1 Foreword

Collaboration between friends is always a worthwhile pursuit. Helen Keller once said, “Alone we can do so little; together we can do so much” and the famous mathematician, Paul Erdös, said, “Another roof, another proof,” whenever he spent time working on projects at a colleague’s house. Being a lifelong educator with the goal of significantly more than conventional teaching, I started a mathematics class for a small group of intellectually curious young students. I wanted to do something unconventional with this group. My goal was to instill not only the value of problem-solving but also that of collaboration and a drive to learn more than what one might be comfortable with so that my students could reach their full potential. As their skills and knowledge grew, they challenged each other with problems and spent much time reviewing, dissecting, and exploring meaningful mathematics. From these roots, their collaboration grew and each brought his own unique approach to mathematics. Out of common desire, dedication, and hard work, *111 Problems in Algebra and Number Theory* by Adrian Andreescu and Vinjai Vale was born. The book, which is a thorough guide for students seeking to expand their mathematical horizons, delves into the fundamentals of algebra and number theory. Through comprehensive theory, well-placed examples, and illustrative problems, the reader learns the value and methodology of problem-solving to prepare students for a broader understanding of mathematics and success on contests such as the USAJMO and beyond. Under my guidance, I saw this book develop through the combined efforts of these two committed students. Their mathematical and personal maturity also grew tremendously while completing this ambitious project. I will always see this book as a culmination of my students’ strenuous efforts and an affirmation of my passion for mathematics and teaching, which I see in Adrian and Vinjai as they give back to the mathematical community.

Dr. Titu Andreescu, March 2016

2 Preface

Algebra and number theory are closely intertwined areas of mathematics. In this book, we explore the fundamentals of junior Olympiad-level algebra and number theory starting from the ground up, with numerous instructive examples positioned along the way. The book begins with the study of inequalities. We then transition to quadratics and polynomials of higher degree, and afterwards present a collection of valuable algebraic techniques that come up time and time again in mathematics competitions. Following is a medley of problems roughly in difficulty order, designed to reinforce the concepts discussed in the theoretical sections. Hints also accompany selected problems. For each problem we provide a complete solution, and for some we present more than one.

In the second half of the book, we discuss some of the foundation of number theory from an algebraic perspective, beginning with divisibility and modular arithmetic. Various other topics, including the standard number theory concepts on junior-level Olympiads, are also covered. Another section of problems follows, this time focused on number theory. Finally, the book ends with a list of all the hints that were included with the problems, in randomized order so the reader need not worry about accidentally spotting a clue for the next problem.

We offer our sincerest thanks to Dr. Titu Andreescu for his generous donation of problems and helpful guidance. Without him, this book would not have been possible. His countless hours of mentoring and insightful advice have greatly shaped this book. We would also like to thank Drs. Gabriel Dospinescu and Richard Stong for proofreading our manuscript and offering many invaluable suggestions. Their thorough remarks have significantly helped elevate the quality of this book. And finally, big thanks are due to our parents for their unfailing support throughout this project, from inception to publication.

Enjoy the problems!

3 How to use this book

Mathematical Olympiads allow contestants many hours to tackle a handful of problems. Hence the problems themselves are of a unique nature; they are meant to challenge one's ability to not only connect known concepts, but also employ creative solutions to demanding problems that stump even the best.

We strongly recommend that the reader takes their time to work on the given problems, including the examples. Most of the time one must attack a problem from several different angles before substantial progress is made as some attempts turn out to be futile. The reader is also encouraged to thoroughly read all provided solutions, including those to problems that he or she has already solved. Every solution is designed to be instructive. Many contain key insights, motivation, and possible false starts. Others focus less on the process of solving, but are good examples for how one should present a solution on an actual contest. We also encourage readers to come back to a problem after reading the solution and attempt to find an alternate path, as most Olympiad problems can be solved in several ways. Ultimately, we hope that the reader becomes more comfortable with the art of solving Olympiad problems and crafting insightful and rigorous solutions.

3.1 Hints

Many of the problems have accompanying hints. Generally, the hint(s) are meant to give the reader a gentle nudge in the right direction. If a problem has only one hint, it is usually such an indication. However, if it has two or three hints, the first one is usually a nudge while the second and third may present more direct advice. We strongly recommend spending some time trying the problem after reading the first hint, before moving on to the second or the third hint. **Only read the hints after wholeheartedly trying the problem alone first!** Also, keep in mind that there are often various possible methods to solve a problem, and the hints may only follow one such track.

