Preface

This book is a sequel to *106 Geometry Problems from the AwesomeMath Summer Program*. It contains 107 geometry questions used in the AwesomeMath Year-Round Program which trains and tests top middle and high-school students from U. S. and around the world.

The book begins with a theoretical chapter, where we review basic facts and familiarize the reader with some more advanced techniques. We then proceed to the main part of the work, the problem sections. The problems are a carefully selected and balanced mix which offers a vast variety of flavors and difficulties, ranging from AMC and AIME levels to high-end IMO problems. Out of thousands of Olympiad problems from around the globe we chose those which best illustrate the featured techniques and their applications. The problems meet our demanding taste and fully exhibit the enchanting beauty of classical geometry. For every problem we provide a detailed solution and strive to pass on the intuition and motivation lying behind. Numerous problems have multiple solutions.

Directly experiencing Olympiad geometry both as contestants and instructors, we are convinced that a neat diagram is essential to efficiently solving a geometry problem. Our diagrams do not contain anything superfluous, yet emphasize the key elements and benefit from a good choice of orientation. Many of the proofs should be legible only from looking at diagrams.

In the theoretical part we discuss some advanced theorems from triangle geometry and develop the theory of transformations, such as homothety, spiral similarity, and inversion. Employing the latter, we demonstrate the effectiveness of dynamic geometric thinking.

True mastery of geometry relies on proficient use of common sense. Therefore, we chose to avoid analytical and computational techniques such as complex numbers, vectors, or barycentric coordinates.

Although the primary audience for this book consists of high-performing students and their teachers, anyone with an interest in Euclidean geometry or recreational mathematics is invited to join this geometric excursion.

Finally, we would like to express our gratitude to Richard Stong and Cosmin Pohoaţă for critiquing the entire manuscript and providing fruitful comments.

We wish you a pleasant reading.

The Authors

Abbreviations and Notation

Notation of geometrical elements

$\angle BAC$	convex angle by vertex A
$\angle(p, q)$	directed angle between lines p and q
$\angle BAC \equiv \angle B'AC'$	angles BAC and $B'AC'$ coincide
AB	line through points A and B, distance between points A and B
$\overline{AB}$	directed segment from point A to point B
$X \in AB$	X lies on the line AB
$X = AC \cap BD$	X is the intersection of the lines AC and BD
$\triangle ABC$	triangle ABC
$[ABC]$	area of $\triangle ABC$
$[A_1 \dots A_n]$	area of polygon $A_1 \dots A_n$
$AB \parallel CD$	lines AB and CD are parallel
$AB \perp CD$	lines AB and CD are perpendicular
$p(X, \omega)$	power of point X with respect to circle ω
$\triangle ABC \cong \triangle DEF$	triangles ABC and DEF are congruent (in this order of vertices)
$\triangle ABC \sim \triangle DEF$	triangles ABC and DEF are similar (in this order of vertices)
$\mathcal{H}(H, k)$	homothety with center H and factor k
$\mathcal{S}(S, k, \varphi)$	spiral similarity with center S, dilation factor k, and angle of rotation φ

Notation of triangle elements

a, b, c	sides or side lengths of $\triangle ABC$
$\angle A, \angle B, \angle C$	angles by vertices A, B, and C of $\triangle ABC$
s	semiperimeter
x, y, z	expressions $\frac{1}{2}(b+c-a)$, $\frac{1}{2}(c+a-b)$, $\frac{1}{2}(a+b-c)$
r	inradius
R	circumradius
K	area
h_a, h_b, h_c	altitudes in $\triangle ABC$
m_a, m_b, m_c	medians in $\triangle ABC$
l_a, l_b, l_c	angle bisectors (segments) in $\triangle ABC$
r_a, r_b, r_c	exradii in $\triangle ABC$

Abbreviations

AMC10	American Mathematics Contest 10
AMC12	American Mathematics Contest 12
AIME	American Invitational Mathematics Examination
USAMTS	United States of America Mathematical Talent Search
USAJMO	United States of America Junior Mathematical Olympiad
USAMO	United States of America Mathematical Olympiad
USA TST	United States of America IMO Team Selection Test
MEMO	Middle European Mathematical Olympiad
IMO	International Mathematical Olympiad
Putnam	William Lowell Putnam Mathematical Competition

