Preface

Although sometimes overlooked in their importance, geometric inequalities are a sharp tool for solving many problems in geometry. They often lead to swift and elegant solutions as algebraic and trigonometric machinery is frequently employed. For the curious reader looking to sharpen their arsenal of mathematical strategies on the Olympiad level, geometric inequalities is a valuable addition. This problem-solving methodology prompts key ideas in other domains such as calculus or complex numbers as the solutions are usually nonstandard in a geometric sense. Nevertheless, trying your hand at these types of inequalities consolidates your math background and geometric reasoning while exposing you to a broad range of problems, all teeming with insightful inequality-type solutions.

The book is organized in a straightforward manner, first starting with the basic geometric principles that come up time and time again, laying the foundation for the essential theorems discussed and necessary for assimilating the harder concepts that follow later. The second chapter is centered around algebraic routines, methods for decomposing geometry problems in their algebraic counterparts. Introductory and advanced problems succeed the theory as a means to reinforce the concepts presented. Every problem has solutions and meaningful discussion about the intuition and development of them from many points of view, not only geometric ones. Numerous problems are presented with more than one solution so that the reader can better grasp the scope and versatility of geometric inequalities. We hope that the expansive variety of the geometry problems and rich theory help the reader in developing a better grasp of the efficacy of geometric inequalities and the typical practices utilized in their solutions.

We would like to thank Chris Jeuell, who thoroughly revised the manuscript, fixing many errors and improving the explanations.

Enjoy the problems!

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Chapter 1

Basic Concepts

In this chapter we consider several examples of geometric inequalities which can be proven using the triangle inequality and its generalization for broken lines as well as some basic examples of area inequalities.

1.1 The Triangle Inequality

Recall that the triangle inequality says that for every three points A, B, C, we have the inequality

$$AB + BC \geq CA.$$

We will also use its vector form, given by

$$|\overrightarrow{AB}| + |\overrightarrow{BC}| \geq |\overrightarrow{AB} + \overrightarrow{BC}|.$$

Note that in both inequalities, equality is attained only if B is a point on line segment AC.

Example 1.1. Let M be a point inside triangle ABC. Prove that:

(a) $MA + MB < CA + CB$;

(b) $MA + MB + MC < \max(AB + BC, BC + CA, CA + AB)$.

Solution. (a) Let N be intersection of lines AM and BC (Fig. 1.1). Then by the triangle inequality, we have $BM < MN + NB$ and $AN < CA + CN$. Hence

$$AM + BM < AM + MN + BN = AN + BN < CA + CN + NB = CA + CB.$$

(b) Let $AB \leq BC \leq CA$. Draw the lines through M parallel to the sides of the triangle and denote by A_1 and A_2, B_1 and B_2, C_1 and C_2 the points where

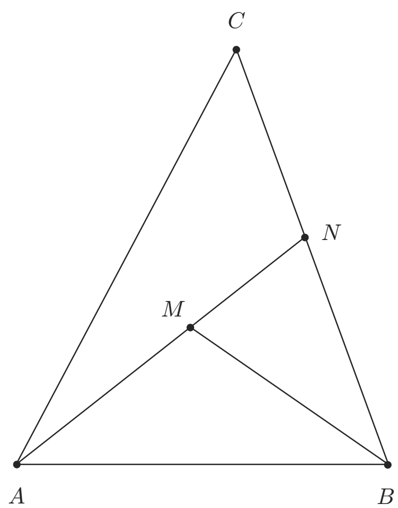

Figure 1.1

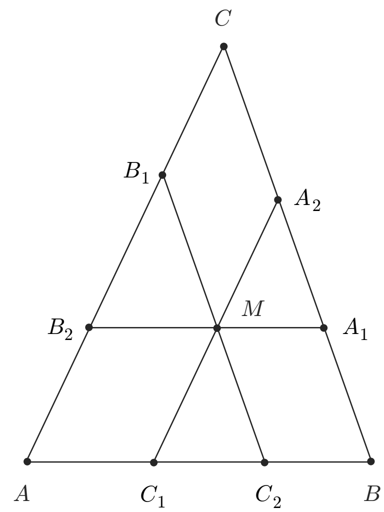

Figure 1.2

they intersect BC, CA, AB, respectively (Fig. 1.2). Then triangles A_1A_2M, MB_1B_2, C_2MC_1 are similar, and their shortest sides are MA_1, MB_2, C_1C_2, respectively. This together with the triangle inequality implies

$$\begin{aligned} MA + MB + MC &< (AB_2 + B_2M) + (MA_1 + A_1B) + (MA_2 + A_2C) \\ &< (AB_2 + B_2B_1) + (A_1A_2 + A_1B) + (CB_1 + A_2C) \\ &= AC + BC. \end{aligned}$$

Here we have used the fact that $MA_2 = B_1C$. (Why?)

