Contents

1 Preface

This book aims to illustrate essential strategies and techniques in elementary algebra from a problem-solving perspective. In algebra especially, knowing theorems is not enough — one must fully understand how and when to apply them. To thoroughly convey this knowledge, numerous examples are provided, in which the motivation and core ideas behind the solutions are explained. Problems of all difficulties are presented, catering to both novice and experienced problem solvers. The methods discussed are important in Olympiad problem-solving as well as in various other domains of mathematics.

The book covers many classical topics in elementary algebra, including factoring, quadratic functions, irrational expressions, Vieta's relations, equations and systems of equations, inequalities, sums and products, and polynomials. Expanding upon the previous work in the series, *105 Problems in Algebra from the AwesomeMath Summer Program*, this book features additional more advanced topics, including exponents and logarithms, complex numbers, and trigonometry. The special section on trigonometric substitutions and more explores seemingly algebraic problems with natural geometric and trigonometric interpretations.

To give the reader practice with the strategies and techniques discussed in each of the chapters, we have included 108 diverse problems, of which 54 are introductory and 54 are advanced. Solutions to all of these problems are provided, in which different approaches are compared.

We would like to sincerely thank Dr. Richard Stong and Dr. Mircea Becheanu, who have helped improve initial drafts of the manuscript. They have uncovered several errors and refined numerous solutions.

Enjoy the problems!

2 Let's factor

Factoring algebraic expressions is a fundamental technique that enables us to solve a broad class of equations, inequalities, and systems of equations. As a familiar example, factoring is often used to find solutions to quadratic equations $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$.

We first recall a few of the most basic identities, beginning with the familiar difference of squares identity. Let a and b be real numbers. Then we have:

$$a^2 - b^2 = (a - b)(a + b),$$

which follows easily from expanding the right-hand side. A similar identity holds for an exponent of 3, the difference of cubes:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

In fact, we can generalize this; we have

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}),$$

for all real numbers a and b, and positive integers n. One way to see this is to note that the expression $a^n - b^n$ equals 0 when $a = b$, which implies that $a - b$ is a factor. If n is odd, we can replace b with $-b$ to obtain a factorization for the sum of nth powers. Indeed, if $n = 2k + 1$ for a positive integer k, we have

$$a^{2k+1} + b^{2k+1} = (a + b)(a^{2k} - a^{2k-1}b + a^{2k-2}b^2 - \cdots - ab^{2k-1} + b^{2k}).$$

Taking the special case $k = 1$ gives us the familiar sum of cubes identity:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

We finally look at the useful algebraic identity

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).$$

Of course, one can in principle simply expand the right-hand side to prove it. Suppose, however, that we were asked to factor the expression $a^3+b^3+c^3-3abc$. To do so, consider the polynomial $P(x)$ with roots a, b, and c:

$$P(x) = (x - a)(x - b)(x - c) = x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc.$$

Since a, b, c are roots, noting that $P(a) = P(b) = P(c) = 0$ gives us three equations:

$$\begin{aligned}
a^3 - (a + b + c)a^2 + (ab + bc + ca)a - abc = 0,\\
b^3 - (a + b + c)b^2 + (ab + bc + ca)b - abc = 0,\\
c^3 - (a + b + c)c^2 + (ab + bc + ca)c - abc = 0.
\end{aligned}$$

Now, adding the three relations and isolating $a^3 + b^3 + c^3 - 3abc$ on one side, we obtain

$$\begin{aligned} a^3 + b^3 + c^3 - 3abc &= (a + b + c)(a^2 + b^2 + c^2) - (ab + bc + ca)(a + b + c) \\ &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca). \end{aligned}$$

We remark that $a^2+b^2+c^2-ab-bc-ca = \frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2] \geq 0$, with equality if and only if $a = b = c$. Thus, $a^3 + b^3 + c^3 = 3abc$ if and only if $a = b = c$ or $a + b + c = 0$. The prequel to this book, *105 Algebra Problems from the AwesomeMath Summer Program*, features a section with numerous problems solved with this identity.

We now turn to examples to explore how these ideas work in practice.