Contents

Chapter 1

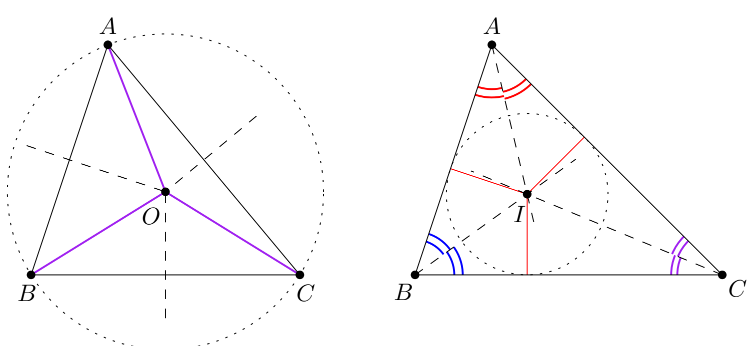
Advanced Topics in Geometry

Overview of Basic Techniques

Let us begin with reviewing some basic facts and techniques. Knowing them is not essential for further reading so don't get discouraged if you have gaps now and then. On the other hand, in order to learn the most from this book, we strongly recommend to get a firm grasp of what is presented in this section. All proofs (and much more) can be found in the preceding book *106 Geometry Problems from the AwesomeMath Summer Program.*

First Triangle Centers

Proposition 1.1 (Existence of the circumcenter)**.** *In triangle ABC the perpendicular bisectors of AB, BC, and CA meet at a single point. This point is called the circumcenter of triangle ABC, is usually denoted by O, and it is the center of the circumscribed circle (or simply circumcircle).*

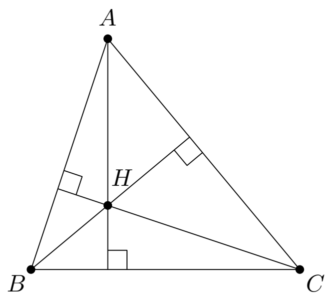


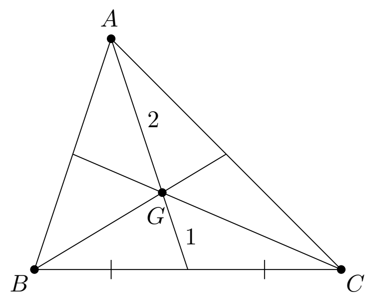
Proposition 1.2 (Existence of the incenter)**.** *In triangle ABC the internal angle bisectors meet at a point. This point is called the incenter of triangle*

ABC, is usually denoted by I, and it is the center of the incircle of triangle ABC.

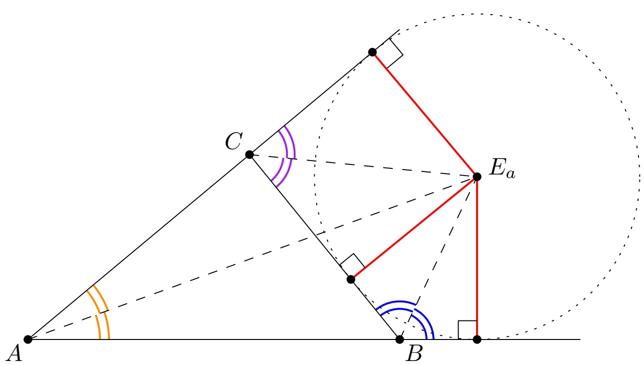
Proposition 1.3 (Existence of the orthocenter). *In triangle ABC the altitudes meet at a single point. This point is called the orthocenter of triangle ABC and is usually denoted by H.*

Proposition 1.4 (Existence of the centroid). *In triangle ABC the medians meet at a point. This point is called the centroid of triangle ABC and is usually denoted by G.*



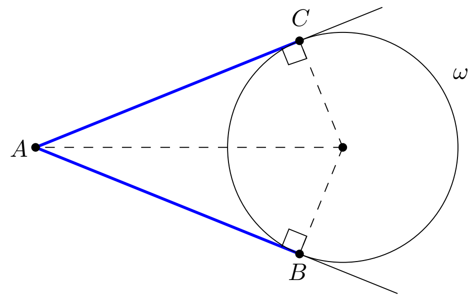


Proposition 1.5 (Existence of the excenter). *In triangle ABC the A-angle bisector and the bisectors of external angle B and C meet at a point. This point is called the A-excenter of triangle ABC, is usually denoted by I_a and it is the center of the A-excircle (circle tangent to the side BC and to the extended sidelines AB and AC). Similarly, we define points I_b and I_c.*



