Preface

This book is an unofficial sequel to the first two geometry books published by XYZ Press, namely 106 Geometry Problems from the AwesomeMath Summer Program and 107 Geometry Problems from the AwesomeMath Year Round Program. Assuming the background presented in these two books, 110 comes as a collection of problems designed for passionate geometers and students preparing for the difficult geometry questions of the IMO.

We know by now that the key to problem solving in geometry is mastering the basics and being able to apply them efficiently. Nonetheless, not all geometry problems can be solved by simply building up from nineteenth century geometry. Some results require deep insight into the configurations, as well as insight that goes well beyond the naive pursuit of common sense techniques. The book we have written is a collection of our favorite such problems.

The solutions and the commentaries that usually follow them represent however the heart of our work. Since a non-trivial portion of the problems were in fact proposed by us in various contests around the world, we often chose to reproduce the way we initially thought about them as the way to present the proofs. This is not only meant to give the reader a glimpse into how authors come up with their problems, but also to induce a structural way of thinking when it comes to intricate configuration debugging. It is also the way we came up with most of the solutions for the non-personal problems that you will encounter in this book.

We wish you a pleasant reading.

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Abbreviations and Notation

Notation of geometrical elements

$\angle BAC$	convex angle by vertex A
$\angle(p,q)$	directed angle between lines p and q
$\angle BAC \equiv \angle B'AC'$	angles BAC and $B'AC'$ coincide
AB	line through points A and B, distance between point A and B
$\overline{AB}$	directed segment from point A to point B
$X \in AB$	X lies on the line AB
$X = AC \cap BD$	X is the intersection of the lines AC and BD
$\triangle ABC$	triangle ABC
$[ABC]$	area of $\triangle ABC$
$[A_1 \dots A_n]$	area of polygon $A_1 \dots A_n$
(ABC)	circumcircle of triangle ABC
$(A_1 \dots A_n)$	area of cyclic polygon $A_1 \dots A_n$
$AB \| CD$	lines AB and CD are parallel
$AB \perp CD$	lines AB and CD are perpendicular
$p(X,\omega)$	power of point X with respect to circle ω
$\triangle ABC \cong \triangle DEF$	triangles ABC and DEF are congruent (in this order of vertices)
$\triangle ABC \sim \triangle DEF$	triangles ABC and DEF are similar (in this order of vertices)
$\mathcal{H}(H,k)$	homothety with center H and factor k
$\Psi(P,r^2)$	inversion with center P and power r^2
$\mathcal{S}(S,k,\varphi)$	spiral similarity with center S, dilation factor k, and angle of rotation φ

Notation of triangle elements

a, b, c	sides or side lengths of $\triangle ABC$
$\angle A, \angle B, \angle C$	angles by vertices A, B, and C of $\triangle ABC$
s	semiperimeter
x, y, z	expressions $\frac{1}{2}(b+c-a)$, $\frac{1}{2}(c+a-b)$, $\frac{1}{2}(a+b-c)$
r	inradius
R	circumradius
K	area
h_a, h_b, h_c	altitudes in $\triangle ABC$
m_a, m_b, m_c	medians in $\triangle ABC$
l_a, l_b, l_c	angle bisectors (segments) in $\triangle ABC$
r_a, r_b, r_c	exradii in $\triangle ABC$

Abbreviations

USAMO	United States of America Mathematical Olympiad
USAJMO	USA Junior Mathematical Olympiad
IMO	International Mathematical Olympiad
Romania IMO TST	Romania IMO Team Selection Test
USA IMO TST	United States of America IMO Team Selection Test
RMM	Romanian Masters of Mathematics

Contents

Problems

Problem 1. Given two nonintersecting chords AB and CD in a circle and a variable point P on the arc AB remote from points C and D, let E and F be the intersections of chords PC, AB, and of PD, AB, respectively. Prove that the value of

$$\frac{AE \cdot BF}{EF}$$

does not depend on the position of P.