Example 2.1. Factor the following expressions:
(a) $x^4 - 3x^2y^2 + y^4$;
(b) $(x + y)(x - y) - 4(y + 1)$;
(c) $4(x^2 + x - y^2) + 1$;
(d) $x(x - 4y) + 4(y^2 - 1)$;
(e) $x^2 - y^2 + 2(x + 3y - 4)$.

Solution. (a) We will try to complete the square first, by writing

$$x^4 - 3x^2y^2 + y^4 = x^4 - 2x^2y^2 + y^4 - x^2y^2 = (x^2 - y^2)^2 - (xy)^2.$$

Using the formula $a^2 - b^2 = (a - b)(a + b)$, it is now easy to obtain the factorization

$$x^4 - 3x^2y^2 + y^4 = (x^2 - y^2 - xy)(x^2 - y^2 + xy).$$

(b) We start by expanding the given expression and separating the variables:

$$(x + y)(x - y) - 4(y + 1) = x^2 - y^2 - 4y - 4 = x^2 - (y^2 + 4y + 4).$$

The expression $y^2 + 4y + 4$ should ring a bell: it is $(y + 2)^2$. Hence using the formula $a^2 - b^2 = (a - b)(a + b)$, we conclude that

$$(x + y)(x - y) - 4(y + 1) = x^2 - (y + 2)^2 = (x - y - 2)(x + y + 2).$$

(c) Again, we start by expanding and separating variables:

$$4(x^2 + x - y^2) + 1 = 4x^2 + 4x - 4y^2 + 1 = (4x^2 + 4x + 1) - 4y^2.$$

We recognize the expansion of $(2x + 1)^2$, hence

$$4(x^2 + x - y^2) + 1 = (2x + 1)^2 - (2y)^2 = (2x - 2y + 1)(2x + 2y + 1).$$

(d) We follow the same strategy as in the previous examples and obtain

$$x(x-4y)+4(y^2-1) = x^2-4xy+4y^2-4 = (x-2y)^2-4 = (x-2y+2)(x-2y-2).$$

(e) This is yet another example where separation of variables and use of fundamental algebraic identities yields a quick and natural solution:

$$x^2 - y^2 + 2(x + 3y - 4) = x^2 + 2x + 1 - y^2 + 6y - 9$$

$$= (x+1)^2 - (y-3)^2 = (x+1-y+3)(x+1+y-3) = (x-y+4)(x+y-2).$$

Example 2.2. Factor the following expressions:
(a) $x^3 + 9x^2 + 27x + 19$;
(b) $x^3 + 3x^2 + 3x - 7$;
(c) $(x - y)(x^2 + xy + y^2 + 3) + 3(x^2 + y^2) + 2$.

Solution. (a) We have many coefficients divisible by 3, so the formula

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

will probably be useful. We are trying to complete the cube to some expression of the form $(x+a)^3$. The previous formula (with $b = x$) shows that a reasonable choice for a such that $(x + a)^3$ is close to the given expression is $a = 3$, since

$$(x + 3)^3 = x^3 + 9x^2 + 27x + 27.$$

So we see that

$$x^3 + 9x^2 + 27x + 19 = (x + 3)^3 - 8 = (x + 3)^3 - 2^3.$$

Using the difference of cubes identity yields

$$(x + 3)^3 - 2^3 = (x + 3 - 2)((x + 3)^2 + 2(x + 3) + 4),$$

and a small computation gives

$$(x + 3)^2 + 2(x + 3) + 4 = x^2 + 6x + 9 + 2x + 6 + 4 = x^2 + 8x + 19,$$

which cannot be factored anymore since the discriminant is negative. Putting everything together gives

$$x^3 + 9x^2 + 27x + 19 = (x + 1)(x^2 + 8x + 19).$$

(b) Following the same strategy as in (a), we write

$$\begin{aligned} x^3 + 3x^2 + 3x - 7 = x^3 + 3x^2 + 3x + 1 - 8 &= (x + 1)^3 - 2^3 \\ &= (x + 1 - 2)((x + 1)^2 + 2(x + 1) + 4) \\ &= (x - 1)(x^2 + 4x + 7). \end{aligned}$$