Metric relations

Proposition 1.6 (Equal Tangents). *Two tangent lines to the given circle ω intersect at A. Denote by B, C the points of tangency with the circle. Then $AB = AC$.*



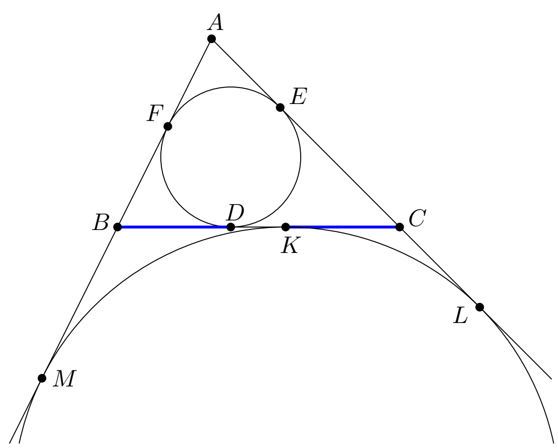
We use the following standard xyz notation in triangle ABC with semiperimeter s:

$$x = s - a = \frac{1}{2}(b+c-a), \quad y = s - b = \frac{1}{2}(c+a-b), \quad z = s - c = \frac{1}{2}(a+b-c),$$

the purpose of which is revealed in the next two propositions.

Proposition 1.7 (Points of contact)**.** *Let ABC be a triangle with semiperimeter s. Denote by D, E, F the points of tangency of the incircle with the sides BC, CA, AB, respectively. Also let the A-excircle touch the lines BC, CA, AB at points K, L, M, respectively. Then the following hold:*

(a) $AE = AF = x, \quad BD = BF = y, \quad CD = CE = z.$
(b) $AL = AM = s.$
(c) Points K and D are symmetric with respect to the midpoint of BC.



Proposition 1.8 (xyz formulas)**.** *In triangle ABC we can find the area K, inradius r, and circumradius R in terms of x, y, z as follows:*

(a)

$$K = \sqrt{(x+y+z)xyz},$$

(b)

$$r = \sqrt{\frac{xyz}{x+y+z}},$$

(c)

$$R = \frac{(y+z)(z+x)(x+y)}{4\sqrt{xyz(x+y+z)}}.$$

Theorem 1.9 (The Extended Law of Sines)**.** *Let ABC be a triangle. Then*

$$\frac{a}{\sin\angle A} = \frac{b}{\sin\angle B} = \frac{c}{\sin\angle C} = 2R,$$

where R is the circumradius of triangle ABC.

Theorem 1.10 (Angle Bisector Theorem)**.** *In triangle ABC let AD, $D \in BC$, be the internal angle bisector. Then*

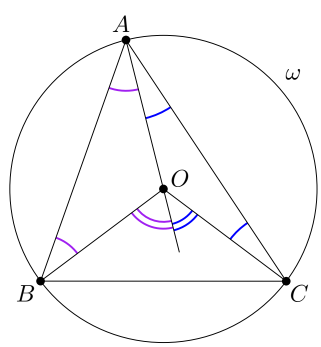
$$\frac{BD}{CD} = \frac{c}{b}, \quad BD = \frac{ac}{b+c}, \quad CD = \frac{ab}{b+c}.$$