Problem 2. Consider a triangle ABC with $a \leq b \leq c$. Denote by X, Y, Z the midpoints of the sides BC, CA, and AB, respectively. Let D, E, F be points on the sides BC, CA, AB satisfying the following two conditions:

(i) D is between X and C, E is between Y and C, and F is between Z and B.

(ii) $\angle CDE \leq \angle BDF$, $\angle CED \leq \angle AEF$, and $\angle BFD \leq \angle AFE$.

Prove that the perimeter of triangle DEF does not exceed the semiperimeter of triangle ABC.

Problem 3. Let A' be the foot of the internal angle bisector of the angle BAC of a given triangle ABC. Let P be an arbitrary point on the cevian AA', different from A', and denote by B', C' the intersections of the lines BP, CP, with the sidelines CA, and AB, respectively. If $BB' = CC'$, prove that $AB = AC$.

Problem 4. Let ABC be a triangle and let A', B', C' be the feet of the altitudes from A, B, C, respectively. Let A_1 be the foot of the perpendicular from A' to AB and let A_2 be the foot of the perpendicular from A' to AC. Furthermore, let Ω_A be the circle centered at vertex A having radius AA'. Analogously, define points B_1, B_2, C_1, C_2, and circles Ω_B and Ω_C. Prove that:

(a) Points A_1, A_2, B_1, B_2, C_1, C_2 are all concyclic.

(b) The center of the circle from (a) has equal powers with respect to Ω_A, Ω_B and Ω_C.

Problem 5. Let $\mathcal{C}$ be a circle and let P be a point in the exterior of $\mathcal{C}$. The tangents from P intersect the circle at A and B, respectively. Let M be the midpoint of segment AP and N the second intersection of the line BM with circle $\mathcal{C}$. Prove that $PN = 2MN$.

Problem 6. Let ℓ be a line in the exterior of a given circle $\rho(O)$. Let A be the foot of the perpendicular from O to ℓ, and let M be an arbitrary point on ρ. Furthermore, let X and Y be the second intersections of the circle with diameter AM with ρ and ℓ, respectively. Prove that the line XY passes through a point which is independent of the position of M.

Problem 7. Let X, Y, Z be the midpoints of the arcs BC, CA, AB, respectively, of triangle ABC containing the vertices of the triangle. Prove that the Simson lines of X, Y, Z with respect to ABC are concurrent.

Problem 8. Let ABC be a triangle with circumcenter O and let D, E, F be any three points lying on the sides BC, CA, AB, respectively. Let D', E', F' be the reflections of D, E, F with respect to the midpoints of BC, CA, AB, respectively. Prove that

(a) The Miquel points M of D, E, F and M' of D', E', F' are equidistant from the circumcenter of ABC.

(b) The centroids of triangles DEF and $D'E'F'$ are symmetric with respect to the centroid of ABC.

(c) Triangles DEF and $D'E'F'$ have the same area.

Problem 9. Let ABC be a triangle with $\angle BAC < \angle ACB$. Let D, E be points on the sides AC and AB, respectively, such that the angles ACB and BED are congruent. If F lies in the interior of the quadrilateral $BCDE$ such that the circumcircle of triangle BCF is tangent to the circumcircle of DEF and the circumcircle of BEF is tangent to the circumcircle of CDF, prove that the points A, C, E, F are concyclic.

Problem 10. Let γ be a circle and l a line in its plane. Let K be a point on l, located outside of γ. Let KA and KB be the tangents from K to γ, where A and B are distinct points on γ. Let P and Q be two points on γ. Lines PA and PB intersect line l in two points R and respectively S. Lines QR and QS intersect the second time circle γ in points C and D. Prove that the tangents from C and D to γ are concurrent on line l.

Problem 11. Given a scalene acute triangle ABC with $AC > BC$ let F be the foot of the altitude from C. Let P be a point on AB, different from A so that $AF = PF$. Let H, O, M be the orthocenter, circumcenter and midpoint of side AC. Let X be the intersection point of BC and HP. Let Y be the intersection point of OM and FX and let OF intersect AC at Z. Prove that F, M, Y, Z are concyclic.