(c) The term $(x-y)(x^2+xy+y^2+3)$ simplifies quite nicely taking into account the identity

$$(x-y)(x^2+xy+y^2) = x^3 - y^3.$$

Hence our expression equals

$$x^3 - y^3 + 3x - 3y + 3x^2 + 3y^2 + 2.$$

We separate the variables to get

$$x^3 + 3x^2 + 3x - y^3 + 3y^2 - 3y + 2.$$

We easily recognize the expansions of $(x+1)^3$ and $(1-y)^3$. Thus our expression equals

$$(x+1)^3 + (1-y)^3 = (x+1+1-y)((x+1)^2 - (x+1)(1-y) + (1-y)^2).$$

Expanding and simplifying the second factor finally yields the factorization

$$(x-y)(x^2+xy+y^2+3)+3(x^2+y^2)+2 = (x-y+2)(x^2+y^2+xy+x-y+1).$$

Example 2.3. Factor the expression $a^4 + 4b^4$.

Solution. Noting that these two terms are in the expansion of $(a^2+2b^2)^2$, we rewrite the expression as

$$a^4 + 4b^4 = (a^2+2b^2)^2 - 4a^2b^2,$$

and since $4a^2b^2 = (2ab)^2$, we can apply the difference of squares identity, giving

$$(a^2+2b^2)^2 - (2ab)^2 = (a^2+2b^2-2ab)(a^2+2b^2+2ab).$$

This identity is known as the *Sophie Germain Identity.*

Example 2.4. Is $2012 \cdot 503^{2011} + 2013^4$ prime?

Solution. Rewrite the expression as

$$2013^4 + 4 \cdot 503^{2012} = 2013^4 + 4 \cdot (503^{503})^4.$$

By the Sophie Germain identity, $a^4 + 4b^4 = (a^2+2b^2-2ab)(a^2+2b^2+2ab)$ with $a = 2013$ and $b = 503^{503}$, this factors as

$$(2013^2 + 2(503^{503})^2 - 2(2013 \cdot 503^{503}))(2013^2 + 2(503^{503})^2 + 2(2013 \cdot 503^{503})).$$

Since both factors are positive integers greater than 1, the number is composite.

Example 2.5. Factor the expression:

$$2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4).$$

Solution. Seeing the terms a^4, b^4, c^4 and $2a^2b^2, 2b^2c^2, 2a^2c^2$, we try to complete the square. Indeed, after some experimentation, we find that the given expression is equivalent to

$$4b^2c^2 - (a^2 - b^2 - c^2)^2.$$

Then, observe that $4b^2c^2 = (2bc)^2$ and use the formula for the difference of squares:

$$\begin{aligned} 4b^2c^2 - (a^2 - b^2 - c^2)^2 &= (2bc)^2 - (a^2 - b^2 - c^2)^2 \\ &= (2bc - a^2 + b^2 + c^2)(2bc + a^2 - b^2 - c^2). \end{aligned}$$

Things are not quite done yet, since we can still factor the expressions above, by rearranging terms and completing squares:

$$2bc - a^2 + b^2 + c^2 = (b + c)^2 - a^2 = (b + c - a)(b + c + a),$$

and

$$2bc + a^2 - b^2 - c^2 = a^2 - (b - c)^2 = (a - b + c)(a + b - c).$$

Putting everything together yields

$$4b^2c^2 - (a^2 - b^2 - c^2)^2 = (b + c - a)(b + c + a)(a - b + c)(a + b - c).$$

The previous exercise is actually closely related to a classical problem in geometry: express the area K of a triangle purely in terms of its side lengths a, b, c. Suppose the angles opposite sides a, b, c are A, B, C respectively. Then

$$K = \frac{bc \sin A}{2},$$

so that by the Law of Cosines

$$\begin{aligned} K^2 = \frac{b^2c^2(1 - \cos^2 A)}{4} &= \frac{b^2c^2}{4}\left(1 - \left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2\right) \\ &= \frac{4b^2c^2 - (a^2 - b^2 - c^2)^2}{16}. \end{aligned}$$