Theorem 1.11 (The Law of Cosines)**.** *Let ABC be a triangle. Then*

$$a^2 = b^2 + c^2 - 2bc\cos\angle A.$$

Circles, Tangents

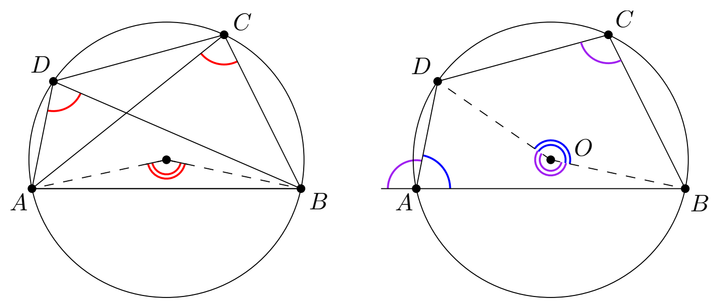
Theorem 1.12 (Inscribed Angle Theorem)**.** *Let BC be a chord of a circle ω centered at O and let $A \in \omega$, $A \neq B, C$. Then the inscribed angle BAC corresponding to arc BC equals one half of the central angle corresponding to the same arc.*



Quadrilaterals which are inscribed in a circle are called *cyclic* and play fundamental role in the technique called *angle-chasing*.

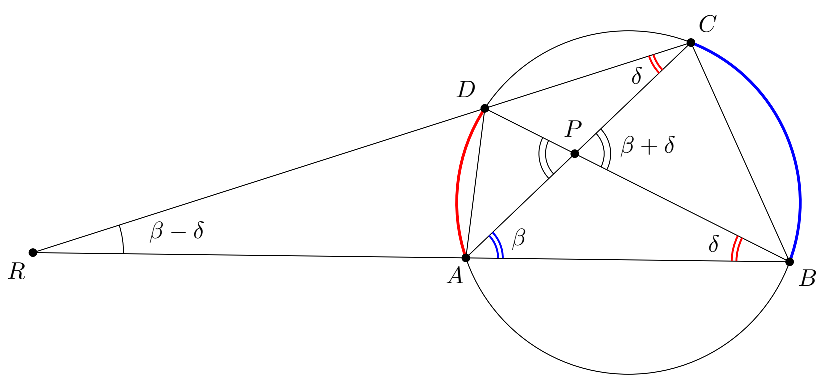
Proposition 1.13 (The key properties of cyclic quadrilaterals)**.** *Let $ABCD$ be a convex quadrilateral. Then:*

(a) If $ABCD$ is cyclic then any of its sides is visible from the other two vertices under the same angle, and any of its diagonals is visible from the other two vertices under angles that sum up to $180°$.
(b) If there is a side of $ABCD$ that is visible from the other two vertices under the same angle, then $ABCD$ is cyclic.
(c) If there is a diagonal of $ABCD$ that is visible from the other two vertices under angles that sum up to $180°$, then $ABCD$ is cyclic.



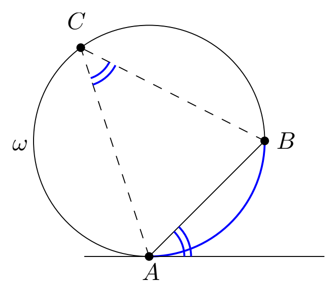
Corollary 1.14 (Angle between chords or secants)**.** *Let $ABCD$ be a quadrilateral inscribed in a circle ω and denote by P the intersection of its diagonals. Suppose that rays BA and CD intersect at R. Finally, denote the inscribed angles corresponding to arcs BC, DA (not containing A, B) by β, δ. Then*

(a) $\angle BPC = \beta + \delta$,
(b) $\angle BRC = \beta - \delta$.



Proposition 1.15 (Angle by tangent)**.** *Let ABC be a triangle inscribed in a circle ω. Let ℓ be a line passing through A different from AB. Let L be a*

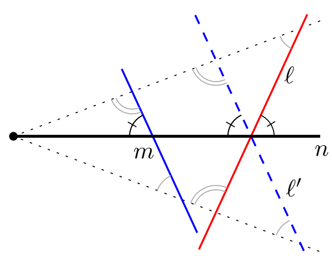
point on ℓ such that AB separates points C, L. Then AL is tangent to ω if and only if $\angle LAB = \angle ACB$.



