Preface

This book contains 106 geometry problems used in the AwesomeMath Summer Program to train and test top middle and high-school students from the U.S. and around the world. Just like the camp offers both introductory and advanced courses, this book also builds up the material gradually. We begin with a theoretical chapter where we familiarize the reader with basic facts and problem-solving techniques. Then we proceed to the main part of the work, the problem sections.

The problems are a carefully selected and balanced mix which offers a vast variety of flavors and difficulties, ranging from AMC and AIME levels to high-end IMO problems. Out of thousands of Olympiad problems from around the globe we chose those which best illustrate the featured techniques and their applications. The problems meet our demanding taste and fully exhibit the enchanting beauty of classical geometry. For every problem we provide a detailed solution and strive to pass on the intuition and motivation lying behind. Many problems have multiple solutions.

Directly experiencing Olympiad geometry both as contestants and instructors, we are convinced that a neat diagram is essential to efficiently solving a geometry problem. Our diagrams do not contain anything superfluous, yet emphasize the key elements and benefit from a good choice of orientation. Many of the proofs should be legible only from looking at the diagrams.

In the theoretical part we cover the basic theorems concerning circles and ratios and conclude with a short excursion to geometric inequalities. However, we feel that most important are the underlying themes that emphasize the unique combination of Eastern European synthetic feel for geometry and the American more computational approach.

True mastery of geometry relies on proficient use of common sense, therefore we chose to avoid analytical and computational techniques such as complex numbers, vectors, or barycentric coordinates. A whole new set of topics will be presented in the sequel to this book: *107 Geometry Problems from the AwesomeMath Year-Round Program*.

Although the primary audience for this book consists of high-performing students and their teachers, anyone with an interest in Euclidean geometry or recreational mathematics is invited to join this geometric excursion.

Finally, we would like to express our gratitude to Richard Stong and Cosmin Pohoaţă for critiquing the entire manuscript and providing fruitful comments.

We wish you a pleasant reading.

The Authors

Abbreviations and Notation

Notation of geometrical elements

$\angle BAC$	convex angle by vertex A
$\angle(p, q)$	directed angle between lines p and q
$\angle BAC \equiv \angle B'AC'$	angles BAC and $B'AC'$ coincide
AB	line through points A and B, distance between points A and B
$\overline{AB}$	directed segment from point A to point B
$X \in AB$	X lies on the line AB
$X = AC \cap BD$	X is the intersection of the lines AC and BD
$\triangle ABC$	triangle ABC
$[ABC]$	area of $\triangle ABC$
$[A_1 \ldots A_n]$	area of polygon $A_1 \ldots A_n$
$AB \parallel CD$	lines AB and CD are parallel
$AB \perp CD$	lines AB and CD are perpendicular
$p(X, \omega)$	power of point X with respect to circle ω
$\triangle ABC \cong \triangle DEF$	triangles ABC and DEF are congruent (in this order of vertices)
$\triangle ABC \sim \triangle DEF$	triangles ABC and DEF are similar (in this order of vertices)

Notation of triangle elements

a, b, c	sides or side lengths of $\triangle ABC$
$\angle A, \angle B, \angle C$	angles by vertices A, B, and C of $\triangle ABC$
s	semiperimeter
x, y, z	expressions $\frac{1}{2}(b+c-a)$, $\frac{1}{2}(c+a-b)$, $\frac{1}{2}(a+b-c)$
r	inradius
R	circumradius
K	area
h_a, h_b, h_c	altitudes in $\triangle ABC$
m_a, m_b, m_c	medians in $\triangle ABC$
l_a, l_b, l_c	angle bisectors (segments) in $\triangle ABC$
r_a, r_b, r_c	exradii in $\triangle ABC$

Abbreviations

AMC10	American Mathematics Contest 10
AMC12	American Mathematics Contest 12
AIME	American Invitational Mathematics Examination
USAJMO	United States of America Junior Mathematical Olympiad
USAMO	United States of America Mathematical Olympiad
USA TST	United States of America IMO Team Selection Test
MEMO	Middle European Mathematical Olympiad
IMO	International Mathematical Olympiad

Contents

Chapter 1

Foundations of Geometry

Preliminaries

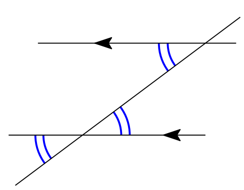
We begin our voyage to the fascinating world of classical geometry by reviewing some elementary facts.