3.2 Notation

We occasionally use the notation LHS and RHS to refer to the left-hand side and right-hand side, respectively, of an equality or inequality. Also, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are used to denote the set of natural, integer, rational, real, and complex numbers, respectively. Other notations, such as $\deg(P)$ for the degree of a polynomial or $\phi(n)$ for Euler's totient function, are defined in the text. The reader might also encounter some new mathematical terminology throughout the book; for example, the word "pairwise" (meaning "when taken in pairs"). For instance, saying that quantites a, b, c are "pairwise distinct" means that $a \neq b$, $b \neq c$, and $c \neq a$.

Contents

Algebra plays a fundamental role not only in mathematics, but also in various other scientific fields. Without algebra there would be no uniform language to express concepts such as arithmetic or a number's properties. Thus, it is important to be well-versed in this domain in order to excel in other mathematical disciplines such as number theory, combinatorics, and even geometry. In this chapter, we will cover algebra as its own branch of mathematics and discuss important techniques that are also applicable in many Olympiad problems.

Chapter 1

Inequalities

The field of inequalities is filled with theory and many beautiful problems. At the Olympiad level, it is sufficient to build a (relatively) basic arsenal of tools and techniques. Inequalities can have "bashy" proofs as well as slick and elegant solutions – this chapter will focus on the latter and will particularly strive to provide the motivation behind these proofs.

We begin by discussing the basics, building our intuition from the ground up. For those who are already somewhat comfortable with inequalities, this section might be a review; yet it is always worthwhile to ensure that all the key concepts are consolidated. We then move on to the classic trio of the Arithmetic Mean – Geometric Mean inequality, the Rearrangement inequality, and the Cauchy-Schwarz inequality. Titu's lemma, a corollary of the Cauchy-Schwarz inequality, is also discussed in detail. Finally we conclude with the less common yet powerful inequalities of Hölder and Schur, both of which have numerous applications in tough Olympiad problems.

1 Basics

Basic algebra deals with fixed results about unknowns – such as $x = 2$, or $a^3 + 3a = 14$. However, what if we only know some information about the result as opposed to its exact value? For example, we might know that it lies in some subset of the real numbers. The easiest subsets to describe are half-lines or intervals, such as $x > 2$ or $1 \le a^3 + 3a \le 14$. Here begins the study of inequalities.

The goal of this section is to build up the intuition behind inequalities from scratch. The reader with some prior experience in inequalities may skip this section.

Example 1.1. Given that $2x + 3 \ge 17$, what can we say about x?

Solution 1.1. If $2x + 3$ is at least 17, we must have that $2x$ is at least 14. That means that x is at least 7. ∎

Example 1.2. Find the minimum possible value of $9x + 4$ if $x \ge 2$.

Solution 1.2. Since $x \ge 2$, we know that $9x \ge 18$, so $9x + 4 \ge 22$. And when $x = 2$, we have $9x + 4 = 22$, so that means the minimum attainable value of $9x + 4$ given $x \ge 2$ is indeed 22. ∎

Remark. Note that $9x + 4 \ge 22$ alone does not imply that 22 is the minimum possible value of $9x + 4$. We need to know that 22 is attainable. In general, proving that a bound is attainable is a vital component of a valid proof.

We see in the last couple of examples that we can do many things with an inequality that we can also do with an equality. For example, we can add or subtract any quantity from both sides, and the inequality will still hold. This is rather intuitive. However, multiplying and dividing are not as straightforward, as we will see in the next problem.

Example 1.3. If two real numbers a and b satisfy $a > b$, then what can we say about $-a$ and $-b$?

Solution 1.3. We might be tempted to say that $-a > -b$, by multiplying both sides by -1. But consider the following argument: subtracting a from both sides yields $0 > b - a$, and subtracting b yields $-b > -a$, which is equivalent to $-a < -b$. Which of these two arguments is correct?

We know for sure that the second one is right, because we have already established that it is perfectly okay to add and subtract quantities from both sides of an inequality. That means that something must be wrong with multiplying by -1. If we consider the example $a = 3$ and $b = 2$, we see that we do indeed have $-3 < -2$ and not $-3 > -2$.