Problem 12. Let ABC be an arbitrary triangle and let I be the incenter of ABC. Let D, E, F be the points on lines BC, CA, AB respectively such that $\angle BID = \angle CIE = \angle AIF = 90^\circ$, and define the following measurements: r_a, r_b, r_c are the exradii of triangle ABC, $[DEF]$ is the area of DEF, and $[ABC]$ is the area of ABC. Prove that

$$\frac{[DEF]}{[ABC]} = \frac{4r(r_a + r_b + r_c)}{(a+b+c)^2}.$$

Problem 13. Let a, b, and c be the lengths of the sides opposite vertices A, B, and C, respectively, in a nonobtuse triangle. Let h_a, h_b and h_c be the corresponding lengths of the altitudes. Show that

$$\left(\frac{h_a}{a}\right)^2 + \left(\frac{h_b}{b}\right)^2 + \left(\frac{h_c}{c}\right)^2 \geq \frac{9}{4},$$

and determine the cases of equality.

Problem 14. Let ABC be an acute-angled triangle with orthocenter H, and let W be a point on the side BC. Denote by M and N the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 which is diametrically opposite to W. Analogously, denote by ω_2 the circumcircle of triangle CWM, and let Y be the point on ω_2 which is diametrically opposite to W. Prove that X, Y and H are collinear.

Problem 15. Let ABC be a triangle and let P be a point in its interior. Let X, Y, Z be the intersections of AP, BP, CP with sides BC, CA, AB, respectively. Prove that

$$\frac{XB}{XY} \cdot \frac{YC}{YZ} \cdot \frac{ZA}{ZX} \leq \frac{R}{2r}.$$

Problem 16. Let $\mathcal{C}_1$ be a circle that is tangent to sides AB and AC of triangle ABC and let $\mathcal{C}_2$ be a circle through B and C that is tangent to $\mathcal{C}_1$ at point D. Prove that the incenter of triangle ABC lies on the internal angle bisector of $\angle BDC$.

Problem 17. Let ABC be a triangle and let P, Q be two points lying on its circumcircle. Prove that their Simson lines meet on the A-altitude of triangle ABC if and only if $PQ \| BC$.

Problem 18. Let ABC be a triangle and let P be a point in its interior with pedal triangle DEF. Suppose that that the lines DE and DF are perpendicular. Prove that the isogonal conjugate of P with respect to ABC is the orthocenter of triangle AEF.

Problem 19. Let τ be a line that is tangent to the circumcircle $\Gamma(O, R)$ of triangle ABC. Let I, I_a, I_b, I_c be the incenter and the three excenters of ABC and let $\delta(P)$ stands for the distance from point P to τ. Prove that there exists a choice of signs so that the following identity is true:

$$\pm\delta(I) \pm \delta(I_a) \pm \delta(I_b) \pm \delta(I_c) = 4R.$$

Problem 20. Prove that the nine-point circle of a triangle is tangent to the incircle and the excircles of the triangle.

Problem 21. In acute triangle ABC, $\angle A < \angle B$ and $\angle A < \angle C$. Let P be a variable point on side BC. Points D and E lie on sides AB and AC, respectively, such that $BP = PD$ and $CP = PE$. Prove that as P moves along side BC, the circumcircle of triangle ADE passes through a fixed point other than A.

Problem 22. Let ABC be a triangle with circumcenter O and let X, Y, Z be the circumcenters of triangles BOC, COA, AOB. Prove that the lines AX, BY, CZ are concurrent.

Problem 23. Let ABC be a triangle and let P be a point in its interior. Let the line passing through P, which is perpendicular to PA, intersect BC at A_1. Analogously, define B_1 and C_1. Prove that A_1, B_1, C_1 are collinear.

