$ and finally $(-2n+1)x^2 = 1$. Since $x^2 \geq 0$ for all real numbers x, we deduce that if the equation has real solutions, then $-2n+1 > 0$, hence $n \leq 0$. We cannot have $n = 0$, since then $\frac{1}{nx}$ wouldn't make sense. Hence the largest n is at most -1. And indeed $n = -1$ gives the real solutions $x = \frac{1}{\sqrt{3}}$ and $x = -\frac{1}{\sqrt{3}}$, so the answer is $n = -1$.

Example 2.3. Solve the system of equations

$$\begin{cases} x - y = 3 \\ x^2 + (x+1)^2 = y^2 + (y+1)^2 + (y+2)^2. \end{cases}$$

Solution. We take advantage of the fact that the first equation is very simple and express $x = y + 3$, replacing this value in the second equation. This yields the quadratic equation

$$(y+3)^2 + (y+4)^2 = y^2 + (y+1)^2 + (y+2)^2.$$

Expanding each term and collecting similar terms, we obtain the equivalent equation

$$2y^2 + 14y + 25 = 3y^2 + 6y + 5 \quad \text{or} \quad y^2 - 8y - 20 = 0.$$

Solving this equation gives $y = -2, 10$ and since $x = y + 3$, we obtain the solutions $(x, y) = (1, -2)$ and $(13, 10)$.

Example 2.4. Evaluate

$$\frac{1}{\sqrt{x + 2\sqrt{x-1}}} + \frac{1}{\sqrt{x - 2\sqrt{x-1}}},$$

where $1 \leq x < 2$.

Solution. We start by simplifying each fraction, by completing the squares at the denominator. We have

$$x + 2\sqrt{x-1} = x - 1 + 2\sqrt{x-1} + 1 = (\sqrt{x-1} + 1)^2$$

and

$$x - 2\sqrt{x-1} = x - 1 - 2\sqrt{x-1} + 1 = (\sqrt{x-1} - 1)^2.$$

Thus, paying attention to the fact that $\sqrt{a^2} = |a|$ and $\sqrt{x-1} - 1 < 0$ (since $x < 2$), we obtain

$$\frac{1}{\sqrt{x+2\sqrt{x-1}}} + \frac{1}{\sqrt{x-2\sqrt{x-1}}} = \frac{1}{1+\sqrt{x-1}} + \frac{1}{1-\sqrt{x-1}}$$
$$= \frac{2}{(1+\sqrt{x-1})(1-\sqrt{x-1})} = \frac{2}{1-(x-1)} = \frac{2}{2-x}.$$

Example 2.5. Solve the system of equations

$$\begin{cases} x + \frac{1}{y} = -1 \\ y + \frac{1}{z} = \frac{1}{2} \\ z + \frac{1}{x} = 2. \end{cases}$$

Solution. The idea is quite simple: we express everything in terms of one variable. Namely, from the first equation we can express x in terms of y, obtaining $x = -1 - \frac{1}{y}$. The second equation gives

$$z = \frac{2}{1-2y}.$$

Replacing these values in the last equation, we obtain

$$\frac{2}{1-2y} - \frac{y}{1+y} = 2.$$

Clearing denominators and simplifying the resulting equation, we arrive at

$$y + 2y^2 = 0.$$

Note that $y \neq 0$, since otherwise $\frac{1}{y}$ wouldn't make sense. We conclude that $y = -\frac{1}{2}$. Coming back to $x = -1 - \frac{1}{y}$ and $z = \frac{2}{1-2y}$, we obtain $x = z = 1$, hence the system has the unique solution $(1, -\frac{1}{2}, 1)$.

Example 2.6. Solve the equation

$$\frac{1}{3x-1} + \frac{1}{4x-1} + \frac{1}{7x-1} = 1.$$

Solution. The algebra would be quite nasty if we tried to clear denominators. Instead, we rewrite the equation as

$$\frac{1}{3x-1} + \frac{1}{4x-1} = 1 - \frac{1}{7x-1}$$

or equivalently

$$\frac{4x-1+3x-1}{(3x-1)(4x-1)} = \frac{7x-1-1}{7x-1}.$$

We remark the common factor $7x-2$, which already gives us the solution $x = \frac{2}{7}$. Suppose that $x \neq \frac{2}{7}$ is another solution. Then dividing the previous relation by $7x-2$ yields

$$\frac{1}{(3x-1)(4x-1)} = \frac{1}{7x-1} \quad \text{or} \quad 12x^2 - 7x + 1 = 7x - 1.$$

This can be further simplified to $6x^2 - 7x + 1 = 0$. Solving this quadratic equation yields the other solutions $x = 1, \frac{1}{6}$ of the equation. Hence the equation has three solutions, given by $\frac{2}{7}, 1, \frac{1}{6}$.