Intuitively, we can think about a number line: we have two points, a and b, and we reflect the entire line across the zero point to get a and b to $-a$ and $-b$, respectively. Then the order of $-a$ and $-b$ will be the opposite of the order of a and b. That means that whenever we multiply by -1, or any negative number for that matter, we must remember to flip the sign. That also means that when we are multiplying by a quantity that could be positive or negative, we need to make sure that we consider both cases separately, as they may require casework due to the potential flip of the sign. The same goes for division, which is essentially multiplying both sides by a number's reciprocal. ■

Example 1.4. Prove that if $a > b > 0$, then $\dfrac{1}{a} < \dfrac{1}{b}$. Why is the condition that a and b are positive necessary?

Solution 1.4. Since $a > b > 0$, $ab > 0$, so we can divide both sides of $a > b$ by ab to get $\frac{1}{b} > \frac{1}{a}$ as desired. The positive condition is important, since we can only divide by ab in this way if $ab > 0$. Otherwise, we can find a counterexample (e.g. $2 > -2$, but it is not true that $\frac{1}{2} < -\frac{1}{2}$). ■

So far we have discussed addition, subtraction, multiplication, and division of a quantity to both sides of an inequality. But there are also other operations we can perform with equalities – such as adding, subtracting, multiplying, or dividing two of them.

Example 1.5. If we have two inequalities, what must be true about them in order for us to successfully add them? What about subtracting them?

Solution 1.5. Looking at some examples, it soon becomes clear that in order to add two inequalities and have the result still hold, they must have the same sign. If $a > b$ and $c > d$, that means that $a + c > b + d$ is definitely true, but we can't say that $a > b$ and $d < c$ implies $a + d > b + c$ – this simply is not always true.

Subtraction is very similar, since it is basically just adding the opposite. So the signs must be opposite – hence we can subtract $a > b$ and $d < c$

to get $a-d > b-c$. (Note this is in fact the same inequality as $a+c > b+d$, except we have moved c and d to the other side.) ■

Example 1.6. What about multiplication and division?

Solution 1.6. If both sides of all the inequalities in question are positive, then we need them to be aligned in the same way. For instance, we can safely multiply $3 > 2$ and $7 > 4$ to get $21 > 8$. But if we attempt to multiply $10 > 3$ and $-2 > -3$, we get $-20 > -9$, which is false. Hence it is safe to multiply two inequalities after making everything positive. Division is similar, because dividing out $a > b$ is the same as multiplying by $\frac{1}{a} < \frac{1}{b}$, by Example 1.4. ■

Remark. There are some other cases – such as when one inequality is $a > b$ for $a > 0$, $b < 0$. Then we can safely multiply $c > d$ for any positive c, d to get $ac > bd$. These other cases are left as an exercise for the curious reader to explore – but the message still stands that one must be extremely careful when multiplying or dividing inequalties.

Now we explore some inequalities that hold without any restrictions on the variables involved.

Theorem. *(Trivial inequality)* Every real number x satisfies $x^2 \geq 0$.

Proof. If x is positive, then $x \cdot x$ is the product of two positive numbers and must also be positive. If x is negative, then $x \cdot x$ is the product of two negative numbers and must be positive. If x is zero, then $x \cdot x$ is also zero. In all cases, we have $x^2 \geq 0$. ■

The Trivial inequality is fundamentally very interesting. In particular, it holds for all real numbers x, without any other conditions. Also, we have already seen several examples of linear inequalities ($x \geq y$, $2x - 3 \geq 5$, and the like), and the Trivial inequality tells us that even the simplest quadratic function is already interesting.

Example 1.7. Prove that for all real numbers a and b, $(a+b)^2 \geq 4ab$.

Solution 1.7. If we expand $(a+b)^2$, we get $a^2 + 2ab + b^2 \geq 4ab$. That is equivalent to $a^2 - 2ab + b^2 \geq 0$, which is $(a-b)^2 \geq 0$. This is clearly true due to the Trivial inequality, so the original inequality must be true as well. ■

Remark. A very useful related result is that for all real numbers a and b, $a^2 + b^2 \geq 2ab$.

Example 1.8. Prove that for any real numbers a and b, $a^2 + 4b^2 \geq 4ab$.

Solution 1.8. If we move the $4ab$ term over to the left-hand side, we see that we can factor, obtaining $(a - 2b)^2 \geq 0$. This is true by the Trivial inequality, so the original desired inequality must also be true. (Alternatively, we could see that this follows from $x^2 + y^2 \geq 2xy$ as $a^2 + (2b)^2 \geq 2a(2b)$.) ■

Example 1.9. Calculate $2^2 + 3^2 + 6^2$ and $2 \cdot 3 + 3 \cdot 6 + 6 \cdot 2$. What about $5^2 + 4^2 + 1^2$ and $5 \cdot 4 + 4 \cdot 1 + 1 \cdot 5$? In general, which is larger: $a^2 + b^2 + c^2$ or $ab + bc + ca$? Why?

