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Preface

109 Inequalities from the AwesomeMath Summer Program explores important theory and techniques involved in proving algebraic inequalities. To expand the reader's mathematical repertoire, we present problems from various mathematical journals and contests from around the globe.

The book is structured into subchapters covering the Trivial Inequality, the AM-GM and Cauchy-Schwarz inequalities, Hölder's Inequality for Sums, Nesbitt's Inequality, as well as the Rearrangement and Chebyshev's inequalities. Knowledge of the above results is not sufficient – the subtleties of these inequalities' efficient applications are just as important. In the exposition, we state and prove several theorems and corollaries related to the topics or methods mentioned above. We then provide numerous examples illustrating how and when to effectively employ the theorems and lemmas discussed.

We conclude with 109 problems that allow the reader to practice the skills honed in the subchapters, of which 54 are introductory and 55 are advanced. Complete solutions to all of these problems are provided. For many problems we offer multiple solutions as well as the motivation behind them.

Inequalities are an essential topic in mathematical Olympiad problem solving and 109 Inequalities serves as an instructive resource for students striving for success at national and international competitions. Inequalities are also of great theoretical interest and pave the way towards advanced topics such as analysis, probability, and measure theory. Most of all, we hope that the reader finds inspiration in both the struggle and beauty of proving interesting algebraic inequalities.

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Enjoy the problems and their solutions!

Titu Andreescu, Adithya Ganesh, January 2015

1 An Introduction

We start with the Trivial Inequality,

$$x^2 \geq 0,$$

which holds for all real numbers x. While this seems obvious, numerous (technically, all) inequalities can be derived using only this fact. When trying to prove that a quantity is nonnegative, we will often seek to write the expression as the square of another expression, or the sum of a number of squares. We turn to some examples to show how this works in practice.

Example 1.1. Let a, b, c be real numbers. Prove that

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

Solution. The inequality is equivalent to the obvious

$$\frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2] \geq 0.$$

This is a basic, but important inequality that we will rely on throughout this book. The equivalent inequality $(a+b+c)^2 \geq 3(ab+bc+ca)$ is also used very often. Equality occurs if and only if $a = b = c$.

Example 1.2. Let a, b, c be positive real numbers. Prove that

$$\max\left(\frac{1}{a} + \frac{b}{4}, \frac{1}{b} + \frac{c}{4}, \frac{1}{c} + \frac{a}{4}\right) \geq 1.$$

Solution. Observe that for all positive real numbers x, we have $\frac{1}{x} + \frac{x}{4} \geq 1$, since this reduces to the obvious $(x-2)^2 \geq 0$. Thus,

$$\left(\frac{1}{a} + \frac{b}{4}\right) + \left(\frac{1}{b} + \frac{c}{4}\right) + \left(\frac{1}{c} + \frac{a}{4}\right)$$

$$= \left(\frac{1}{a} + \frac{a}{4}\right) + \left(\frac{1}{b} + \frac{b}{4}\right) + \left(\frac{1}{c} + \frac{c}{4}\right) \geq 1 + 1 + 1 = 3.$$

The desired result now follows. Equality holds if and only if $a = b = c = 2$.

Example 1.3. Let a and b be positive real numbers. Prove that

$$\frac{8}{a+b} - \frac{9}{a+b+ab} \leq 1.$$

Solution. Let $s = a+b$ and $p = ab$. It follows that $s^2 \geq 4p$, since this reduces to $(a-b)^2 \geq 0$. After some algebraic manipulations, the desired inequality reduces to

$$8p \leq s^2 + s(p+1).$$

Note that $p+1 \geq 2\sqrt{p}$, since this reduces to $(\sqrt{p}-1)^2 \geq 0$. The desired inequality follows from $s^2 \geq 4p$ and $s(p+1) \geq (2\sqrt{p})(2\sqrt{p}) = 4p$.