Example 2.7. Find all pairs (a, b) of positive real numbers such that

$$4a + 9b = \frac{9}{a} + \frac{4}{b} = 12.$$

Solution. We write the second equation as

$$\frac{9b + 4a}{ab} = 12$$

and we observe that the numerator equals 12 by hypothesis. Thus $ab = 1$, that is $b = \frac{1}{a}$. Replacing this value of b in the equation $4a + 9b = 12$ we obtain $4a + \frac{9}{a} = 12$. Clearing denominators, we obtain a quadratic equation $4a^2 - 12a + 9 = 0$, which has the unique solution $a = \frac{3}{2}$. Going back to the system, we obtain $b = \frac{2}{3}$.

If you found the first step (establishing that $ab = 1$) tricky, we can work more directly as follows: from the equation $4a + 9b = 12$ we express b in terms of a. We replace this value of b in the equation $\frac{9}{a} + \frac{4}{b} = 12$, obtaining a quadratic equation in a, with the unique solution $a = \frac{3}{2}$.

Example 2.8. If a is a real number such that $a - \frac{1}{a} = 1$, find $a^4 + \frac{1}{a^4}$.

Solution. It is easier to realize what you shouldn't do in this exercise: you should not solve the equation $a - \frac{1}{a} = 1$ and then plug in the values you get to compute $a^4 + \frac{1}{a^4}$ (of course, with a lot of nasty computations one would obtain the desired answer, but this is far from being an elegant approach). Let us take the square of the given relation $a - \frac{1}{a} = 1$, and obtain

$$a^2 + \frac{1}{a^2} - 2 = 1, \quad \text{that is} \quad a^2 + \frac{1}{a^2} = 3.$$

Now, all we have to do is to repeat the process: we take the square of the last relation and obtain

$$a^4 + \frac{1}{a^4} + 2 = 9, \quad \text{hence} \quad a^4 + \frac{1}{a^4} = 9 - 2 = 7.$$

Example 2.9. Solve the equation

$$x^4 - 97x^3 + 2012x^2 - 97x + 1 = 0.$$

Solution. The key point is that the equation is symmetric. Dividing by x^2, we obtain the equivalent equation

$$x^2 - 97x + 2012 - \frac{97}{x} + \frac{1}{x^2} = 0$$

We reduce this to a quadratic equation by setting

$$x + \frac{1}{x} = y.$$

Then $x^2 + \frac{1}{x^2} + 2 = y^2$, hence the previous equation becomes

$$y^2 - 97y + 2010 = 0, \quad \text{or} \quad (y - 30)(y - 67) = 0.$$

Thus $y = 30$ or $y = 67$. Now, remember that $y = x + \frac{1}{x}$, hence we obtain the quadratic equation $x^2 - xy + 1 = 0$. Solving the two equations $x^2 - 30x + 1 = 0$ and $x^2 - 67x + 1 = 0$ gives the solutions

$$x = \frac{67 \pm \sqrt{4485}}{2}, \quad \text{and} \quad x = 15 \pm \sqrt{224}.$$

Example 2.10. Let a, b, c be real numbers such that $a \geq b \geq c$. Prove that

$$a^2 + ac + c^2 \geq 3b(a - b + c).$$

Solution. Let us rewrite the inequality as

$$3b^2 - 3b(a + c) + a^2 + ac + c^2 \geq 0.$$

This is a quadratic inequality in b. The discriminant is

$$9(a + c)^2 - 12(a^2 + ac + c^2) = -3(a^2 - 2ac + c^2) = -3(a - c)^2 \leq 0,$$

thus the polynomial $3x^2 - 2x(a + c) + a^2 + ac + c^2$ takes only nonnegative values (its leading coefficient 3 is positive). In particular, its value at b is nonnegative, and the result follows.