Hence the previous exercise recovers the famous Heron's formula

$$16K^2 = (a + b + c)(b + c - a)(c + a - b)(a + b - c).$$

Example 2.6. Find all pairs (x, y) of integers satisfying

$$\begin{cases} x^2 + 11 = xy + y^4 \\ y^2 - 30 = xy. \end{cases}$$

Solution. The easiest way to solve this problem is to notice that y must divide 30, as the second equation shows. By checking all divisors of 30, we obtain corresponding values for x using the second equation and then we test whether these values satisfy the first equation. This approach requires quite a few cases and thus is fairly computational.
For a more indirect, and also more elegant approach, let us add up the equations. We obtain

$$x^2 + y^2 - 19 = 2xy + y^4,$$

which is equivalent to

$$(x - y)^2 - (y^2)^2 = 19.$$

This factors as a difference of squares:

$$(y^2 - (x - y))(y^2 + (x - y)) = -19.$$

And now we are in good shape, since 19 is a prime. Thus the two factors $y^2 - (x - y)$ and $y^2 + (x - y)$ must be one of the pairs $(19, -1)$, $(-1, 19)$, $(-19, 1)$, or $(1, -19)$. We can eliminate half of these cases by noting that the sum of the factors is nonnegative:

$$y^2 - (x - y) + y^2 + (x - y) = 2y^2 \geq 0.$$

Thus the two factors must be $(19, -1)$ or $(-1, 19)$. In both cases, noting that the sum of the factors is $2y^2$ gives

$$2y^2 = 19 - 1 = 18,$$

and $y = \pm 3$. Plugging in both possible values of y into the equations yields the solutions

$$(x, y) = (-7, 3) \quad \text{or} \quad (7, -3).$$

Example 2.7. Factor the expression

$$(x^2y + y^2z + z^2x) + (xy^2 + yz^2 + zx^2) + 2xyz.$$

Solution. Observe that setting $x = -y$ makes the expression equal 0, since doing so gives

$$y^3 + y^2z - z^2y - y^3 + yz^2 + zy^2 - 2y^2z = 0.$$

Similarly, letting $y = -z$ and $z = -x$ make the expression vanish as well.
It follows that $(x+y), (y+z)$ and $(z+x)$ are factors of the expression. Looking at the degree of the terms in the given expression, we conclude that the expression is of the form $k(x+y)(y+z)(z+x)$. Taking the special case $x = y = z = 1$ yields $8k = 8$, which implies $k = 1$. Thus, the expression factors as $(x+y)(y+z)(z+x)$.

Example 2.8. Simplify the expressions:

(a) $\dfrac{1}{x-1} - \dfrac{3}{x^3-1}$;

(b) $\dfrac{x^4 - 2x^3 - x^2 + 2x + 1}{x^4 - 2x^3 + x^2 - 1}$.

Solution. (a) Note that the factor $x - 1$ appears in both denominators, since

$$x^3 - 1 = (x-1)(x^2 + x + 1).$$

Hence

$$\frac{1}{x-1} - \frac{3}{x^3-1} = \frac{1}{x-1}\left(1 - \frac{3}{x^2+x+1}\right).$$

There is no mystery now: we compute

$$1 - \frac{3}{x^2+x+1} = \frac{x^2+x-2}{x^2+x+1},$$

and we observe (by solving the equation $x^2 + x - 2 = 0$) that the numerator can be factored as $(x-1)(x+2)$. Thus

$$\frac{1}{x-1} - \frac{3}{x^3-1} = \frac{1}{x-1} \cdot \frac{(x-1)(x+2)}{x^2+x+1} = \frac{x+2}{x^2+x+1}.$$