Example 1.2. (Heron's problem) Points A and B lie on one side of a line l. Find a point C on l such that $CA + CB$ is minimized.

Solution. Denote by B' be the reflection of B in l (Fig. 1.3). The triangle inequality for triangle ACB' implies that

$$CA + CB = CA + CB' \geq AB'.$$

Equality occurs when C is the intersection of l and the segment AB' (i.e., point C_0 in Fig. 1.3).

Example 1.3. Let $ABCD$ be a cyclic quadrilateral. Prove that

(a) $|AB - CD| + |AD - BC| \geq 2|AC - BD|$;

(b) $AB + BD \leq AC + CD$ if $\angle A \geq \angle D$.

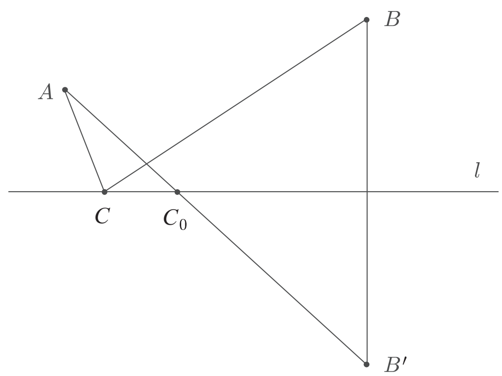

Figure 1.3

Solution. (a) Let M be the intersection of the diagonals AC and BD. Then triangles ABM and DCM are similar and

$$\begin{aligned}|AC - BD| &= |AM + MC - BM - DM| \\ &= \left|AM + BM \cdot \frac{CD}{AB} - BM - AM \cdot \frac{CD}{AD}\right| \\ &= \frac{|AM - BM|}{AB} \cdot |AB - CD| \leq |AB - CD|.\end{aligned}$$

Similarly,

$$|AC - BD| \leq |AD - BC|$$

and so

$$|AB - CD| + |AD - BC| \geq 2|AC - BD|.$$

(b) Note first that the given condition is equivalent to $\angle MAD \geq \angle MDA$, hence $MD \geq MA$. On the other hand, we know that

$$\frac{CD}{AB} = \frac{CM}{MB} = \frac{DM}{MA} = k \geq 1,$$

and therefore

$$AC + CD - AB - BD = (k-1)(AB + BM - AM) \geq 0.$$

Example 1.4. Let M be a point on a segment AB and K a point in the plane. Prove that:

(a) If M is the midpoint of AB, then

$$KM \leq \frac{KA + KB}{2};$$

(b) If $\dfrac{MB}{AB} = \lambda$, $0 < \lambda < 1$, then

$$KM \leq \lambda KA + (1-\lambda)KB.$$

(c) If G is the centroid of triangle ABC, then

$$KG \leq \frac{KA + KB + KC}{3}.$$

Solution. (a) Consider the point N such that $ANBK$ is a parallelogram. Then

$$KM = \frac{1}{2}KN \leq \frac{1}{2}(KB + BN) = \frac{1}{2}(KB + KA).$$

Note also that this inequality is a special case of (b) for $\lambda = \dfrac{1}{2}$.

(b) We have

$$\overrightarrow{KM} = \lambda\,\overrightarrow{KA} + (1-\lambda)\,\overrightarrow{KB}.$$

Then the triangle inequality for vectors implies

$$|\overrightarrow{KM}| \leq \lambda\,|\overrightarrow{KA}| + (1-\lambda)\,|\overrightarrow{KB}|.$$

(c) We know that $\dfrac{GM}{CG} = \dfrac{1}{3}$. Hence from (b) and (a), it follows that

$$KG \leq \frac{1}{3}(KC + 2KM) < \frac{1}{3}(KC + KA + KB).$$

This inequality also follows from the identity

$$\overrightarrow{KA} + \overrightarrow{KB} + \overrightarrow{KC} = 3\overrightarrow{KG}$$

and the triangle inequality for vectors.