We can also try to complete squares, writing the inequality as

$$12b^2 - 12b(a + c) + 4(a^2 + ac + c^2) \geq 0,$$

then

$$3(2b - (a + c))^2 + (a - c)^2 \geq 0.$$

Example 2.11. Prove that $3(x+y+1)^2+1 \geq 3xy$ for all $x, y \in \mathbf{R}$.

Solution. Let us write $x+y=a$ and $xy=b$. We need to prove that $3(a+1)^2+1 \geq 3b$. Now, the equation $t^2-at+b=0$ has the real solutions x, y, hence its discriminant is nonnegative, that is $a^2 \geq 4b$ (of course, we can also give a direct proof, since the inequality is equivalent to $(x-y)^2 \geq 0$). Thus $b \leq \frac{a^2}{4}$ and it suffices to prove that

$$3(a+1)^2+1 \geq \frac{3}{4}a^2.$$

Multiplying by 4, expanding $(a+1)^2$ and rearranging terms reduces the inequality to

$$9a^2+24a+16 \geq 0,$$

equivalent to $(3a+4)^2 \geq 0$, thus true.

We note that an alternative solution consists in completing the square, which allows us to rewrite the inequality as

$$3\left(x+\frac{1}{2}y+1\right)^2+\left(\frac{3}{2}y+1\right)^2 \geq 0.$$

Example 2.12. Find all pairs (x, y) of real numbers such that

$$4x^2+9y^2+1=12(x+y-1).$$

Solution. Let us separate the variables by writing the equation in the form

$$4x^2-12x+9y^2-12y+13=0.$$

Next, we complete the square to obtain

$$(2x-3)^2+(3y-2)^2=0.$$

Since a sum of squares equals zero if and only if each square is zero, it follows that $2x-3=0$ and $3y-2=0$. Thus there is only one solution, given by $x=\frac{3}{2}$ and $y=\frac{2}{3}$.

Example 2.13. Prove that if $a \geq b > 0$, then

$$\frac{(a-b)^2}{8a} \leq \frac{a+b}{2}-\sqrt{ab} \leq \frac{(a-b)^2}{8b}.$$

Solution. We complete the square to obtain

$$\frac{a+b}{2}-\sqrt{ab}=\frac{a-2\sqrt{a}\sqrt{b}+b}{2}=\frac{(\sqrt{a}-\sqrt{b})^2}{2}.$$

On the other hand, we have

$$a - b = \sqrt{a}^2 - \sqrt{b}^2 = (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}).$$

Dividing by $(\sqrt{a} - \sqrt{b})^2$, we are therefore reduced to proving the inequalities

$$\frac{(\sqrt{a} + \sqrt{b})^2}{8a} \le \frac{1}{2} \le \frac{(\sqrt{a} + \sqrt{b})^2}{8b}.$$

The inequality on the left is equivalent (after multiplication by $8a$ and taking square roots) to

$$\sqrt{a} + \sqrt{b} \le 2\sqrt{a}$$

and is an immediate consequence of $a \ge b$. We proceed similarly with the inequality on the right.

Example 2.14. Simplify the expression

$$\frac{4}{4x^2 + 12x + 9} - \frac{12}{6x^2 + 5x - 6} + \frac{9}{9x^2 - 12x + 4}.$$

Solution. Here it is easier to say what you should **not** do: clear denominators! Indeed, that would give a terrible mess and chances to solve the exercises with this approach are close to zero. Instead, let us analyze a little bit each term in the sum, more precisely its denominator. Each denominator is a quadratic polynomial in x, so a natural approach would be to see whether it can itself be factored. Of course, a sum of products is not something very enlightening, but one might hope that the denominators have a common factor. Well, solving the quadratic equations $4x^2 + 12x + 9 = 0$, $6x^2 + 5x - 6 = 0$ and $9x^2 - 12x + 4 = 0$, or by completing the square we end up with

$$\begin{aligned} 4x^2 + 12x + 9 &= (2x + 3)^2, \\ 6x^2 + 5x - 6 &= (2x + 3)(3x - 2) \\ 9x^2 - 12x + 4 &= (3x - 2)^2. \end{aligned}$$

It turns out that the don't have a common factor, but they have a quite good shape: if $a = 2x + 3$ and $b = 3x - 2$, then our expression is simply

$$\frac{4}{a^2} - \frac{12}{ab} + \frac{9}{b^2} = \frac{4b^2 - 12ab + 9a^2}{(ab)^2}.$$