Solution 1.9. By simple computation, we obtain $2^2+3^2+6^2 = 4+9+36 = 49$ and $2 \cdot 3 + 3 \cdot 6 + 6 \cdot 2 = 6 + 18 + 12 = 36$. Also, $5^2 + 4^2 + 1^2 = 25 + 16 + 1 = 42$ and $5 \cdot 4 + 4 \cdot 1 + 1 \cdot 5 = 20 + 4 + 5 = 29$. It certainly seems like $a^2 + b^2 + c^2$ is larger than $ab + bc + ca$ in general – now we must prove it.

We don't know much about the sum of three squares, but we do know about the sum of two squares: $a^2 + b^2 \geq 2ab$. This doesn't really help too much in its current form, because it excludes the c^2 on the left-hand side. To fix that, we consider the two similar inequalities $b^2 + c^2 \geq 2bc$, $c^2 + a^2 \geq 2ca$. Then we add all three, to get $2a^2 + 2b^2 + 2c^2 \geq 2ab + 2bc + 2ca$. Dividing both sides by 2 yields the desired. ■

Remark. Note that equality (that is, $a^2 + b^2 + c^2 = ab + bc + ca$) holds if and only if $a = b = c$, since the three inequalities we added reach equality when $a = b$, $b = c$, and $c = a$.

Example 1.10. Prove that for all real numbers a, b, c,

$$3(a^2 + b^2 + c^2) \geq (a + b + c)^2 \geq 3(ab + bc + ca).$$

Solution 1.10. First we prove that $3(a^2 + b^2 + c^2) \geq (a + b + c)^2$. Expanding both sides gives the equivalent form

$$3a^2 + 3b^2 + 3c^2 \geq a^2 + b^2 + c^2 + 2ab + 2bc + 2ca.$$

This can be reduced to $a^2 + b^2 + c^2 \geq ab + bc + ca$, which we just proved, so the first part is complete. Now for $(a + b + c)^2 \geq 3(ab + bc + ca)$. Once again, we expand:

$$a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \geq 3ab + 3bc + 3ca.$$

As before, this reduces to $a^2 + b^2 + c^2 \geq ab + bc + ca$, so we are done. ■

Example 1.11. Prove that for all positive real numbers a, b, c, we have $a^2(b+c) + b^2(c+a) + c^2(a+b) \geq 6abc$.

Solution 1.11. Note that we can rearrange the left-hand side to $a(b^2+c^2) + b(c^2 + a^2) + c(a^2 + b^2)$. Then we apply the inequality $x^2 + y^2 \geq 2xy$ to get

$$a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2) \geq 2abc + 2abc + 2abc = 6abc. \blacksquare$$

Example 1.12. Prove that for all positive real numbers a, b, c,

$$(a + b)(b + c)(c + a) \geq 8abc.$$

Solution 1.12. *First Solution*: We expand the left-hand side and obtain the equivalent inequality:

$$2abc + a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 \geq 8abc.$$

Then we apply the previous example and we reach the desired conclusion. ■

Second Solution: Let $x = \sqrt{a}$, $y = \sqrt{b}$, $z = \sqrt{c}$. Then the inequality is equivalent to

$$(x^2 + y^2)(y^2 + z^2)(z^2 + x^2) \geq 8x^2y^2z^2,$$

which follows from multiplying the three inequalities $x^2 + y^2 \geq 2xy$, $y^2 + z^2 \geq 2yz$, $z^2 + x^2 \geq 2zx$. (We can perform this multiplication freely because both sides of all three inequalities are positive.) ■

Example 1.13. Prove that for all real numbers $a_1, a_2, \ldots, a_n$, we have

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq a_1a_2 + a_2a_3 + \cdots + a_na_1.$$

Solution 1.13. This inequality is very similar to $a^2 + b^2 + c^2 \geq ab + bc + ca$, which we saw in Example 1.9. It follows directly by doubling and then rewriting it as

$$(a_1 - a_2)^2 + (a_2 - a_3)^2 + \cdots + (a_n - a_1)^2 \geq 0. \blacksquare$$