Problem 24. Given a quadrilateral $ABCD$ with $\angle B = \angle D = 90°$. Point M is chosen on segment AB so that $AD = AM$. Rays DM and CB intersect at point N. Points H and K are the feet of the perpendiculars from D and C to lines AC and AN, respectively. Prove that $\angle MHN = \angle MCK$.

Problem 25. Let P be a point in the plane of a triangle ABC and let Q be its isogonal conjugate with respect to ABC. Prove that

$$\frac{AP \cdot AQ}{AB \cdot AC} + \frac{BP \cdot BQ}{BA \cdot BC} + \frac{CP \cdot CQ}{CA \cdot CB} = 1.$$

Problem 26. In a triangle ABC, let A_1, B_1, C_1 be the points where the excircles touch the sides BC, CA and AB respectively. Prove that AA_1, BB_1 and CC_1 are the side lenghts of a triangle.

Problem 27. Prove that the three points of intersections of the adjacent internal angle trisectors of a triangle form an equilateral triangle.

Problem 28. Let Ω be the circumcircle of triangle ABC and let D be the tangency point of its incircle $\rho(I)$ with the side BC. Let ω be the circle internally tangent to Ω at T, and to BC at D. Prove that $\angle ATI = 90°$.

Problem 29. Let ABC be a triangle and let ℓ be a tangent to its incircle. Let x, y, z be the signed distances from A, B, C to ℓ, respectively. Prove that

$$ax + by + cz = 2[ABC],$$

where $[ABC]$ denotes the area of triangle ABC.

Problem 30. Let a, b, c, and x, y, z be the side lengths of two given triangles ABC, XYZ with areas $[ABC]$, and $[XYZ]$, respectively. Then,

$$a^2\left(y^2+z^2-x^2\right)+b^2\left(z^2+x^2-y^2\right)+c^2\left(x^2+y^2-z^2\right) \geq 16[ABC][XYZ],$$

with equality if and only if the triangles ABC and XYZ are similar.

Problem 31. Diagonals of a cyclic quadrilateral $ABCD$ intersect at point K. The midpoints of segments AC and BD are M and N, respectively. The circumcircles of triangles ADM and BCM intersect at points M and L. Prove that K, L, M and N all lie on a circle.

Problem 32. Let ABC be a triangle with incenter I, incircle γ and circumcircle Γ. Let M, N, P be the midpoints of sides BC, CA, AB and let E, F be the tangency points of γ with CA and AB, respectively. Let U, V be the intersections of EF with MN and MP, respectively, and let X be the midpoint of arc BAC of Γ. Prove that XI bisects UV.

Problem 33. Let ABC be a triangle and let M, N, P be the midpoints of sides BC, CA, AB, respectively. Denote by X, Y, Z the midpoints of the altitudes emerging from the vertices A, B, C, respectively. Prove that the radical center of the circles AMX, BNY, CPZ is the nine-point center of triangle ABC.

Problem 34. Let $A_1A_2A_3A_4$ be a quadrilateral with no pair of parallel sides. For each $i = 1, 2, 3, 4$, define ω_1 to be the circle touching the quadrilateral externally, and which is tangent to the lines $A_{i-1}A_i, A_iA_{i+1}$, and $A_{i+1}A_{i+2}$ (indices are considered modulo 4 so $A_4 = A_0, A_5 = A_1$, and $A_6 = A_2$). Let T_i be the point of tangency of ω_i with the side A_iA_{i+1}. Prove that the lines A_1A_2, A_3A_4, and T_2T_4 are concurrent if and only if the lines A_2A_3, A_4A_1, and T_1T_3 are concurrent.

Problem 35. Let P be a point inside an arbitrary triangle ABC. Prove that

$$\frac{1}{PA} + \frac{1}{PB} + \frac{1}{PC} \geq \frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c},$$

where R_a, R_b, R_c are the circumradii of triangles PBC, PCA, PAB, respectively.