Again, it is easy to recognize that the numerator is the square of $(2b - 3a)^2$. Since

$$2b - 3a = 2(3x - 2) - 3(2x + 3) = -13,$$

we obtain the nice equality

$$\frac{4}{4x^2 + 12x + 9} - \frac{12}{6x^2 + 5x - 6} + \frac{9}{9x^2 - 12x + 4} = \left(\frac{13}{6x^2 + 5x - 6}\right)^2.$$

Example 2.15. Solve in real numbers the equation

$$x^4 + 16x - 12 = 0.$$

Solution. We will try to find a, b, c such that the left hand-side can be written as $(x^2 + a)^2 - (bx + c)^2$. If we can find such numbers a, b, c, then solving the equation will come down to solving two quadratic equations $x^2 + a = bx + c$ and $x^2 + a + bx + c = 0$.

The identity

$$x^4 + 16x - 12 = (x^2 + a)^2 - (bx + c)^2$$

is equivalent to the chain of equalities

$$2a = b^2, \quad 16 = -2bc, \quad a^2 - c^2 = -12.$$

Thus $a = \frac{b^2}{2}$, $c = -\frac{8}{b}$ and replacing these in the last equation yields

$$\frac{b^4}{4} - \frac{64}{b^2} = -12.$$

Let $b^2 = 4d$. The equation becomes $4d^2 - \frac{16}{d} = -12$ and we easily recognize the root $d = 1$. Thus we can take $b = 2$ and then $a = \frac{b^2}{2} = 2$ and $c = -\frac{8}{b} = -4$.

Now, it remains to solve the equations $x^2 + 2 = 2x - 4$ and $x^2 + 2 = -2x + 4$. The first one has no real solutions since it can be written as $(x - 1)^2 + 5 = 0$, while the second one can be written $(x + 1)^2 = 3$ and has the solutions

$$x_1 = -1 - \sqrt{3}, \quad x_2 = \sqrt{3} - 1.$$

Example 2.16. The equation $x^4 - 4x = 1$ has two real roots. Find their product.

Solution. Let us add $2x^2 + 1$ to both terms, in order to complete the square in the left hand-side. We obtain the equivalent equation

$$(x^2 + 1)^2 = 2x^2 + 4x + 2 = 2(x + 1)^2.$$

This is equivalent to $x^2 + 1 = \sqrt{2}(x + 1)$ or $x^2 + 1 = -\sqrt{2}(x + 1)$. The first equation is $x^2 - \sqrt{2}x + 1 - \sqrt{2} = 0$ and its discriminant is $\Delta = 4\sqrt{2} - 2 > 0$. Hence it has two solutions x_1, x_2 and their product is given by Vieta's formulae: $x_1 x_2 = 1 - \sqrt{2}$. Since we already know that the initial equation has two real roots, they must be x_1, x_2 and we are done. Of course, it would be very easy to check that the equation $x^2 + 1 = -\sqrt{2}(x + 1)$ has no real root, since its discriminant is negative. Thus the answer of the problem is $1 - \sqrt{2}$.

We can also solve this as follows we look for a, b, c such that

$$x^4 - 4x - 1 = (x^2 + a)^2 - (bx + c)^2$$

for all x, which is equivalent to

$$2a = b^2, \quad bc = 2, \quad a^2 - c^2 = -1.$$

Replacing $a = \frac{b^2}{2}$ and $c = \frac{2}{b}$ in the last equation yields

$$\frac{b^4}{4} - \frac{4}{b^2} = -1.$$

Setting $b^2 = 4d$, this gives us the third degree equation $4d^3 + d - 1 = 0$, with the apparent solution $d = \frac{1}{2}$. Thus $b^2 = 2$ and we can take $b = \sqrt{2}$, then $a = 1$ and $c = \sqrt{2}$. The original equation is therefore reduced to the resolution of the equations $x^2 + 1 = \sqrt{2}x + \sqrt{2}$ and $x^2 + 1 = -\sqrt{2}x - \sqrt{2}$. As above, we obtain the product of the real roots equals $1 - \sqrt{2}$.