Antiparallel lines

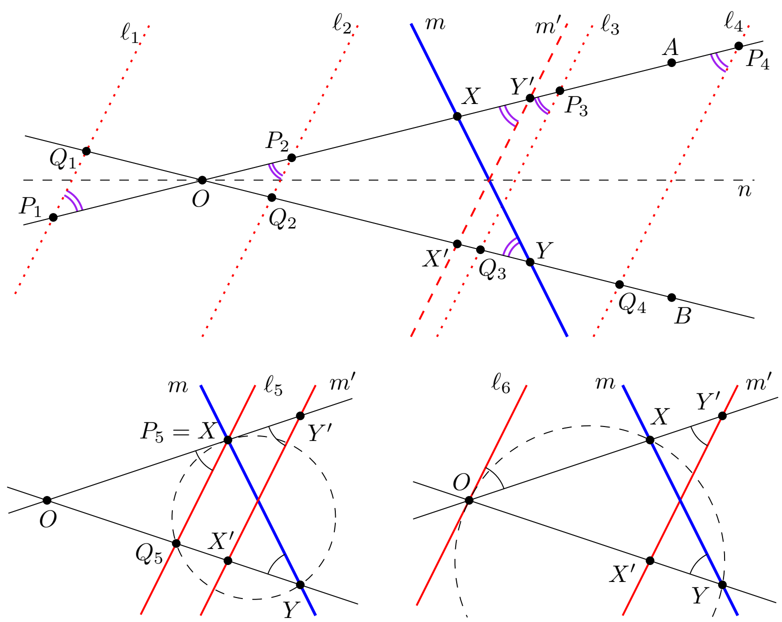
Given a line n we say that lines ℓ and m (neither parallel to n) are **antiparallel** with respect to line n if the reflection ℓ' of ℓ about n is parallel to m. Observe that the following holds:

(a) If ℓ is antiparallel to m then it is antiparallel to all lines parallel to m.
(b) (Symmetry) If ℓ is antiparallel to m then m is antiparallel to ℓ.
(c) Given a line n and a set of mutually parallel lines, then lines antiparallel to all of these with respect to n form again a set of mutually parallel lines.



Proposition 1.16. *Let line m intersect rays OA, OB of angle AOB at distinct points X, Y, respectively. Let line ℓ, ($\ell \neq m$) intersect lines OA, OB of angle AOB at (not necessarily distinct) points P, Q, respectively. Then ℓ and m are antiparallel with respect to the angle bisector of angle AOB if and only if one of the following (based on the configuration) holds:*

(a) Points X, Y, P, Q are concyclic (if they are pairwise distinct).
(b) Line OA is tangent to the circumcircle of triangle XYQ (if $X = P$). A similar result holds if $Y = Q$.



(c) Line ℓ is tangent to the circumcircle of triangle XYO (if ℓ passes through O).

Since antiparallel lines are usually taken with respect to the angle bisector of some angle, let us in that case call these lines *antiparallel with respect to* that angle or simply *antiparallel in* that angle. Of particular interest are antiparallel lines that both pass through the vertex of an angle – such lines are called *isogonal*. One pair of isogonal lines is especially worth emphasizing.

Proposition 1.17 (H and O are friends)**.** *In triangle ABC points H (the orthocenter) and O (the circumcenter) lie on isogonal lines in each of the angles $\angle A$, $\angle B$, $\angle C$.*

Directed angles mod[1] 180°

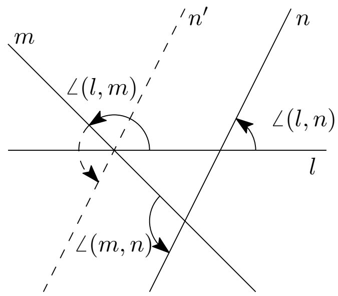
The magnitude of an angle between lines l, m intersecting at vertex O can be viewed as a number from interval $[0, 180)$ describing (in degrees) the amount of counter-clockwise rotation around O which takes l to m. Let us call this quantity **the directed measure of an angle** and denote it by $\angle(l, m)$. Note that order of lines in brackets matters – in fact $\angle(l, m) + \angle(m, l) = 180°$. This notion will be our main weapon for simplifying angle-chasing casework throughout the book.

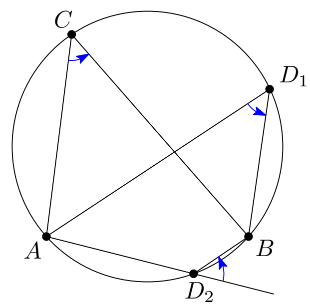
[1]This means, we shall work with remainders after division by 180. For example, instead of 200°, we shall work with 20°.

