Preface

This book is a sequel to *106 Geometry Problems from the AwesomeMath* *Summer Program*. It contains 107 geometry questions used in the AwesomeMath Year-Round Program which trains and tests top middle and high-school students from U. S. and around the world.

The book begins with a theoretical chapter, where we review basic facts and familiarize the reader with some more advanced techniques. We then proceed to the main part of the work, the problem sections. The problems are a carefully selected and balanced mix which offers a vast variety of flavors and difficulties, ranging from AMC and AIME levels to high-end IMO problems. Out of thousands of Olympiad problems from around the globe we chose those which best illustrate the featured techniques and their applications. The problems meet our demanding taste and fully exhibit the enchanting beauty of classical geometry. For every problem we provide a detailed solution and strive to pass on the intuition and motivation lying behind. Numerous problems have multiple solutions.

Directly experiencing Olympiad geometry both as contestants and instructors, we are convinced that a neat diagram is essential to efficiently solving a geometry problem. Our diagrams do not contain anything superfluous, yet emphasize the key elements and benefit from a good choice of orientation. Many of the proofs should be legible only from looking at diagrams.

In the theoretical part we discuss some advanced theorems from triangle geometry and develop the theory of transformations, such as homothety, spiral similarity, and inversion. Employing the latter, we demonstrate the effectiveness of dynamic geometric thinking.

True mastery of geometry relies on proficient use of common sense. Therefore, we chose to avoid analytical and computational techniques such as complex numbers, vectors, or barycentric coordinates.
Although the primary audience for this book consists of high-performing students and their teachers, anyone with an interest in Euclidean geometry or recreational mathematics is invited to join this geometric excursion.

Finally, we would like to express our gratitude to Richard Stong and Cosmin Pohoată for critiquing the entire manuscript and providing fruitful comments.

We wish you a pleasant reading.

The Authors
Abbreviations and Notation

Notation of geometrical elements

\( \angle BAC \) convex angle by vertex \( A \)

\( \angle (p, q) \) directed angle between lines \( p \) and \( q \)

\( \angle BAC \equiv \angle B'AC' \) angles \( BAC \) and \( B'AC' \) coincide

\( AB \) line through points \( A \) and \( B \), distance between points \( A \) and \( B \)

\( \overline{AB} \) directed segment from point \( A \) to point \( B \)

\( X \in AB \) \( X \) lies on the line \( AB \)

\( X = AC \cap BD \) \( X \) is the intersection of the lines \( AC \) and \( BD \)

\( \triangle ABC \) triangle \( ABC \)

\( [ABC] \) area of \( \triangle ABC \)

\( [A_1 \ldots A_n] \) area of polygon \( A_1 \ldots A_n \)

\( AB \parallel CD \) lines \( AB \) and \( CD \) are parallel

\( AB \perp CD \) lines \( AB \) and \( CD \) are perpendicular

\( p(X, \omega) \) power of point \( X \) with respect to circle \( \omega \)

\( \triangle ABC \cong \triangle DEF \) triangles \( ABC \) and \( DEF \) are congruent (in this order of vertices)

\( \triangle ABC \sim \triangle DEF \) triangles \( ABC \) and \( DEF \) are similar (in this order of vertices)

\( \mathcal{H}(H, k) \) homothety with center \( H \) and factor \( k \)

\( \mathcal{S}(S, k, \varphi) \) spiral similarity with center \( S \), dilation factor \( k \), and angle of rotation \( \varphi \)
Notation of triangle elements

\begin{align*}
a, b, c & \quad \text{sides or side lengths of } \triangle ABC \\
\angle A, \angle B, \angle C & \quad \text{angles by vertices } A, B, \text{ and } C \text{ of } \triangle ABC \\
s & \quad \text{semiperimeter} \\
x, y, z & \quad \text{expressions } \frac{1}{2}(b + c - a), \frac{1}{2}(c + a - b), \frac{1}{2}(a + b - c) \\
r & \quad \text{inradius} \\
R & \quad \text{circumradius} \\
K & \quad \text{area} \\
h_a, h_b, h_c & \quad \text{altitudes in } \triangle ABC \\
m_a, m_b, m_c & \quad \text{medians in } \triangle ABC \\
l_a, l_b, l_c & \quad \text{angle bisectors (segments) in } \triangle ABC \\
r_a, r_b, r_c & \quad \text{exradii in } \triangle ABC
\end{align*}