Example 1.5. Four points A, B, C, D are given in the plane and let E and F be the respective midpoints of the segments AB and CD. Prove that

$$EF \leq \frac{AD + BC}{2}.$$

Solution. Let M be the midpoint of DB. Then

$$EF \leq EM + MF = \frac{1}{2}AD + \frac{1}{2}BC.$$

Example 1.6. (Ptolemy's inequality) For every four points A, B, C, D in the plane, we have

$$AC \cdot BD \leq AB \cdot CD + BC \cdot AD.$$

Equality holds if and only if $ABCD$ is a cyclic quadrilateral.

Solution. We may assume that B lies in $\angle ADC$. On rays $\overrightarrow{DA}$, $\overrightarrow{DB}$, $\overrightarrow{DC}$, consider the points A_1, B_1, C_1, respectively, such that

$$DA_1 = \frac{1}{DA}, \quad DB_1 = \frac{1}{DB}, \quad DC_1 = \frac{1}{DC}.$$

Then $\triangle ABC \sim \triangle A_1B_1C_1$ and so

$$A_1B_1 = \frac{AB}{DA \cdot DB}, \quad B_1C_1 = \frac{BC}{DB \cdot DC}, \quad C_1A_1 = \frac{CA}{DC \cdot DA}.$$

The desired inequality follows from the triangle inequality:

$$A_1B_1 + B_1C_1 \geq A_1C_1.$$

Equality holds if and only if B_1 lies on the segment A_1C_1, that is, when

$$\angle BAD + \angle BCD = \angle A_1B_1D + \angle C_1B_1D = 180^\circ.$$

Example 1.7. On side AB of triangle ABC, a square with center O is constructed externally to the triangle. Let M and N be the respective midpoints of sides AC and BC. Prove that

$$OM + ON \leq \left(\frac{\sqrt{2}+1}{2}\right)(AC + BC).$$

Show that equality holds if and only if $\angle ACB = 135^\circ$.

Solution. Let K be the midpoint of AB. Then by Ptolemy's inequality (Example 1.6), we have

$$NO \cdot AK \leq AO \cdot NK + AN \cdot OK,$$

which can be written as

$$NO \leq \frac{AC}{2} + \frac{\sqrt{2}}{2}BC.$$

Similarly,

$$MO \leq \frac{BC}{2} + \frac{\sqrt{2}}{2}AC.$$

Adding these inequalities yields

$$OM + ON \leq \left(\frac{\sqrt{2}+1}{2}\right)(AC + BC).$$

Equality holds if and only if $\angle ANK = \angle BMK = 135^\circ$, i.e., $\angle ACB = 135^\circ$.

Example 1.8. (Pompeiu's theorem) Let ABC be an equilateral triangle and let M be a point in its plane. Prove that the segments AM, BM, CM are sides of a triangle. Also prove that this triangle is degenerate if and only if M lies on the circumcenter of triangle ABC.

First Solution. By Ptolemy's inequality for points A, M, B, C, it follows that

$$AB \cdot CM \leq AM \cdot BC + BM \cdot AC.$$

Since $AB = BC = CA$, we get $CM \leq AM+BM$. Similarly, $BM \leq CM+AM$ and $AM \leq BM + CM$. We have equality in one of these inequalities, say in the first one, if and only if $AMBC$ is a cyclic quadrilateral.

Second Solution. Consider the rotation of 60° about A, and let M_1 be the image of M (Fig. 1.4). Then $AM = MM_1$, $CM_1 = BM$, and $\triangle MM_1C$ is the desired triangle.

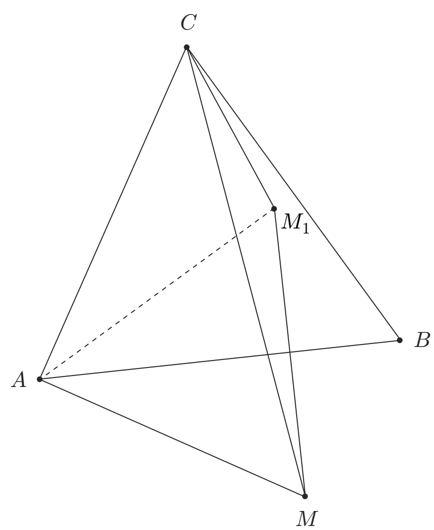

Figure 1.4

Note that it degenerates if and only if the points M_1, C, M are collinear which implies that M lies on the circumcircle of triangle ABC. (Why?)