Example 2.17. Solve in real numbers the equation

$$(x + 1)(x + 2)(x + 3)(x + 4) = 360.$$

Solution. Expanding the product is out of the question, so there is certainly some trick here. We try to pair the factors in the product defining the left hand-side. If we pair the first two and then the last two we obtain the equation $(x^2+3x+2)(x^2+7x+12) = 360$, which is not simpler. The same thing happens if we pair the first and the third factor, but a miracle happens if we pair the first and the last factor: we obtain the equation

$$(x^2 + 5x + 4)(x^2 + 5x + 6) = 360,$$

which is fourth degree equation in x, but a **quadratic** equation in $y = x^2 + 5x$. And while solving quartic equations is hard, solving quadratic ones is straightforward. Namely, the equation $(y + 4)(y + 6) = 360$ is equivalent to $y^2 + 10y - 336 = 0$, with solutions $y = 14$ and $y = -24$. Next, we have to solve the equations $x^2 + 5x = 14$ and $x^2 + 5x = -24$. The discriminant of the second one is negative, so it does not give real solutions. On the other hand, the equation $x^2 + 5x = 14$ has solutions $x = -7$ and $x = 2$. Hence these two numbers are the solutions of the initial equation.

Example 2.18. Find all $n > 1$ such that

$$x_1^2 + x_2^2 + ... + x_n^2 \geq x_n(x_1 + x_2 + ... + x_{n-1})$$

for all real numbers $x_1, ..., x_n$.

Solution. We write the inequality as

$$x_1^2 - x_1x_n + x_2^2 - x_2x_n + ... + x_{n-1}^2 - x_{n-1}x_n + x_n^2 \geq 0.$$

We complete the squares to get the equivalent inequality

$$\left(x_1 - \frac{x_n}{2}\right)^2 + ... + \left(x_{n-1} - \frac{x_n}{2}\right)^2 - \frac{n-1}{4}x_n^2 + x_n^2 \geq 0,$$

that is

$$\left(x_1 - \frac{x_n}{2}\right)^2 + ... + \left(x_{n-1} - \frac{x_n}{2}\right)^2 \geq \frac{n-5}{4}x_n^2.$$

If $n \leq 5$, then the right hand-side is nonpositive and the left hand-side is nonnegative, hence the inequality holds. On the other hand, if $n > 5$, then we can choose

$$x_1 = \frac{x_n}{2}, ..., x_{n-1} = \frac{x_n}{2}, \quad x_n = 1$$

and the inequality is no longer true. Hence the answer is $n = 2, 3, 4, 5$.

3 Factorizations and algebraic identities

There are a few classical algebraic identities that play a crucial role in almost all branches of mathematics. In this section we recall some of them and give many examples of applications to factorization of algebraic expressions. Being able to recognize factorizations of (sometimes complicated) algebraic expressions is fundamental, since this often plays an important role in solving equations, systems of equations or proving inequalities.

A first fundamental identity is

$$a^2 - b^2 = (a-b)(a+b).$$

This holds for all real numbers a, b (and is actually much more general than that, but we will stick to real numbers from now on) and follows easily by expanding the right hand-side and canceling the terms ab and $-ba$. Though very simple, this identity is crucial when factoring or simplifying more complicated algebraic expressions. It is also a special case of a more general, and also very handy identity, which describes a partial factorization of $a^n - b^n$. Note that $a^n - b^n$ vanishes when $a = b$, hence it must have a factor of $a - b$. We actually have

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + ... + ab^{n-2} + b^{n-1})$$

for all real numbers a, b and all positive integers n. Indeed, we have

$$(a-b)(a^{n-1} + a^{n-2}b + ... + ab^{n-2} + b^{n-1}) =$$

$$a(a^{n-1} + a^{n-2}b + ... + b^{n-1}) - b(a^{n-1} + ... + b^{n-1}) =$$

$$a^n + a^{n-1}b + ... + ab^{n-1} - a^{n-1}b - ... - ab^{n-1} - b^n = a^n - b^n,$$

by canceling equal terms. For instance, if $n = 3$ we get the very useful identity

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

and for $n = 4$ we obtain

$$a^4 - b^4 = (a-b)(a^3 + a^2b + ab^2 + b^3).$$

One may wonder if we can still factor a^2+ab+b^2 and $a^3+a^2b+ab^2+b^3$. This is not the case for a^2+ab+b^2, but the answer is positive for $a^3+a^2b+ab^2+b^3$, since

$$a^3 + a^2b + ab^2 + b^3 = a^2(a+b) + b^2(a+b) = (a+b)(a^2+b^2).$$

We obtain therefore

$$a^4 - b^4 = (a-b)(a+b)(a^2+b^2).$$