Problem 36. Through the vertex A of a triangle ABC, a straight line AD is drawn, cutting the side BC at D. Let I be the incenter of triangle ABC, let P be the center of the circle which touches DC, DA, and the circumcircle of ABC internally, and let Q be the center of the circle which touches DB, DA, and the circumcircle of ABC internally. Prove that P, I, Q are collinear.

Problem 37. Let ABC be a triangle with circumcircle Γ and let A' be a point on the side BC. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the circles tangent simultaneously to AA', BA', Γ and AA', CA', Γ, respectively. Prove that

(a) If A' is the foot of the internal angle bisector from A, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are tangent to each other at the incenter of triangle ABC.

(b) If A' is the tangency point of the A-excircle with BC, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are congruent.

Problem 38. Suppose M and N are distinct points in the plane of triangle ABC such that $AM : BM : CM = AN : BN : CN$. Prove that the line MN contains the circumcenter of triangle ABC.

Problem 39. The points M, N, P are chosen on the sides BC, CA, AB of a triangle ABC, such that the triangle MNP is acute-angled. We denote with x the length of the shortest altitude of the triangle ABC, and with y the length of the longest altitude of the triangle MNP. Prove that $x \leq 2y$.

Problem 40. Let ABC be a nonisosceles triangle, for which denote by X, Y, Z the tangency points of its incircle with the sides BC, CA and AB, respectively. Let D be the intersection of OI with the sideline BC, where O, I are the circumcenter and incenter, respectively. The line perpendicular to YZ through X cuts AD at E. Prove that the line YZ is the perpendicular bisector of EX.

Problem 41. Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

Problem 42. Given triangle ABC, its centroid G, and its incenter I, construct, using only an unmarked straightedge, its orthocenter H.

Problem 43. Let ABC be a scalene triangle and let Ω be a circle that intersects the side BC at A_1, A_2, the side CA at B_1, B_2, and the side AB at C_1, C_2. Let the tangents at A_1 and A_2 with respect to Ω meet at P, and similarly define R and S. Prove that lines AP, BQ, CR are concurrent.

Problem 44. Let the sides AD and BC of the quadrilateral $ABCD$ (such that AB is not parallel to CD) intersect at point P. Points O_1 and O_2 are circumcenters and points H_1 and H_2 are orthocenters of triangles ABP and CDP, respectively. Denote the midpoints of segments O_1H_1 and O_2H_2 by E_1 and E_2, respectively. Prove that the perpendicular from E_1 on CD, the perpendicular from E_2 on AB and the lines H_1H_2 are concurrent.

Problem 45. Let ABC be an acute-angled triangle with circumcircle $\Gamma(O)$ and let ℓ be a line in the plane which intersects the lines BC, CA, AB at X, Y, Z, respectively. Let ℓ_A, ℓ_B, ℓ_C be the reflections of ℓ across BC, CA, AB, respectively. Furthermore, let M be the Miquel point of triangle ABC with respect to line ℓ.

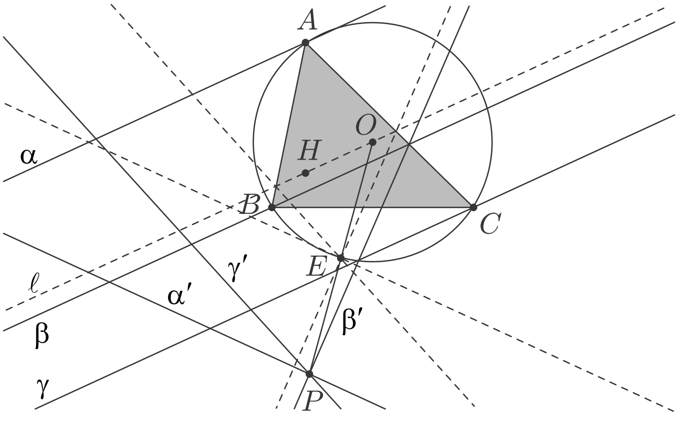
(a) Prove that lines ℓ_A, ℓ_B, ℓ_C determine a triangle whose incenter lies on the circumcircle of triangle ABC.