Example 1.9. Let E and F be two points outside a convex quadrilateral $ABCD$ such that triangles ABE and CDF are equilateral. Prove that for all points M and N in the plane,

$$AM + BM + MN + CN + DN \geq EF.$$

Solution. From Pompeiu's inequality (Example 1.8) for points M, A, E, B and N, C, F, D, it follows that

$$AM + BM + MN + CN + DN \geq EM + MN + FN \geq EF.$$

1.2 Broken Lines

In this section we will use the so-called generalized triangle inequality which says that for any points $A_1, A_2, \ldots, A_n$, $n \geq 3$ in the plane (Fig. 1.5), the following inequality is true:

$$A_1A_2 + A_2A_3 + \ldots + A_{n-1}A_n \geq A_1A_n.$$

This inequality follows by induction on n using the triangle inequality.

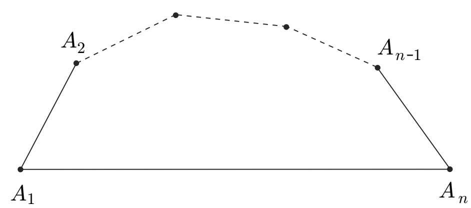

Figure 1.5

Note that equality occurs if and only if points $A_2, \ldots, A_{n-1}$ lie on the segment A_1A_n in this order.

Example 1.10. Given a convex polygon P, consider the polygon P' whose vertices are the midpoints of the sides of P. Prove that the perimeter of P' is not less than half the perimeter of P.

Solution. If $n = 3$, then the perimeter of triangle P' is half the perimeter of triangle P. Let $n \geq 4$ and let $A_1, A_2, \ldots, A_n$ be the vertices of P. Denote by $B_1, B_2, \ldots, B_n$ the midpoints of $A_1A_2, A_3A_4, \ldots, A_nA_1$, respectively. Then

$$\begin{aligned} &2B_1B_2 + 2B_2B_3 + \ldots + 2B_nB_1 \\ &= \frac{1}{2}(A_1A_3 + A_2A_4) + \frac{1}{2}(A_2A_4 + A_3A_5) + \ldots + \frac{1}{2}(A_nA_2 + A_1A_3) \\ &> \frac{1}{2}(A_1A_2 + A_3A_4) + \frac{1}{2}(A_2A_3 + A_4A_5) + \ldots + \frac{1}{2}(A_nA_1 + A_2A_3) \\ &= A_1A_2 + A_2A_3 + \ldots + A_nA_1. \end{aligned}$$

Example 1.11. Let $ABCDEF$ be a convex hexagon with $\angle A \geq 90°$ and $\angle D \geq 90°$. Prove that the perimeter of quadrilateral $BCEF$ is not less than $2AD$.

Solution. Denote by M, N, K the respective midpoints of BF, BE, CE (Fig. 1.6). Note that point A lies inside the circle with diameter BF, since $\angle A \geq 90°$.
Hence $AM \leq \dfrac{BF}{2}$, and similarly, $DK \leq \dfrac{CE}{2}$. Thus

$$BF + FE + CB + EC \geq 2AM + 2MN + 2NK + 2KD \geq 2AD.$$

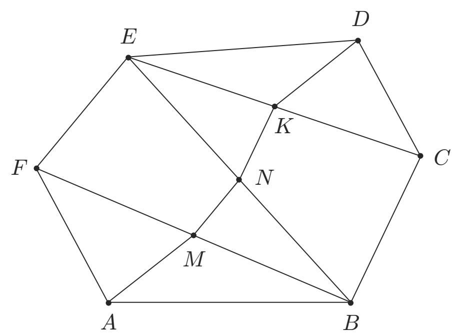

Figure 1.6

Example 1.12. Among all quadrilaterals $ABCD$ with $AB = 3, CD = 2$, and $\angle AMB = 120°$, where M is the midpoint of CD, find the one of minimal perimeter.

Solution. Let C' and D' be the reflections of C and D in the lines BM and AM, respectively (Fig. 1.7).