Example 1.4. Let a, b, c be distinct real numbers. Prove that

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2} \geq 2.$$

Solution. Observe that

$$\begin{aligned}
&\frac{bc}{(a-b)(a-c)} + \frac{ca}{(b-c)(b-a)} + \frac{ab}{(c-a)(c-b)} \\
&= \frac{bc(b-c)}{(a-b)(b-c)(a-c)} + \frac{ca(c-a)}{(a-b)(b-c)(a-c)} + \frac{ab(a-b)}{(a-b)(b-c)(a-c)} \\
&= \frac{(a-b)(b-c)(a-c)}{(a-b)(b-c)(a-c)} = 1.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2} &= \left(\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b}\right)^2 \\
&+ \frac{2bc}{(a-b)(a-c)} + \frac{2ca}{(b-c)(b-a)} + \frac{2ab}{(c-a)(c-b)} \\
&= \left(\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b}\right)^2 + 2 \geq 2.
\end{aligned}$$

Equality occurs if and only if $\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b} = 0$.

Example 1.5. (Roberto Bosch Cabrera, Mathematical Reflections) Let a, b, c be distinct real numbers. Prove that

$$\left(\frac{a}{a-b} + 1\right)^2 + \left(\frac{b}{b-c} + 1\right)^2 + \left(\frac{c}{c-a} + 1\right)^2 \geq 5.$$

Solution. From the identity in the previous example,

$$\begin{aligned}
\sum_{\text{cyc}} \frac{a}{a-b}\left(1 - \frac{b}{b-c}\right) &= -\sum_{\text{cyc}} \frac{ca}{(a-b)(b-c)} \\
&= -\sum_{\text{cyc}} \frac{ab(a-b)}{(a-b)(b-c)(c-a)} = 1.
\end{aligned}$$

This implies that

$$\sum_{\text{cyc}} \frac{a}{a-b} = 1 + \sum_{\text{cyc}} \left(\frac{a}{a-b} \cdot \frac{b}{b-c} \right).$$

Applying this identity to simplify the left side of the desired inequality,

$$\begin{aligned} \sum_{\text{cyc}} \left(\frac{a}{a-b} + 1 \right)^2 &= 3 + \sum_{\text{cyc}} \left(\frac{a}{a-b} \right)^2 + 2 \sum_{\text{cyc}} \frac{a}{a-b} \\ &= 3 + \sum_{\text{cyc}} \left(\frac{a}{a-b} \right)^2 + 2 \left[1 + \sum_{\text{cyc}} \frac{a}{a-b} \cdot \frac{b}{b-c} \right] \\ &= 5 + \left(\sum_{\text{cyc}} \frac{a}{a-b} \right)^2 \geq 5. \end{aligned}$$

This completes the proof.

Example 1.6. (Titu Andreescu) Let a, b, c be positive real numbers. Prove that at least one of the inequalities

$$\frac{3}{a} \geq 2 - b; \quad \frac{2}{b} \geq 4 - c; \quad \frac{6}{c} \geq 6 - a$$

is true.

Solution. Suppose for sake of contradiction that

$$a(2-b) > 3, \quad b(4-c) > 2, \quad c(6-a) > 6.$$

Then $6 - a > 0, \quad 2 - b > 0, \quad 4 - c > 0$, and

$$abc(2-b)(4-c)(6-a) > 36.$$

But $0 < a(6-a) \leq 9, 0 < b(2-b) \leq 1$, and $0 < c(4-c) \leq 4$, since these reduce to the obvious inequalities $(a-3)^2 \geq 0$, $(b-1)^2 \geq 0$, and $(c-2)^2 \geq 0$. Hence $abc(2-b)(4-c)(6-a) \leq 36$, which is a contradiction. The conclusion now follows.

Example 1.7. (Vasile Cîrtoaje, Mircea Lascu, Romania Junior TST 2003) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$1 + \frac{3}{a+b+c} \geq \frac{6}{ab+ac+bc}.$$

Solution. Let $x = \frac{1}{a}$, $y = \frac{1}{b}$, $z = \frac{1}{c}$. It follows that $xyz = 1$. The desired inequality is equivalent to

$$1 + \frac{3}{xy + yz + zx} \geq \frac{6}{x + y + z}.$$

From Example 1.1, we have that $(x+y+z)^2 \geq 3(xy+yz+zx)$, which implies that

$$1 + \frac{3}{xy + yz + zx} \geq 1 + \frac{9}{(x + y + z)^2}.$$

Finally, the inequality $\left(1 - \frac{3}{x+y+z}\right)^2 \geq 0$ implies

$$1 + \frac{9}{(x + y + z)^2} \geq \frac{6}{x + y + z}.$$

This proves the desired result. Equality holds if and only if $x = y = z = 1$, that is, $a = b = c = 1$.