It is clear that this expression cannot be simplified further, since $x + 2$ is not a factor of the denominator (as -2 is not a root of $x^2 + x + 1$).
(b) It is easier to analyze the denominator of the fraction, since we easily recognize

$$x^4 - 2x^3 + x^2 = x^2(x^2 - 2x + 1) = x^2(x-1)^2.$$

Then we can factor the denominator as a difference of squares:

$$x^4 - 2x^3 + x^2 - 1 = (x^2 - x)^2 - 1^2 = (x^2 - x - 1)(x^2 - x + 1).$$

Now, it is natural to ask whether the numerator is divisible by one of $x^2 - x - 1$ or $x^2 - x + 1$, since otherwise our chances of being able to simplify the expression are rather small. Long division shows that indeed

$$x^4 - 2x^3 - x^2 + 2x + 1 = (x^2 - x - 1)^2,$$

hence

$$\frac{x^4-2x^3-x^2+2x+1}{x^4-2x^3+x^2-1} = \frac{(x^2-x-1)^2}{(x^2-x-1)(x^2-x+1)} = \frac{x^2-x-1}{x^2-x+1}.$$

Here is an alternative method of dealing with the numerator: observe that it has quite a lot of common terms with the denominator. Hence we write it as

$$x^4-2x^3+x^2-1+(-2x^2+2x+2) = (x^2-x-1)(x^2-x+1)-2(x^2-x-1)$$
$$= (x^2-x-1)(x^2-x+1-2) = (x^2-x-1)^2.$$

Example 2.9. Find the sum of the absolute values of the roots of

$$x^4-4x^3-4x^2+16x-8.$$

Solution. Vieta's relations, which we will discuss in Chapter 5, easily tell us that sum of the roots is 4. However, it is not as easy to obtain the sum of the absolute values of the roots as requested by the problem statement.
To simplify the quartic, we try to complete the square. Indeed, noting that the first three terms $x^4-4x^3-4x^2$ look similar to $x^4-4x^3+4x^2 = x^2(x-2)^2$, we can rewrite the given expression as

$$\begin{aligned} x^4-4x^3-4x^2+16x-8 &= (x^4-4x^3+4x^2)-(8x^2-16x+8) \\ &= x^2(x-2)^2-8(x-1)^2 \\ &= (x^2-2x)^2-(2\sqrt{2}x-2\sqrt{2})^2 \end{aligned}$$

and from here we can apply difference of squares. Our expression is equal to

$$= (x^2-(2+2\sqrt{2})x+2\sqrt{2})(x^2-(2-2\sqrt{2})x-2\sqrt{2}).$$

We can repeat a similar process by completing the square again, using the fact that $(1+\sqrt{2})^2 = 3+2\sqrt{2}$. Indeed,

$$\begin{aligned} x^2-(2+2\sqrt{2})x+2\sqrt{2} &= x^2-(2+2\sqrt{2})x+3+2\sqrt{2}-3 \\ &= (x-(1+\sqrt{2}))^2-(\sqrt{3})^2 \\ &= (x-1-\sqrt{2}+\sqrt{3})(x-1-\sqrt{2}-\sqrt{3}). \end{aligned}$$

Proceeding similarly for the other factor, we have

$$x^2-(2-2\sqrt{2})x-2\sqrt{2} = (x-1+\sqrt{2}+\sqrt{3})(x-1+\sqrt{2}-\sqrt{3})$$

Our roots are thus $1+\sqrt{2}+\sqrt{3}, 1+\sqrt{2}-\sqrt{3}, 1-\sqrt{2}+\sqrt{3}$, and $1-\sqrt{2}-\sqrt{3}$. Using the approximations $\sqrt{2} \approx 1.414$, and $\sqrt{3} \approx 1.732$, we easily see that the only negative root is $1-\sqrt{2}-\sqrt{3}$. The requested sum is therefore

$$\begin{aligned}(1+\sqrt{2}+\sqrt{3}) &+ (1+\sqrt{2}-\sqrt{3}) + (1-\sqrt{2}+\sqrt{3}) + (-1)(1-\sqrt{2}-\sqrt{3}) \\ &= 2+2\sqrt{2}+2\sqrt{3}.\end{aligned}$$

Example 2.10. Prove that for any positive integer n, $(n+1)^5+n$ is not a prime.