Abbreviations

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<th>Abbreviation</th>
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<tbody>
<tr>
<td>AMC10</td>
<td>American Mathematics Contest 10</td>
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<td>AMC12</td>
<td>American Mathematics Contest 12</td>
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<tr>
<td>AIME</td>
<td>American Invitational Mathematics Examination</td>
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<td>USAMTS</td>
<td>United States of America Mathematical Talent Search</td>
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<td>USAJMO</td>
<td>United States of America Junior Mathematical Olympiad</td>
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<td>USAMO</td>
<td>United States of America Mathematical Olympiad</td>
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<tr>
<td>USA TST</td>
<td>United States of America IMO Team Selection Test</td>
</tr>
<tr>
<td>MEMO</td>
<td>Middle European Mathematical Olympiad</td>
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<td>International Mathematical Olympiad</td>
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Chapter 1

Advanced Topics in Geometry

Overview of Basic Techniques

Let us begin with reviewing some basic facts and techniques. Knowing them is not essential for further reading so don’t get discouraged if you have gaps now and then. On the other hand, in order to learn the most from this book, we strongly recommend to get a firm grasp of what is presented in this section. All proofs (and much more) can be found in the preceding book 106 Geometry Problems from the AwesomeMath Summer Program.

First Triangle Centers

Proposition 1.1 (Existence of the circumcenter). In triangle ABC the perpendicular bisectors of AB, BC, and CA meet at a single point. This point is called the circumcenter of triangle ABC, is usually denoted by O, and it is the center of the circumscribed circle (or simply circumcircle).

Proposition 1.2 (Existence of the incenter). In triangle ABC the internal angle bisectors meet at a point. This point is called the incenter of triangle
ABC, is usually denoted by $I$, and it is the center of the incircle of triangle ABC.

**Proposition 1.3** (Existence of the orthocenter). In triangle ABC the altitudes meet at a single point. This point is called the orthocenter of triangle ABC and is usually denoted by $H$.

**Proposition 1.4** (Existence of the centroid). In triangle ABC the medians meet at a point. This point is called the centroid of triangle ABC and is usually denoted by $G$.

**Proposition 1.5** (Existence of the excenter). In triangle ABC the $A$-angle bisector and the bisectors of external angle $B$ and $C$ meet at a point. This point is called the $A$-excenter of triangle ABC, is usually denoted by $I_a$ and it is the center of the $A$-excircle (circle tangent to the side BC and to the extended sidelines $AB$ and $AC$). Similarly, we define points $I_b$ and $I_c$.

**Metric relations**

**Proposition 1.6** (Equal Tangents). Two tangent lines to the given circle $\omega$ intersect at $A$. Denote by $B, C$ the points of tangency with the circle. Then $AB = AC$. 
We use the following standard $xyz$ notation in triangle $ABC$ with semiperimeter $s$:

\[
x = s - a = \frac{1}{2}(b + c - a), \quad y = s - b = \frac{1}{2}(c + a - b), \quad z = s - c = \frac{1}{2}(a + b - c),
\]

the purpose of which is revealed in the next two propositions.

**Proposition 1.7 (Points of contact).** Let $ABC$ be a triangle with semiperimeter $s$. Denote by $D, E, F$ the points of tangency of the incircle with the sides $BC, CA, AB$, respectively. Also let the $A$-excircle touch the lines $BC, CA, AB$ at points $K, L, M$, respectively. Then the following hold:

(a) $AE = AF = x, \quad BD = BF = y, \quad CD = CE = z$.

(b) $AL = AM = s$.

(c) Points $K$ and $D$ are symmetric with respect to the midpoint of $BC$.

**Proposition 1.8 ($xyz$ formulas).** In triangle $ABC$ we can find the area $K$, inradius $r$, and circumradius $R$ in terms of $x, y, z$ as follows:

(a) \[ K = \sqrt{(x + y + z)xyz}, \]
(b) 

\[ r = \sqrt{\frac{xyz}{x+y+z}}, \]

(c) 

\[ R = \frac{(y+z)(z+x)(x+y)}{4\sqrt{xyz(x+y+z)}}. \]

**Theorem 1.9** (The Extended Law of Sines). Let \( ABC \) be a triangle. Then

\[ \frac{a}{\sin \angle A} = \frac{b}{\sin \angle B} = \frac{c}{\sin \angle C} = 2R, \]

where \( R \) is the circumradius of triangle \( ABC \).

**Theorem 1.10** (Angle Bisector Theorem). In triangle \( ABC \) let \( AD, D \in BC \), be the internal angle bisector. Then

\[ \frac{BD}{CD} = \frac{c}{b}, \quad BD = \frac{ac}{b+c}, \quad CD = \frac{ab}{b+c}. \]

**Theorem 1.11** (The Law of Cosines). Let \( ABC \) be a triangle. Then

\[ a^2 = b^2 + c^2 - 2bc \cos \angle A. \]

**Circles, Tangents**

**Theorem 1.12** (Inscribed Angle Theorem). Let \( BC \) be a chord of a circle \( \omega \) centered at \( O \) and let \( A \in \omega, A \neq B, C \). Then the inscribed angle \( BAC \) corresponding to arc \( BC \) equals one half of the central angle corresponding to the same arc.