Example 1.8. (Czech and Slovak Republics 2005) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a}{(a+1)(b+1)} + \frac{b}{(b+1)(c+1)} + \frac{c}{(c+1)(a+1)} \geq \frac{3}{4}.$$

Solution. After clearing denominators, we obtain the equivalent inequality

$$ab + ac + bc + a + b + c \geq 3(abc + 1),$$

that is

$$ab + ac + bc + a + b + c \geq 6.$$

The desired result follows from the fact that

$$ab + ac + bc + a + b + c - 6 = \frac{1}{c} + \frac{1}{b} + \frac{1}{a} + a + b + c - 6$$

$$= \left(\frac{1}{\sqrt{a}} - \sqrt{a}\right)^2 + \left(\frac{1}{\sqrt{b}} - \sqrt{b}\right)^2 + \left(\frac{1}{\sqrt{c}} - \sqrt{c}\right)^2 \geq 0.$$

Equality occurs if and only if $a = b = c = 1$.

Example 1.9. (Dorin Andrica, Mathematical Reflections) Let x, y, z be positive real numbers such that $x + y + z = 1$. Find the maximum of

$$E(x, y, z) = \frac{xy}{x + y} + \frac{yz}{y + z} + \frac{zx}{z + x}.$$

Solution. Observe that $\frac{(x+y)^2}{4} \geq xy$, since this reduces to $(x-y)^2 \geq 0$. Thus

$$E(x,y,z) \leq \frac{\frac{(x+y)^2}{4}}{x+y} + \frac{\frac{(y+z)^2}{4}}{y+z} + \frac{\frac{(z+x)^2}{4}}{z+x}$$
$$= \frac{x+y}{4} + \frac{y+z}{4} + \frac{z+x}{4} = \frac{x+y+z}{2} = \frac{1}{2}.$$

Consequently, the maximum value of $E(x,y,z)$ is $\dfrac{1}{2}$. Equality holds if and only if $x=y=z=\frac{1}{3}$.

Example 1.10. (Titu Andreescu, Mathematical Reflections) Let a and b be positive real numbers such that

$$|a-2b| \leq \frac{1}{\sqrt{a}} \quad \text{and} \quad |2a-b| \leq \frac{1}{\sqrt{b}}.$$

Prove that $a+b \leq 2$.

Solution. First, square both of the given inequalities to obtain the results

$$a^2-4ab+4b^2 \leq \frac{1}{a} \quad \text{and} \quad 4a^2-4ab+b^2 \leq \frac{1}{b}.$$

It follows that

$$a^3-4a^2b+4ab^2 \leq 1 \quad \text{and} \quad 4a^2b-4ab^2+b^3 \leq 1.$$

Summing the two inequalities, we obtain

$$a^3+b^3 \leq 2.$$

To conclude, observe that

$$2 \geq a^3+b^3 = (a+b)(a^2-ab+b^2) \geq (a+b)\frac{(a+b)^2}{4},$$

implying $(a+b)^3 \leq 8$, and the desired result follows. The equality case is achieved if and only if $a=b=1$.

Example 1.11. (Titu Andreescu) Let a be a positive real number. Prove that

$$a + \frac{1}{a^2} + \frac{1}{4a^3} \geq 2.$$

Solution. Clearing the denominators, observe that the inequality becomes

$$4a^4 - 8a^3 + 4a + 1 \geq 0,$$

or, equivalently,

$$4a^4 - 8a^3 + 4a \geq -1.$$

Now the expression on the left side is easy to factor. We obtain

$$a(4a^3 - 8a^2 + 4) \geq -1,$$

and since plugging in $a = 1$ makes the left side vanish, we see that $a - 1$ must divide $4a^3 - 8a^2 + 4$. Indeed, after factoring, the inequality reduces to

$$4a(a-1)(a^2 - a - 1) = (2a^2 - 2a)(2a^2 - 2a - 2) \geq -1.$$

Applying the difference of squares identity, this is equivalent to

$$(2a^2 - 2a)(2a^2 - 2a - 2) + 1 = (2a^2 - 2a - 1)^2 - 1^2 + 1 = (2a^2 - 2a - 1)^2 \geq 0,$$

which is clear. Equality occurs if and only if $a = \frac{1+\sqrt{3}}{2}$ (since $a > 0$).