(b) If S is the incenter from (a) and O_a, O_b, O_c denote the circumcenters of triangles AYZ, BZX, CXY, respectively, prove that the circumcircles of triangles SOO_a, SOO_b, SOO_c are concurrent at a second point, which lies on Γ.

Problem 46. Let $ABCDEF$ be a convex hexagon, all of whose sides are tangent to a circle ω with center O. Suppose that the circumcircle of triangle ACE is concentric with ω. Let J be the foot of the perpendicular from B to CD. Suppose that the perpendicular from B to DF intersects the line EO at a point K. Let L be the foot of the perpendicular from K to DE. Prove that $DJ = DL$.

Problem 47. A non-isosceles acute triangle ABC is given. Let O, I, H be the circumcenter, the incenter, and the orthocenter of the triangle ABC, respectively. Prove that $\angle OIH > 135^\circ$.

Problem 48. Let ABC be a triangle with circumcenter O and orthocenter H. Parallel lines α, β, γ are drawn through the vertices A, B, C, respectively. Let α', β', γ' be the reflections of α, β, γ in the sides BC, CA, AB, respectively. Prove that these reflections are concurrent if and only if α, β, γ are parallel to the Euler line OH of triangle ABC.



Problem 49. Let ABC be a triangle with circumcircle Γ and nine-point circle γ. Let X be a point on γ, and let Y, Z be on Γ so that the midpoints of segments XY and XZ are on γ.

(a) Prove that the midpoint of YZ is on γ.

(b) Find the locus of the symmedian point of triangle XYZ, as X moves along Γ.

Problem 50. Let ABC be a triangle with circumcircle ω and ℓ a line not intersecting ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove the circumcircles of triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.

Problem 51. Let ABC be an acute-angled triangle and let H be its orthocenter. Let t_a, t_b, t_c be the inradii of triangles HBC, HCA, and HAB, respectively. Prove that

$$t_a + t_b + t_c \leq (6\sqrt{3} - 9)r,$$

where r is the inradius of triangle ABC.

Problem 52. Let ABC be an acute angled triangle satisfying the conditions $AB > BC$ and $AC > BC$. Denote by O and H the circumcenter and orthocenter, respectively, of the triangle ABC. Suppose that the circumcircle of the triangle AHC intersects the line AB at M different from A, and the circumcircle of the triangle AHB intersects the line AC at N different from A. Prove that the circumcenter of the triangle MNH lies on the line OH.

Problem 53. Let $ABCD$ be a rectangle and let ω be an arbitrary circle passing through vertices A and C. Let Γ_1, Γ_2 be the circles inside $ABCD$ so that Γ_1 is tangent to AB, BC, and Γ, and Γ_2 is tangent to CD, DA, and Γ. Prove that the sum of the radii of Γ_1 and Γ_2 is independent of the choice of ω.

Problem 54. Given is an acute-angled triangle ABC and the points A_1, B_1, C_1, which are the feet of its altitudes from A, B, C, respectively. A circle passes through A_1 and B_1 and touches the smaller arc AB of the circumcircle of ABC at point C_2. Points A_2 and B_2 are defined analogously. Prove that the lines A_1A_2, B_1B_2, C_1C_2 have a common point, which lies on the Euler line of triangle ABC.

Problem 55. Let ABC be a triangle with nine-point circle ω. Prove that

(a) There exist precisely three points on the circumcircle of triangle ABC whose Simson lines with respect to the triangle ABC are tangent to ω.

(b) The triangle formed by the three points from (a) is equilateral.

Problem 56. Let ABC be an acute triangle with circumcircle ω. Let ℓ be a tangent line to ω. Let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle ω.

Problem 57. Let M be a point inside triangle ABC with circumcircle (O). Let A_1, B_1, C_1 be the intersections of AM, BM, CM with (O) which are different from the vertices of the triangle. Furthermore, let A_2, B_2, C_2 be the reflections of A_1, B_1, C_1 across the sidelines BC, CA, and AB, respectively. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are similar.