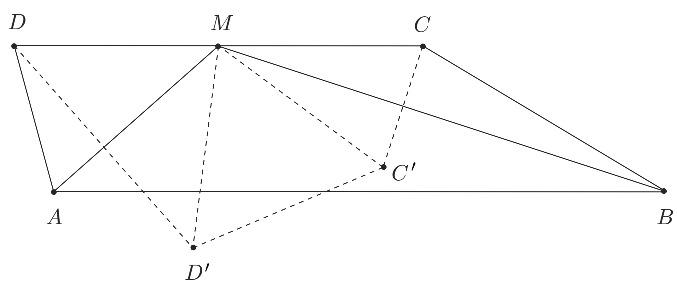

Figure 1.7

Then triangle $C'MD'$ is equilateral because $C'M = D'M = \frac{1}{2}CD$ and

$$\angle C'MD' = 180° - 2\angle CMB - 2\angle DMA = 60°.$$

Hence

$$AD + \frac{1}{2}CD + CB = AD' + D'C' + C'B \geq AB.$$

It follows that $AD + CB \geq AB - \frac{1}{2}CD = 2$. Thus $AB + BC + CD + DA \geq 7$, with equality if and only if C' and D' lie on AB.

In the latter case, $\angle ADM = \angle AD'M = 120°$, $\angle BCM = \angle BC'M = 120°$, and $\angle AMD = 60° - \angle CMB = \angle CBM$. Hence triangles AMD and MBC are similar, implying

$$AD \cdot BC = \left(\frac{CD}{2}\right)^2 = 1.$$

On the other hand, $AD + BC = 2$, and we conclude that $AD = BC = 1$. Therefore the quadrilateral $ABCD$ of minimum perimeter is an isosceles trapezoid with sides $AB = 3, BC = AD = 1$, and $CD = 2$ (Fig. 1.8).

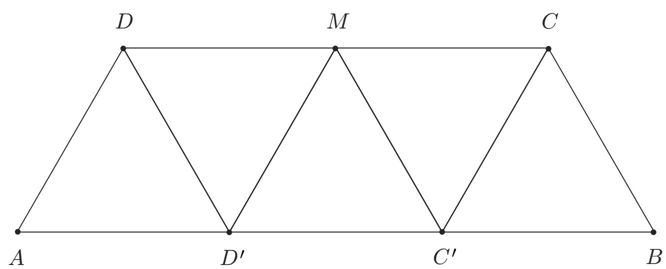

Figure 1.8

Example 1.13. (Fagnano's problem) Prove that of all triangles inscribed in a given acute triangle, the orthic triangle has the least perimeter.

Solution. Let ABC be the given triangle and let M, N, P be arbitrary points on the sides AB, BC, CA, respectively. Denote by E and F the respective feet of the perpendiculars from M to AC and BC. Then the quadrilateral $MFCE$ is inscribed in the circle with diameter CM and therefore $EF = CM \sin\angle C$. Let Q and R be the respective midpoints of MP and MN. Then

$$MN + NP + PM = 2FR + 2QR + 2QE \geq 2EF = 2CM \sin\angle C.$$

Let AA_1, BB_1, CC_1 be the altitudes of triangle ABC, and let E_1 and F_1 be the feet of the perpendiculars from C_1 to AC and BC, respectively. Then $E_1F_1 = CC_1 \sin\angle C$. Denote by Q_1 and R_1 the respective midpoints of C_1B_1 and C_1A_1. Then

$$\angle E_1Q_1B_1 = 2\angle E_1C_1B_1 = 2\angle C_1B_1B = \angle C_1B_1A_1,$$

which shows that $E_1Q_1 \parallel A_1B_1$. Similarly, $F_1R_1 \parallel A_1B_1$. Hence the points E_1, Q_1, R_1, F_1 are collinear, and we obtain

$$A_1B_1 + B_1C_1 + C_1A_1 = 2Q_1R_1 + 2Q_1E_1 + 2R_1F_1 = 2E_1F_1 = 2CC_1 \sin\angle C.$$

Thus

$$MN + NP + PM = 2CM \sin\angle C \geq 2CC_1 \sin\angle C = A_1B_1 + B_1C_1 + C_1A_1.$$

Remark. Fagnano's problem can also be solved in the case when the given triangle is not acute-angled. Assume, for example, that $\angle ACB \geq 90°$. It is not difficult to see that in this case, the triangle MNP with minimal perimeter occurs when $N = P = C$ and M is the foot of the altitude of triangle ABC through C. In this case, triangle MNP is degenerate.