Quadrilaterals which are inscribed in a circle are called **cyclic** and play fundamental role in the technique called **angle-chasing**.
Proposition 1.13 (The key properties of cyclic quadrilaterals). Let $ABCD$ be a convex quadrilateral. Then:

(a) If $ABCD$ is cyclic then any of its sides is visible from the other two vertices under the same angle, and any of its diagonals is visible from the other two vertices under angles that sum up to $180^\circ$.

(b) If there is a side of $ABCD$ that is visible from the other two vertices under the same angle, then $ABCD$ is cyclic.

(c) If there is a diagonal of $ABCD$ that is visible from the other two vertices under angles that sum up to $180^\circ$, then $ABCD$ is cyclic.

Corollary 1.14 (Angle between chords or secants). Let $ABCD$ be a quadrilateral inscribed in a circle $\omega$ and denote by $P$ the intersection of its diagonals. Suppose that rays $BA$ and $CD$ intersect at $R$. Finally, denote the inscribed angles corresponding to arcs $BC$, $DA$ (not containing $A$, $B$) by $\beta$, $\delta$. Then

(a) $\angle BPC = \beta + \delta$,

(b) $\angle BRC = \beta - \delta$.

Proposition 1.15 (Angle by tangent). Let $ABC$ be a triangle inscribed in a circle $\omega$. Let $\ell$ be a line passing through $A$ different from $AB$. Let $L$ be a
point on \( \ell \) such that \( AB \) separates points \( C, L \). Then \( AL \) is tangent to \( \omega \) if and only if \( \angle LAB = \angle ACB \).

\[
\begin{array}{c}
\text{Antiparallel lines} \\
\text{Given a line } n \text{ we say that lines } \ell \text{ and } m (\text{neither parallel to } n) \text{ are antiparallel with respect to line } n \text{ if the reflection } \ell' \text{ of } \ell \text{ about } n \text{ is parallel to } m. \text{ Observe that the following holds:}
\end{array}
\]

(a) If \( \ell \) is antiparallel to \( m \) then it is antiparallel to all lines parallel to \( m \).
(b) (Symmetry) If \( \ell \) is antiparallel to \( m \) then \( m \) is antiparallel to \( \ell \).
(c) Given a line \( n \) and a set of mutually parallel lines, then lines antiparallel to all of these with respect to \( n \) form again a set of mutually parallel lines.

\[
\begin{array}{c}
\text{Proposition 1.16. Let line } m \text{ intersect rays } OA, OB \text{ of angle } AOB \text{ at distinct points } X, Y, \text{ respectively. Let line } \ell, (\ell \neq m) \text{ intersect lines } OA, OB \text{ of angle } AOB \text{ at (not necessarily distinct) points } P, Q, \text{ respectively. Then } \ell \text{ and } m \text{ are antiparallel with respect to the angle bisector of angle } AOB \text{ if and only if one of the following (based on the configuration) holds:}
\end{array}
\]

(a) Points \( X, Y, P, Q \) are concyclic (if they are pairwise distinct).
(b) Line \( OA \) is tangent to the circumcircle of triangle \( XYQ \) (if \( X = P \)). A similar result holds if \( Y = Q \).
(c) Line $\ell$ is tangent to the circumcircle of triangle $XYO$ (if $\ell$ passes through $O$).

Since antiparallel lines are usually taken with respect to the angle bisector of some angle, let us in that case call these lines antiparallel with respect to that angle or simply antiparallel in that angle. Of particular interest are antiparallel lines that both pass through the vertex of an angle – such lines are called isogonal. One pair of isogonal lines is especially worth emphasizing.

**Proposition 1.17** ($H$ and $O$ are friends). In triangle $ABC$ points $H$ (the orthocenter) and $O$ (the circumcenter) lie on isogonal lines in each of the angles $\angle A$, $\angle B$, $\angle C$.

**Directed angles mod$^1$ 180°**

The magnitude of an angle between lines $l$, $m$ intersecting at vertex $O$ can be viewed as a number from interval $[0, 180)$ describing (in degrees) the amount of counter-clockwise rotation around $O$ which takes $l$ to $m$. Let us call this quantity the directed measure of an angle and denote it by $\angle(l,m)$. Note that order of lines in brackets matters – in fact $\angle(l,m) + \angle(m,l) = 180^\circ$. This notion will be our main weapon for simplifying angle-chasing casework throughout the book.