Example 1.12. Let a and b be positive real numbers such that $a + b = 1$. Prove that

$$2 < \left(a - \frac{1}{a}\right)\left(b - \frac{1}{b}\right) \leq \frac{9}{4}$$

and

$$3 < \left(a^2 - \frac{1}{a}\right)\left(b^2 - \frac{1}{b}\right) \leq \frac{49}{16}.$$

Solution. The first inequality is equivalent to

$$2 < \frac{(a-1)(a+1)(b-1)(b+1)}{ab} \leq \frac{9}{4},$$

which reduces to

$$2 < (a+1)(b+1) \leq \frac{9}{4}.$$

The left inequality becomes $0 < ab$, while the right inequality reduces to $ab \leq \frac{1}{4}$, which follows from $ab \leq \frac{(a+b)^2}{4}$ via $0 \leq (a-b)^2$.
For the second inequality, observe that

$$\left(a^2 - \frac{1}{a}\right)\left(b^2 - \frac{1}{b}\right) = \frac{(a-1)(a^2+a+1)(b-1)(b^2+b+1)}{ab}$$

$$= (a^2 + a + 1)(b^2 + b + 1) = a^2b^2 + 3,$$

where we have used that $a + b = 1$ several times. Then, clearly,

$$3 < a^2b^2 + 3 \leq \frac{49}{16},$$

because the right inequality reduces to the same $ab \leq \frac{1}{4}$, proven earlier. Equality in the two upper bounds will occur if and only if $a = b = \frac{1}{2}$.

Example 1.13. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove the inequality

$$\frac{a^3}{a^2+b^2} + \frac{b^3}{b^2+c^2} + \frac{c^3}{c^2+a^2} \geq \frac{1}{2}.$$

Solution. The obvious inequality $(a-b)^2 \geq 0$ implies that $\frac{ab}{a^2+b^2} \leq \frac{1}{2}$. Consequently,

$$\frac{a^3}{a^2+b^2} = a - b \cdot \frac{ab}{a^2+b^2} \geq a - \frac{b}{2}.$$

By a similar argument, it follows that

$$\frac{b^3}{b^2+c^2} \geq b - \frac{c}{2}; \quad \frac{c^3}{c^2+a^2} \geq c - \frac{a}{2}.$$

Putting the above results together and using the given condition $a+b+c = 1$, we conclude that

$$\frac{a^3}{a^2+b^2} + \frac{b^3}{b^2+c^2} + \frac{c^3}{c^2+a^2} \geq a + b + c - \frac{a+b+c}{2} = \frac{1}{2}.$$

The desired result follows. Equality holds if and only if $a = b = c = \frac{1}{3}$.

Example 1.14. Let a, b, c be positive real numbers such that $abc = 1$. Prove the following inequality

$$\frac{1}{a^2+a+1} + \frac{1}{b^2+b+1} + \frac{1}{c^2+c+1} \geq 1.$$

Solution. For fans of brute force, clear the denominators by multiplying both sides by $(a^2+a+1)(b^2+b+1)(c^2+c+1)$. Our inequality is equivalent to

$$\sum_{\text{cyc}} (a^2+a+1)(b^2+b+1) \geq (a^2+a+1)(b^2+b+1)(c^2+c+1).$$

A short computation reveals that this inequality reduces to our old friend, Example 1.1: $a^2 + b^2 + c^2 \geq ab + bc + ca$.