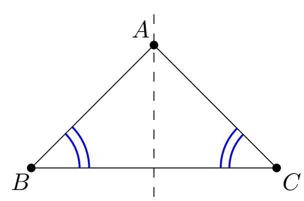
Basic Angles

We state the following:

- Vertical angles are equal.
- A line subtends the same angle with any two parallel lines. In other words, alternate angles are equal.
- In triangle ABC we have $AB = AC$ if and only if $\angle B = \angle C$.

The last two parts of the previous statement need to be taken seriously. The second one offers an efficient way to deal with parallel lines and the third one is one of the very few which translates angles into distances and vice versa.



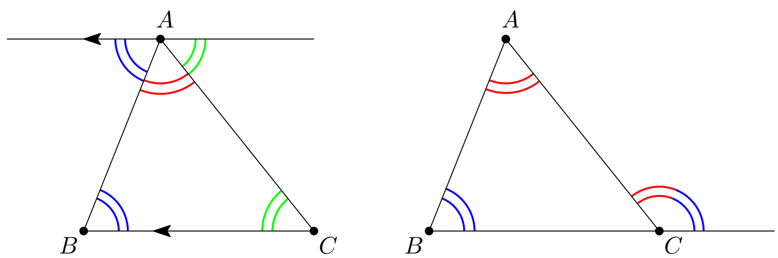


We are ready to prove the universally known theorem on the sum of internal angles in a triangle. In addition, we prove a slight extension, which often offers tiny but pleasant shortcuts in angle calculations.

Proposition 1.1. *Let ABC be a triangle with angles $\angle A$, $\angle B$, $\angle C$. Then:*

(a) $\angle A + \angle B + \angle C = 180°$.
(b) The external angle by vertex C equals $\angle A + \angle B$.

Proof. For (a) draw a line through point A parallel with BC. Since the three angles by vertex A add up to $180°$, we arrive at the result by using alternate angles.

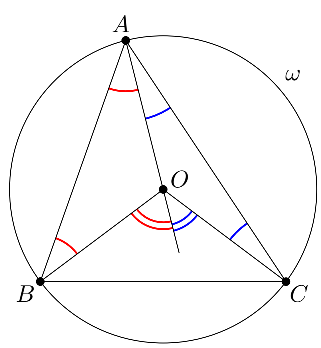


In order to prove (b) we just note that the external angle by vertex C is supplementary to $\angle C$ as well as the sum $\angle A + \angle B$ (by part (a)). □

Also, we know all it takes to prove the Inscribed Angle Theorem, which will later form our understanding of circles.

Theorem 1.2 (Inscribed Angle Theorem)**.** *Let BC be a chord of a circle ω centered at O and let $A \in \omega$, $A \neq B, C$. Then the inscribed angle BAC corresponding to arc BC equals one half of the central angle corresponding to the same arc.*

Proof. Assume first O lies inside triangle ABC.



From isosceles triangles OAB and OAC (radii are equal!) we infer $\angle OAB = \angle OBA$ and $\angle OAC = \angle OCA$. Then if we extend ray AO beyond O we can find $\angle BOC$ as sum of two external angles. We see that

$$\angle BOC = 2\angle BAO + 2\angle OAC = 2\angle BAC$$

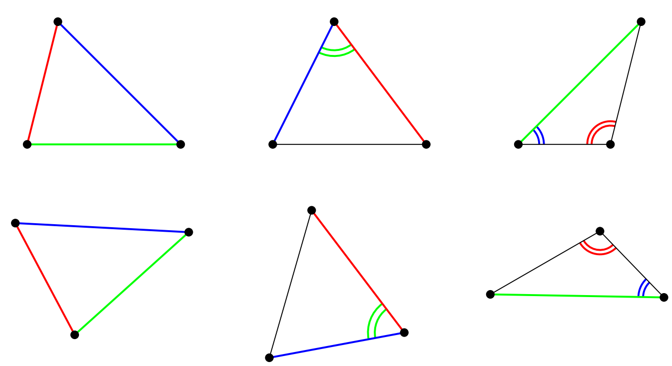
which is exactly what we wanted.