Problem 58. Let P be a given point inside the triangle ABC. Suppose L, M, N are the midpoints of BC, CA, AB, respectively, and

$$PL : PM : PN = BC : CA : AB.$$

The extensions of AP, BP, CP meet again the circumcircle of ABC at points D, E, F, respectively. Prove that the circumcenters of triangles PBF, PCE, PCD, PAF, PAE, PBD are concyclic.

Problem 59. Let ABC be an acute-angled triangle and let τ be the inradius of its orthic triangle. Prove that

$$r \geq \sqrt{R\tau},$$

where r and R are the inradius and circumradius of triangle ABC, respectively.

Problem 60. Let ABC be a fixed triangle, and let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. Let P be a variable point on the circumcircle. Let lines PA_1, PB_1, PC_1 meet the circumcircle again at A', B', C', respectively. Assume that the points A, B, C, A', B', C' are distinct, and lines AA', BB', CC' form a triangle. Prove that the area of this triangle does not depend on P.

Problem 61. Let P be a point in the plane of a triangle ABC. The lines AP, BP, CP intersect the lines BC, CA, AB at the points A', B', C'. Let Q be the isogonal conjugate of the point P with respect to triangle ABC. Prove that the reflections of the lines AQ, BQ, CQ in the lines $B'C'$, $C'A'$, $A'B'$ are concurrent.

Problem 62. Let triangle ABC be a triangle with circumcircle (O) and incircle (I). Let D, E, F be the tangency points of (I) with BC, CA, AB, respectively. The line EF intersects the circumcircle (O) at X_1, X_2. Similarly, we define Y_1, Y_2 and Z_1, Z_2. Prove that the radical center of the circumcircles of triangles DX_1X_2, EY_1Y_2, FZ_1Z_2 is the orthocenter of triangle DEF.

Problem 63. Let M be an arbitrary point on the circumcircle of triangle ABC and let the tangents from this point to the incircle of the triangle meet the sideline BC at X_1 and X_2. Prove that the second intersection of the circumcircle of triangle MX_1X_2 with the circumcircle of ABC coincides with the tangency point of the circumcircle with the A-mixtilinear incircle.

Problem 64. Let A_1, B_1, C_1 be points on the sides BC, CA, AB of triangle ABC such that the lines AA_1, BB_1, CC_1 are concurrent. Construct three circles Γ_1, Γ_2, and Γ_3 outside the triangle such that they are tangent to the sides of ABC at A_1, B_1, and C_1, respectively, and also tangent to the circumcircle of ABC. Prove that the circle tangent externally to these three circles is also tangent to the incircle of triangle ABC.

Problem 65. Let ABC be a triangle and let P be a point in its plane. Let D, E, F be three points on the lines through P perpendicular to the lines BC, CA, and AB, respectively. Prove that if triangle DEF is equilateral and if P lies on the Euler line of ABC, then the centroid of DEF also lies on the Euler line of ABC.

Problem 66. Let $ABCD$ be a cyclic quadrilateral with circumcircle Γ and let E be an arbitrary point on the side AB. Let DE meet BC at F, DE meet again Γ at P, BP meet AF at Q, and QE meet CD at V. Prove that point V is independent of the position of E.

Problem 67. Let ABC be a triangle and let I and O denote its incenter and circumcenter, respectively. Let ω_A be the circle through B and C that is tangent to the incircle of the triangle ABC; the circles ω_B and ω_C are defined similarly. The circles ω_B and ω_C meet at a point A' distinct from A; the points B' and C' are defined similarly. Prove that the lines AA', BB' and CC' are concurrent at a point on the line IO.

Problem 68. Let ABC be a triangle with circumcircle Ω. Points X and Y are on Ω such that XY meets AB and AC at D and E, respectively. Prove that the midpoints of segments XY, BE, CD, and DE are concyclic.