Proposition 1.18. *(a)* $\angle(l,m)+\angle(m,n)=\angle(l,n)$, *with addition mod* $180°$.

(b) For any point P $\angle(PA,AB)=\angle(PA,AC)$ *if and only if points* A, B, C *lie on a single line in some order.*

(c) $\angle(AC,CB)=\angle(AD,DB)$ *if and only if points* A, B, C, D *lie on one circle in some order.*





Power of a Point

Proposition 1.19. *(a) Let* $ABCD$ *be a convex quadrilateral and let* $P = AC \cap BD$. *Then the points* A, B, C, D *are concyclic if and only if*

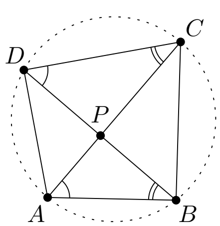
$$PC \cdot PA = PB \cdot PD.$$

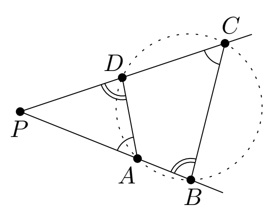
(b) Let $ABCD$ *be a convex quadrilateral and let* $P = AB \cap CD$. *Then the points* A, B, C, D *are concyclic if and only if*

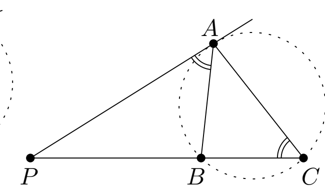
$$PA \cdot PB = PC \cdot PD.$$

(c) Assume points P, B, C *are collinear in this order and point* A *does not lie on this line. Then the line* PA *is tangent to the circumcircle of triangle* ABC *if and only if*

$$PA^2 = PB \cdot PC.$$







Theorem 1.20 (Power of a Point)**.** *Given point* P *and circle* ω, *let* ℓ *be an arbitrary line passing through* P *and intersecting* ω *at points* A *and* B. *Then*

the value of $PA \cdot PB$ does not depend on the choice of ℓ. Also, if P lies outside of ω and PT, $T \in \omega$, is a tangent to ω then $PA \cdot PB = PT^2$.

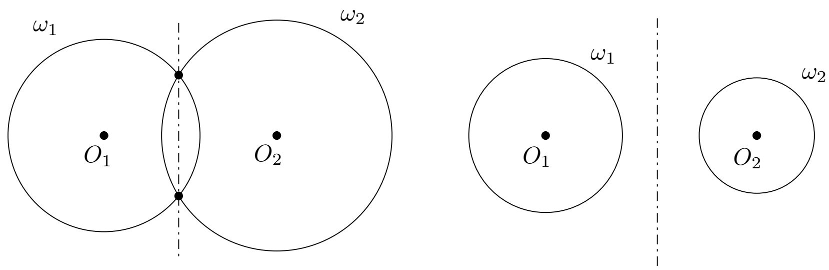
If we denote the center of ω by O and its radius by R then $PA \cdot PB = |OP^2 - R^2|$. The quantity

$$p(P, \omega) = OP^2 - R^2$$

is called the power of point P with respect to circle ω.

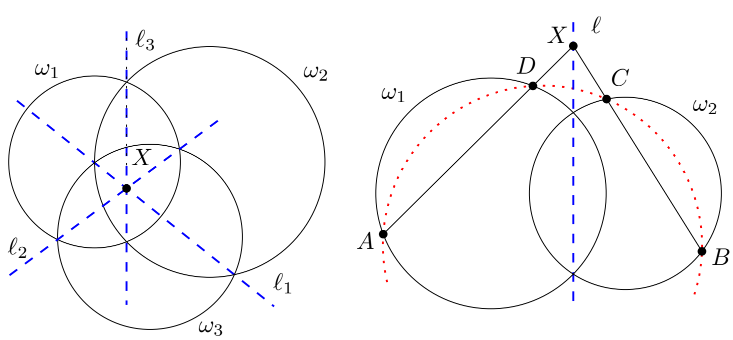
Note that the number $p(P, \omega)$ is negative when P lies inside ω, zero when it lies on ω, and positive otherwise.