The case when O lies outside or on the boundary of triangle ABC is treated in the same fashion with a few of the additions becoming subtractions. □

Triangle Congruence and Similarity

Informally, we say that two triangles are congruent if they have the same shape and size. Of course, once two triangles are congruent, their corresponding parts (sides, angles, altitudes, ...) are equal. For proving congruence, we have the following criteria:

- (SSS criterion) If three pairs of sides of two triangles are equal in length, then the triangles are congruent.
- (SAS criterion) If two pairs of sides of two triangles are equal in length, and the included angles are equal, then the triangles are congruent.
- (ASA criterion) If two pairs of angles of two triangles are equal, and two corresponding sides are equal in length, then the triangles are congruent.



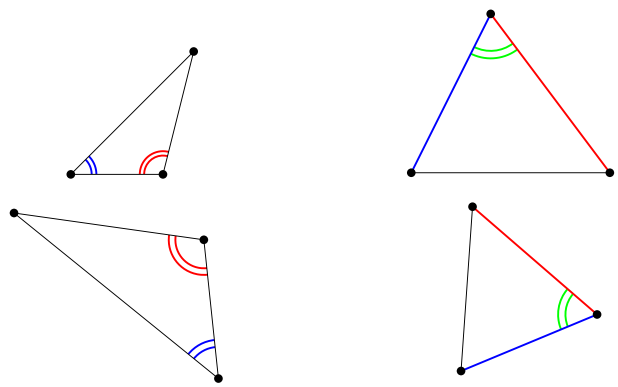
And finally, one criterion which was designed especially for right triangles.

- (HL criterion) If two right triangles have equal hypotenuses and one pair of equal legs, then they are congruent.

For similarity, it is enough for two triangles to have the same shape (i.e. internal angles). Again, similarity implies that all elements of one triangle are just scaled versions of the same elements of the other triangle. Therefore, the ratio of lengths of any corresponding segments is constant. Namely, it is the factor of similarity.

The similarity criteria are the following:

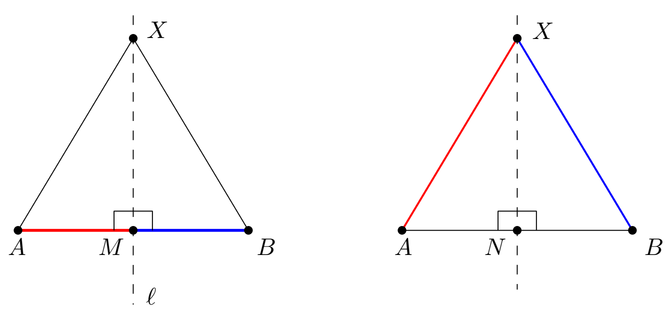
- (AA criterion) If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.
- (SAS criterion) If an angle of one triangle is congruent to the corresponding angle of another triangle and the sides that include this angle are proportional, then the two triangles are similar.



Congruence is most frequently used to give rigorous proofs for very natural claims. Here we prove that a line of symmetry of a segment or an angle indeed has the expected property of being the locus of equidistant points.

Proposition 1.3. *Let A and B be distinct points in the plane. Then the locus of points X for which $XA = XB$ is precisely the perpendicular bisector of AB.*

Proof. Denote by M the midpoint of AB (which is obviously the only satisfying point on AB) and by ℓ the perpendicular bisector of AB.

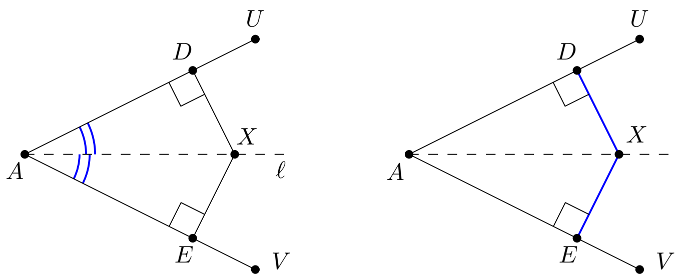


Now if $X \in \ell$ the right triangles AMX and BMX are congruent (SAS: $\angle AMX = \angle BMX = 90°$, $AM = BM$, and XM they have in common) and so $AX = BX$.