Example 1.15. Let x and y be positive real numbers in the interval $(0, 1)$. Prove that

$$\frac{2}{1-xy} \leq \frac{1}{1-x^2} + \frac{1}{1-y^2}.$$

Solution. This inequality rewrites as

$$0 \le \frac{1}{1-x^2} - \frac{1}{1-xy} + \frac{1}{1-y^2} - \frac{1}{1-xy},$$

which is equivalent to

$$0 \le \frac{x(x-y)}{(1-x^2)(1-xy)} - \frac{y(x-y)}{(1-y^2)(1-xy)}.$$

The last inequality reduces to

$$0 \le \frac{(x-y)^2(1+xy)}{(1-x^2)(1-y^2)(1-xy)},$$

which is clear.

Example 1.16. Let a, b, c be positive real numbers less than 1. Prove that

$$\frac{1}{1-\sqrt{ab}} + \frac{1}{1-\sqrt{bc}} + \frac{1}{1-\sqrt{ca}} \le \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}.$$

Solution. Applying the result from the previous example, we obtain

$$\frac{2}{1-\sqrt{ab}} \le \frac{1}{1-a} + \frac{1}{1-b},$$

and its two counterparts. Adding these three inequalities yields the conclusion.

Example 1.17. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2(b+c)}{b^2+c^2} + \frac{b^2(c+a)}{c^2+a^2} + \frac{c^2(a+b)}{a^2+b^2} \ge a+b+c.$$

Solution. Note that

$$\sum_{\text{cyc}} \left(\frac{a^2(b+c)}{b^2+c^2} - a \right) = \sum_{\text{cyc}} \frac{ab(a-b)+ac(a-c)}{b^2+c^2}$$

$$= \sum_{\text{cyc}} \frac{ab(a-b)}{b^2+c^2} + \sum_{\text{cyc}} \frac{ba(b-a)}{c^2+a^2} =$$

$$= \sum_{\text{cyc}} \frac{ab(a-b)(c^2+a^2-b^2-c^2)}{(b^2+c^2)(c^2+a^2)} = \sum_{\text{cyc}} \frac{ab(a+b)(a-b)^2}{(b^2+c^2)(c^2+a^2)} \ge 0.$$

Equality occurs when $a = b = c$, as well as when $a = 0$ and $b = c$, or $b = 0$ and $c = a$, or $c = 0$ and $a = b$.

Example 1.18. Let a, b, c be nonnegative real numbers. Prove that

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge (ab+bc+ca)^3.$$

Solution. Observe that

$$4(a^2+ab+b^2)-3(a+b)^2=(a-b)^2\ge 0.$$

Multiplying the inequality $4(a^2+ab+b^2)\ge 3(a+b)^2$ with its cyclic counterparts, we obtain

$$64(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)\ge 27(a+b)^2(b+c)^2(c+a)^2.$$

To prove the desired inequality, it remains to show that

$$27(a+b)^2(b+c)^2(c+a)^2\ge 64(ab+bc+ca)^3.$$

But from the inequality $(a+b+c)^2\ge 3(ab+bc+ca)$, we need only show that

$$81(a+b)^2(b+c)^2(c+a)^2\ge 64(a+b+c)^2(ab+bc+ca)^2.$$

This inequality is equivalent to

$$9(a+b)(b+c)(c+a)\ge 8(a+b+c)(ab+bc+ca),$$

which reduces to

$$a(b-c)^2+b(c-a)^2+c(a-b)^2\ge 0.$$

Equality occurs if and only if $a=b=c=1$, or when two of the variables are equal to 0.

Example 1.19. Let x and y be nonnegative real numbers. Prove that

$$\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2}\ge\frac{1}{1+xy}.$$

Solution. Observe that

$$\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2}-\frac{1}{1+xy}$$

$$=\frac{xy(x^2+y^2)-x^2y^2-2xy+1}{(1+x)^2(1+y)^2(1+xy)}=\frac{xy(x-y)^2+(xy-1)^2}{(1+x)^2(1+y)^2(1+xy)}\ge 0.$$

The desired result follows.

Example 1.20. (Vasile Cîrtoaje) Let a,b,c,d be positive real numbers such that $abcd=1$. Prove the inequality

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}+\frac{1}{(1+d)^2}\ge 1.$$