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$^1$This means, we shall work with remainders after division by 180. For example, instead of $200^\circ$, we shall work with $20^\circ$. 
Proposition 1.18. (a) \( \angle(l,m) + \angle(m,n) = \angle(l,n) \), with addition mod 180°.
(b) For any point \( P \), \( \angle(PA, AB) = \angle(PA, AC) \) if and only if points \( A, B, C \) lie on a single line in some order.
(c) \( \angle(AC, CB) = \angle(AD, DB) \) if and only if points \( A, B, C, D \) lie on one circle in some order.

Power of a Point

Proposition 1.19. (a) Let \( ABCD \) be a convex quadrilateral and let \( P = AC \cap BD \). Then the points \( A, B, C, D \) are concyclic if and only if
\[ PC \cdot PA = PB \cdot PD. \]
(b) Let \( ABCD \) be a convex quadrilateral and let \( P = AB \cap CD \). Then the points \( A, B, C, D \) are concyclic if and only if
\[ PA \cdot PB = PC \cdot PD. \]
(c) Assume points \( P, B, C \) are collinear in this order and point \( A \) does not lie on this line. Then the line \( PA \) is tangent to the circumcircle of triangle \( ABC \) if and only if
\[ PA^2 = PB \cdot PC. \]

Theorem 1.20 (Power of a Point). Given point \( P \) and circle \( \omega \), let \( \ell \) be an arbitrary line passing through \( P \) and intersecting \( \omega \) at points \( A \) and \( B \). Then
the value of $PA \cdot PB$ does not depend on the choice of $\ell$. Also, if $P$ lies outside of $\omega$ and $PT$, $T \in \omega$, is a tangent to $\omega$ then $PA \cdot PB = PT^2$.

If we denote the center of $\omega$ by $O$ and its radius by $R$ then $PA \cdot PB = |OP^2 - R^2|$. The quantity

$$p(P, \omega) = OP^2 - R^2$$

is called the power of point $P$ with respect to circle $\omega$.

Note that the number $p(P, \omega)$ is negative when $P$ lies inside $\omega$, zero when it lies on $\omega$, and positive otherwise.

**Proposition 1.21** (Radical axis). Let $\omega_1, \omega_2$ be two circles with distinct centers $O_1, O_2$ and radii $R_1, R_2$, respectively. Then the locus of points $X$ for which $p(X, \omega_1) = p(X, \omega_2)$ is a line perpendicular to $O_1O_2$. This line is called the radical axis of the two circles.

The radical axis is a powerful tool in many problems involving intersecting circles since in that case the radical axis is the line joining their intersections, which both have equal (namely zero) power with respect to the two circles.

**Proposition 1.22** (Radical center). Let $\omega_1, \omega_2, \omega_3$ be circles with pairwise distinct centers. Then their pairwise radical axes are either parallel or concurrent. The point of concurrence is called the radical center of the three circles.
**Proposition 1.23** (Radical Lemma). Let line \( \ell \) be radical axis of the circles \( \omega_1, \omega_2 \). Let \( A, D \) be distinct points on \( \omega_1 \) and let \( B, C \) be distinct points on \( \omega_2 \) such that the lines \( AD \) and \( BC \) are not parallel. Then the lines \( AD \) and \( BC \) intersect at \( \ell \) if and only if \( ABCD \) is cyclic.

**Theorem 1.24** (Menelaus' Theorem). Let \( ABC \) be a triangle and let points \( D, E, F \) lie on the lines \( BC, CA, AB \), respectively, so that either none or two of them lie on the triangle sides. Then the points \( D, E, F \) are collinear if and only if
\[
\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.
\]

Segments which connect vertex of a triangle with a point on the opposite side are called *cevians*.

**Theorem 1.25** (Ceva's Theorem). Let \( ABC \) be a triangle, and let \( P, Q, R \) be points on the sides \( BC, CA, AB \), respectively. Then the lines \( AP, BQ, CR \) are concurrent if and only if
\[
\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = 1.
\]

**Theorem 1.26** (Existence of isogonal conjugate). Let cevians \( AP, BQ, CR \) concur at point \( X \). Now construct cevians \( AP', BQ', CR' \) which are isogonal to \( AP, BQ, CR \), respectively, in the respective angles. Then the cevians \( AP', BQ', CR' \) are concurrent. The point of concurrence is called the isogonal conjugate of \( X \).

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2Menelaus of Alexandria (c. 70–140) was a Greek mathematician and astronomer.
3Giovanni Ceva (1647–1734) was an Italian mathematician.
Directed segments

A **directed segment** emanating from $A$ with endpoint $B$ will be denoted by $\overrightarrow{AB}$.

The important property of directed segments is that the ratio or the product of two directed segments, which are part of the same line, is assigned a sign. The sign is positive if the directed segments have the same orientation and negative otherwise. By the same logic we have

$$\overrightarrow{AB} = -\overrightarrow{BA}.$$