Proposition 1.21 (Radical axis)**.** *Let ω_1, ω_2 be two circles with distinct centers O_1, O_2 and radii R_1, R_2, respectively. Then the locus of points X for which $p(X, \omega_1) = p(X, \omega_2)$ is a line perpendicular to O_1O_2. This line is called the radical axis of the two circles.*



The radical axis is a powerful tool in many problems involving intersecting circles since in that case the radical axis is the line joining their intersections, which both have equal (namely zero) power with respect to the two circles.

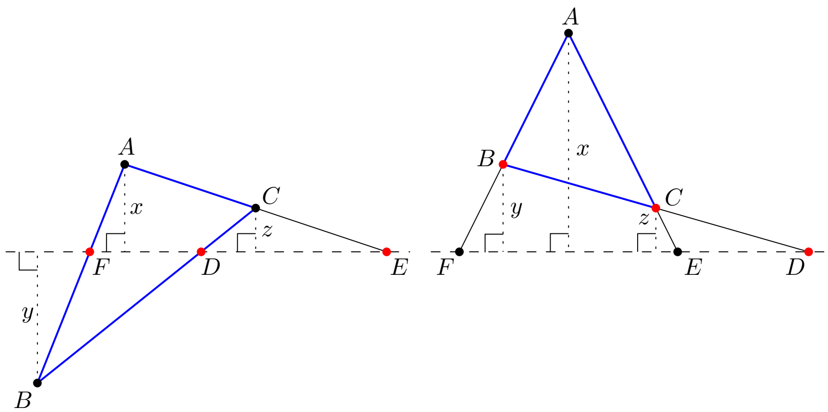
Proposition 1.22 (Radical center)**.** *Let $\omega_1, \omega_2, \omega_3$ be circles with pairwise distinct centers. Then their pairwise radical axes are either parallel or concurrent. The point of concurrence is called the radical center of the three circles.*



Proposition 1.23 (Radical Lemma)**.** *Let line ℓ be radical axis of the circles ω_1, ω_2. Let A, D be distinct points on ω_1 and let B, C be distinct points on ω_2 such that the lines AD and BC are not parallel. Then the lines AD and BC intersect at ℓ if and only if $ABCD$ is cyclic.*

Theorem 1.24 (Menelaus'[2] Theorem)**.** *Let ABC be a triangle and let points D, E, F lie on the lines BC, CA, AB, respectively, so that either none or two of them lie on the triangle sides. Then the points D, E, F are collinear if and only if*

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$



Segments which connect vertex of a triangle with a point on the opposite side are called *cevians*.

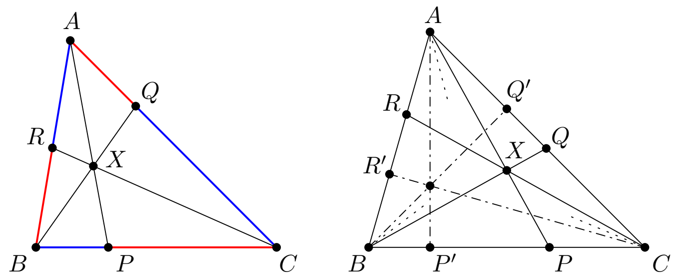
Theorem 1.25 (Ceva[3]'s Theorem)**.** *Let ABC be a triangle, and let P, Q, R be points on the sides BC, CA, AB, respectively. Then the lines AP, BQ, CR are concurrent if and only if*

$$\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = 1.$$

Theorem 1.26 (Existence of isogonal conjugate)**.** *Let cevians AP, BQ, CR concur at point X. Now construct cevians AP', BQ', CR' which are isogonal to AP, BQ, CR, respectively, in the respective angles. Then the cevians AP', BQ', CR' are concurrent. The point of concurrence is called the isogonal conjugate of X.*

[2]Menelaus of Alexandria (c. 70–140) was a Greek mathematician and astronomer.

[3]Giovanni Ceva (1647–1734) was an Italian mathematician.



Directed segments

A **directed segment** emanating from A with endpoint B will be denoted by $\overline{AB}$.

The important property of directed segments is that the ratio or the product of two directed segments, which are part of the same line, is assigned a sign. The sign is positive if the directed segments have the same orientation and negative otherwise. By the same logic we have

$$\overline{AB} = -\overline{BA}.$$