On the other hand if $AX = BX$, then let N be the foot of perpendicular from X to AB. Now $\triangle ANX \cong \triangle BNX$ (HL) and thus $AN = NB$ which implies $X \in \ell$. □

Proposition 1.4. *Rays AU and AV form an angle. The locus of points X which have the same distance from the rays AU and AV and lie inside angle UAV is precisely the bisector of $\angle UAV$.*

Proof. Let D and E be the projections of X onto AU and AV, respectively, and let ℓ be the bisector of angle UAV. If $X \in \ell$, then $\triangle ADX \cong \triangle AEX$ (ASA), hence $XD = XE$.

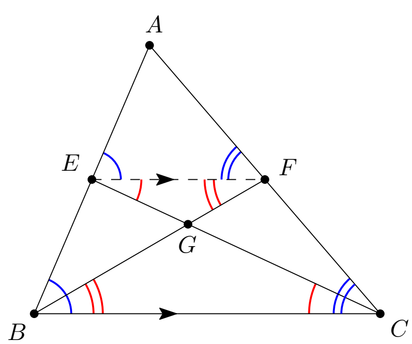


Conversely, if $XD = XE$, we have $\triangle ADX \cong \triangle AEX$ (HL), from which it follows that $\angle XAD = \angle XAE$ and so $X \in \ell$. □

Unlike congruence, similarity has much more striking applications. One of them is that medians divide each other in the ratio $2 : 1$.

Proposition 1.5. *Let ABC be a triangle and let E and F be the midpoints of the sides AB and AC, respectively. Denote by G the intersection of BF and CE. Then $BG = 2GF$ and $CG = 2GE$.*

Proof. First, observe that $\triangle AEF \sim \triangle ABC$ (SAS).

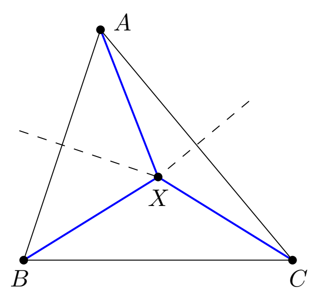


Since the factor of similarity is 2 it follows that $EF = \frac{1}{2}BC$. Moreover, we have $\angle FEA = \angle CBA$, thus $EF \parallel BC$. But then $\angle BCE = \angle CEF$ and we find that $\triangle BCG \sim \triangle FEG$ (AA). Since $EF = \frac{1}{2}BC$, the factor of similarity is $\frac{1}{2}$ and we arrive at the desired equalities $BG = 2GF$ and $CG = 2GE$. □

First Triangle Centers

Despite being such a simple object, the triangle hides perhaps an infinite number of surprising results, many of which are connected to some of its important points. Those are called triangle centers and nowadays over five thousand of them are recognized. Luckily, in olympiad math, it is usually enough to be acquainted with just a small fraction.

Proposition 1.6 (Existence of the Circumcenter)**.** *In triangle ABC the perpendicular bisectors of AB, BC, and CA meet at a single point. This point is called the circumcenter of triangle ABC, is usually denoted by O, and it is the center of the circumscribed circle (or simply circumcircle).*



Proof. Let X be the intersection of the perpendicular bisectors of AB and AC. From this we learn $XA = XB$ and $XA = XC$, which gives us $XB = XC$ and this implies that X lies on the perpendicular bisector of BC (if in doubts, see Proposition 1.3).

We have proved that all perpendicular bisectors pass through X. Of course, a circle with center X and radius $XA = XB = XC$ is the circumcircle of triangle ABC. □

Proposition 1.7 (Existence of the Incenter)**.** *In triangle ABC the internal angle bisectors meet at a point. This point is called the incenter of triangle ABC, is usually denoted by I, and it is the center of the incircle of triangle ABC.*

Proof. As expected we denote by X the intersection of the bisectors of $\angle B$ and $\angle C$. Then we know that X is equidistant from the sides AB and BC and also from the sides AC and BC (see Proposition 1.4 if necessary). It follows that is is also equidistant from AB and AC. In other words, it lies on the A-angle bisector. We have found a common point of all three internal angle bisectors.

The circle centered at X having for its radius the common distance from X to the lines BC, CA, and AB is then the incircle of triangle ABC. □