Solution. Let $n+1=m$. Then, the number of terms becomes much smaller than what we would obtain if we expanded $(n+1)^5$.

$$(n+1)^5+n = m^5+m-1.$$

We try to factor this to show that the quantity is always composite.

$$m^5+m-1 = m^5+m^2-m^2+m-1 = m^2(m^3+1)-m^2+m-1.$$

We see that we have obtained a sum of cubes in the last expression. Factoring this, we find that the m^2-m+1 term is common:

$$\begin{gathered} m^2(m^3+1)-m^2+m-1 \\ = m^2(m+1)(m^2-m+1)-(m^2-m+1) = (m^2-m+1)(m^3+m^2-1). \end{gathered}$$

To conclude, we need to check that neither factor is 1, since in this case the product could be prime. Since n is a positive integer, we have $m=n+1 \geq 2$. For the first factor, $m^2-m+1=1$ implies $m^2-m=m(m-1)=0$, and hence the first factor is 1 when $m=0$ or 1, which will not occur.
For the second factor, $m^3+m^2-1=1$ implies $m^3+m^2-2=0$. Since 1 is a root of this polynomial, we know $m-1$ is a factor, and we obtain $m^3+m^2-2=(m-1)(m^2+2m+2)=0$. The quadratic has no real roots as its discriminant Δ is negative: $\Delta = 2^2-4\cdot 2<0$. Hence the second factor is 1 only when $m=1$, which will not occur. Since both factors will never equal 1, the desired result follows.

Example 2.11. Simplify the fraction

$$\frac{a^5+(a-1)^4}{(a-1)^5-a^4}.$$

Solution. Call the given fraction $F(a)$. Adding the quantity a to the expression introduces a common factor of $(a-1)^4$:

$$a+F(a) = \frac{a(a-1)^5+(a-1)^4}{(a-1)^5-a^4}.$$

Factoring out $(a-1)^4$ from the numerator gives

$$\frac{a(a-1)^5+(a-1)^4}{(a-1)^5-a^4}=\frac{(a-1)^4(a^2-a+1)}{(a-1)^5-a^4}.$$

Clearly, the denominator is not divisible by $a-1$, so we try a^2-a+1 as factor. Indeed, by long division, we obtain $(a-1)^5-a^4=(a^2-a+1)(a^3-5a^2+4a-1)$. Therefore

$$a+F(a)=\frac{(a-1)^4}{a^3-5a^2+4a-1}.$$

Solving for $F(a)$, it follows that

$$\begin{aligned}F(a)=\frac{(a-1)^4}{a^3-5a^2+4a-1}-a&=\frac{a^4-4a^3+6a^2-4a+1}{a^3-5a^2+4a-1}-a\\&=\frac{a^3+2a^2-3a+1}{a^3-5a^2+4a-1}.\end{aligned}$$

Example 2.12. Let a,b,c be distinct real numbers such that

$$a^2(1-b+c)+b^2(1-c+a)+c^2(1-a+b)=ab+bc+ca.$$

Prove that

$$\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}=1.$$

Solution. First, observe that

$$\begin{aligned}\left(\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}\right)^2&=\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}\\&+2\left(\frac{1}{(a-b)(b-c)}+\frac{1}{(b-c)(c-a)}+\frac{1}{(c-a)(a-b)}\right)\\&=\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+2\cdot\frac{c-a+a-b+b-c}{(a-b)(b-c)(c-a)}\\&=\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}.\end{aligned}$$

Now, it suffices to prove that

$$\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}=1 \text{ or } \frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}=-1.$$

We now show that the first of these equalities is equivalent to the given condition. Indeed, the given condition can be written as

$$a^2(-b+c)+b^2(-c+a)+c^2(-a+b)=-a^2-b^2-c^2+ab+bc+ca,$$

which can be factored as

$$(a-b)(b-c)(c-a) = (b-c)(c-a) + (c-a)(a-b) + (a-b)(b-c).$$

This is equivalent to

$$1 = \frac{1}{a-b} + \frac{1}{b-c} + \frac{1}{c-a},$$

as desired.

Example 2.13. Factor the following expressions:
(a) $(x-y)^3 + (y-z)^3 + (z-x)^3$;
(b) $(x-a)^3(b-c)^3 + (x-b)^3(c-a)^3 + (x-c)^3(a-b)^3$.