Problem 69. If two perpendicular straight lines are drawn through the orthocenter of a triangle, they intercept a segment on each of the sidelines. Prove that the midpoints of these three segments are collinear.

Problem 70. Suppose that ABC is an equilateral triangle and that P is a point in the plane of ABC. The perpendicular from P to BC meets AB at X, the perpendicular from P to CA meets BC at Y, and the perpendicular from P to AB meets CA at Z.

(a) If P is in the interior of triangle ABC, prove that the area of XYZ is not greater than the area of ABC.

(b) If P lies on the circumcircle of ABC, prove that X, Y, and Z are collinear.

Problem 71. A triangle ABC is inscribed in a circle ω. A variable line ℓ chosen parallel to BC meets segments AB, AC at points D, E, respectively, and meets ω at points K, L (where D lies between K and E). Circle γ_1 is tangent to the segments KD and BD and also tangent to ω, while circle γ_2 is tangent to the segments LE and CE and also tangent to ω. Determine the locus, as ℓ varies, of the meeting point of the common inner tangents to γ_1 and γ_2.

Problem 72. Let $ABCD$ be a cyclic quadrilateral with circumcircle Γ. Let E be the intersection of AB and CD, F the intersection of AD and BC, and

M, N the midpoints of AC and BD, respectively. Prove that EF is tangent to the circumcircle of triangle MNF.

Problem 73. Consider triangle ABC, and three squares $BCDE$, $CAFG$ and $ABHI$ constructed on its sides, outside the triangle. Let XYZ be the triangle enclosed by the lines EF, DI and GH. Prove that $[XYZ] \leq (4-2\sqrt{3})[ABC]$.

Problem 74. A triangle is divided by its three medians into six smaller triangles. Show that the circumcenters of these smaller triangles lie on a circle.

Problem 75. Let H be the orthocenter of an acute triangle ABC with circumcircle Γ. Let P be a point on the arc AB (not containing C) of Γ, and let M be a point on the arc CA (not containing B) of Γ such that H lies on the segment PM. Let K be another point on Γ such that KM is parallel to the Simson line of P with respect to triangle ABC. Let Q be another point on Γ such that $PQ \parallel AB$. Segments AB and KQ intersect at a point J. Prove that triangle KJM is an isosceles triangle.

Problem 76. Let ABC be a triangle and let D, E, F be the tangency points of the incircle with BC, CA, AB, respectively. Let EF meet the circumcircle Γ of ABC at X and Y. Furthermore, let T be the second intersection of the circumcircle of DXY with the incircle. Prove that AT passes through the tangency point A' of the A-mixtilinear incircle with Γ.

Problem 77. Let ABC be a triangle with incircle γ and circumcircle Γ. Let Ω be the circle tangent to the rays AB, AC and to Γ externally, and let A' be the tangency point of Ω with Γ. Furthermore, draw the tangents from A' with respect to incircle γ and let B', C' be their other intersections with Γ. If X denotes the tangency point of the chord $B'C'$ with γ, prove that the circumcircle of triangle BXC is tangent to γ.

Problem 78. In triangle ABC, let D, E, F be the feet of the altitudes from A, B, and C, respectively. Let H be the orthocenter of triangle ABC and let I_1, I_2, I_3 be the incenters of triangles EHF, FHD, and DHE, respectively. Prove that the lines AI_1, BI_2, CI_3 are concurrent.

Problem 79. Let ABC be a triangle with incircle Γ and let D, E, F be the tangency points of Γ with BC, CA, AB, respectively. Let M, N, P be the midpoints of BC, CA, AB, and let X, Y, Z be points on the lines AI, BI, CI, respectively. Prove that the lines XD, YE, ZF are concurrent if and only if the lines XM, YN, ZP are concurrent.

Problem 80. Consider a convex quadrilateral and the incircles of the triangles determined by one of its diagonals. Prove that the tangency points of the incircles with the diagonal are symmetric with respect to the midpoint of the diagonal if and only if the diagonals and the line through the incenters are concurrent.