Solution. (a) Since $(x-y)+(y-z)+(z-x)=0$, we immediately find the factorization $(x-y)^3+(y-z)^3+(z-x)^3 = 3(x-y)(y-z)(z-x)$ from the fact that $a^3+b^3+c^3 = 3abc$ when $a+b+c=0$.
Alternatively, observe that plugging in $x=y$ makes the expression equal 0. Similarly, plugging in $y=z$ and $z=x$ makes the expression vanish.
We write

$$(x-y)^3 + (y-z)^3 + (z-x)^3 = a(x-y)(y-z)(z-x),$$

for some constant a. Equating coefficients, we find that $3xy^2 = axy^2$, revealing that $a=3$. Indeed, to check, we expand the right-hand side which indeed equals the left-hand side. Thus

$$(x-y)^3 + (y-z)^3 + (z-x)^3 = 3(x-y)(y-z)(z-x).$$

(b) Let $a' = (x-a)(b-c)$, $b' = (x-b)(c-a)$, $c' = (x-c)(a-b)$. Expanding, we verify that $a'+b'+c'=0$. It follows that

$$a'^3 + b'^3 + c'^3 = 3a'b'c',$$

or

$$\begin{aligned}&(x-a)^3(b-c)^3 + (x-b)^3(c-a)^3 + (x-c)^3(a-b)^3\\ &\quad = 3(a-b)(b-c)(c-a)(x-a)(x-b)(x-c).\end{aligned}$$

Example 2.14. Let a, b, c be distinct, nonzero real numbers. If two fractions among

$$\frac{a^2-bc}{a(1-bc)}; \quad \frac{b^2-ca}{b(1-ca)}; \quad \frac{c^2-ab}{c(1-ab)}$$

are equal, then prove that all of these are equal, and that their common value equals $a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

Solution. We use a basic property of proportions: if $\frac{x}{y} = \frac{z}{w}$ and $y \neq w$, then $\frac{x}{y} = \frac{z}{w} = \frac{x-z}{y-w}$. Hence if we assume, without loss of generality, that

$$\frac{a^2 - bc}{a(1-bc)} = \frac{b^2 - ca}{b(1-ca)},$$

then

$$\frac{a^2 - bc}{a(1-bc)} = \frac{b^2 - ca}{b(1-ca)} = \frac{a^2 - bc - b^2 + ca}{a - abc - b + abc}.$$

We recognize the difference of squares $a^2 - b^2$, and writing $a^2 - bc - b^2 + ca = (a-b)(a+b) + c(a-b)$ allows us to factor the numerator:

$$\frac{a^2 - bc}{a(1-bc)} = \frac{a^2 - bc - b^2 + ca}{a - abc - b + abc} = \frac{(a-b)(a+b+c)}{a-b} = a + b + c.$$

The previous equality shows that $a^2 - bc = a(1-bc)(a+b+c)$, implying

$$-bc - ca - ab = -abc(a+b+c).$$

Dividing both sides by $-abc$ gives

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = a + b + c.$$

We are almost done. We have shown that the first two fractions equal $a+b+c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, and now we have to show that the third fraction equals this common value. Now, observe that

$$\begin{aligned}\frac{c^2 - ab}{c(1-ab)} - a - b - c &= \frac{c^2 - ab - ac - bc - c^2 + abc(a+b+c)}{c(1-ab)} \\ &= \frac{-(ab + bc + ca) + abc(a+b+c)}{c(1-bc)} = 0,\end{aligned}$$

where this equality follows from $-bc - ca - ab = -abc(a+b+c)$, a fact we proved earlier. Hence,

$$\frac{c^2 - ab}{c(1-ab)} = a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c},$$

which is equal to the other two fractions, as desired.

Example 2.15. Prove that for any positive integers m and n, the number

$$8m^6 + 27m^3n^3 + 27n^6$$

is composite.

Solution. Seeing two terms divisible by 3, and many cubes, we think of the identity $x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$. We try to rewrite the expression in a way that will allow us to use this factorization. Rewriting two terms as cubes and splitting $27m^3n^3$, we have

$$8m^6+27m^3n^3+27n^6=(2m^2)^3+(3n^2)^3-27m^3n^3+54m^3n^3$$
$$=(2m^2)^3+(3n^2)^3+(-3mn)^3-3(2m^2)(3n^2)(-3mn).$$

Now, this is in the form $x^3+y^3+z^3-3xyz$, and we can use the identity noted above with $x=2m^2$, $y=3n^2$, and $z=-3mn$. This gives us

$$(2m^2)^3+(3n^2)^3+(-3mn)^3-3(2m^2)(3n^2)(-3mn)$$
$$=(2m^2+3n^2-3mn)(4m^4+9n^4+9m^2n^2-6m^2n^2+9mn^3+6m^3n).$$

Therefore, $2m^2+3n^2-3mn$ is always a divisor of $8m^6+27m^3n^3+27n^6$. To finish, using the fact that m and n are positive integers, it suffices to show that $1<2m^2+3n^2-3mn<8m^6+27m^3n^3+27n^6$. This ensures that the product is not prime as a result of equaling 1 times a prime number.
Indeed, since $3mn>0$ we have

$$2m^2+3n^2-3mn<2m^2+3n^2<8m^6+27m^3n^3+27n^6.$$

On the other hand,

$$2m^2+3n^2-3mn=2(m-n)^2+n^2+mn>1.$$

It follows that $8m^6+27m^3n^3+27n^6$ is composite, as claimed.

Example 2.16. Write

$$\frac{x^{24}}{24}+8\cdot 3^{11}$$

as a product of four nonconstant polynomials with rational coefficients.

Solution. Clearing denominators, we need to factor

$$x^{24}+24\cdot 8\cdot 3^{11}=x^{24}+2^6\cdot 3^{12}.$$

Using the sum of cubes formula, we may write

$$x^{24}+2^6\cdot 3^{12}=(x^8)^3+(2^2\cdot 3^4)^3=(x^8+2^2\cdot 3^4)(x^{16}-2^2\cdot 3^4x^8+2^4\cdot 3^8).$$

Recall the Sophie Germain identity $a^4+4b^4=(a^2+2b^2-2ab)(a^2+2b^2+2ab)$. We can factor the 8th degree factor by applying the Sophie Germain identity with $a=x^2$ and $b=3$. Indeed, we have

$$x^{24}+2^6\cdot 3^{12}=(x^4-6x^2+18)(x^4+6x^2+18)(x^{16}-2^2\cdot 3^4x^8+2^4\cdot 3^8).$$

The problem asks for one more factor; we will obtain it from the 16th degree polynomial. One way to do so is to note that the original polynomial can be factored directly using the Sophie Germain identity with $a = x^6$ and $b = 2\cdot 3^3$. We obtain

$$\begin{aligned} x^{24} + 2^6 \cdot 3^{12} &= (x^6)^4 + 4\cdot(2\cdot 3^3)^4 \\ &= (x^{12} - 2^2\cdot 3^3 x^6 + 2^3\cdot 3^6)(x^{12} + 2^2\cdot 3^3 x^6 + 2^3 \cdot 3^6). \end{aligned}$$

Now, the quartic factors from before are factors of these 12th degree polynomials. We use long division to obtain

$$(x^{12} - 2^2\cdot 3^3 x^6 + 2^3\cdot 3^6) = (x^4 + 6x^2 + 18)(x^8 - 6x^6 + 18x^4 - 108x^2 + 324),$$

and

$$(x^{12} + 2^2\cdot 3^3 x^6 + 2^3\cdot 3^6) = (x^4 - 6x^2 + 18)(x^8 + 6x^6 + 18x^4 + 108x^2 + 324).$$

Therefore

$$\frac{x^{24}}{24} + 8\cdot 3^{11} = \frac{1}{24}\left(x^{24} + 2^6\cdot 3^{12}\right) = \frac{1}{24}A(x)B(x)C(x)D(x),$$

where

$$A(x) = (x^4 - 6x^2 + 18); \quad B(x) = (x^4 + 6x^2 + 18);$$
$$C(x) = (x^8 - 6x^6 + 18x^4 - 108x^2 + 324); \quad D(x) = (x^8 + 6x^6 + 18x^4 + 108x^2 + 324).$$