## Preface

This book delves into the topic of exponents and logarithms. These are important functions in algebra, calculus, and beyond; a solid foundation in this material will serve the reader in their mathematical training for years to come. Of course, it is not enough to simply understand the basics of exponents and logarithms and know how to solve rote exercises with them. A deeper understanding of the intricacies of this subject, along with the ability to solve difficult problems, is necessary. This work aims to give the reader this understanding by providing both theory and a wealth of problems.

The first six chapters cover the theoretical background. Starting from the basics, the reader will gain familiarity with the exponential and logarithmic functions and learn how to solve different problems with them. Each chapter comes with a variety of examples that illustrate the concepts and techniques discussed. The latter portion of the book contains 114 carefully chosen problems (accompanied by solutions) to practice with. This gives the reader the opportunity to test their understanding of the concepts from the preceding six chapters and gain insight in solving problems with exponents and logarithms. We believe that this challenge will be a great experience for any reader who enjoys problem-solving.

We would like to thank Chris Jeuell, who revised the initial draft of this manuscript, fixing many errors and improving the explanations. We also would like to thank Navid Safei, who provided an abundance of great problems.

We hope you enjoy!

Titu Andreescu and Sean Elliott

# Contents

Preface . . . . . . . . . . v
1 Basics of Exponents and Logarithms . . . . . . . . . . 1
2 Algebraic Techniques in Exponent and Logarithm Problems . . 13
3 Equations and Systems of Equations Involving Exponents and Logarithms . . . . . . . . . . 23
4 Inequalities Involving Exponents and Logarithms . . . . . . . . 33
5 Exponents and Logarithms in Number Theory . . . . . . . . . 43
6 Exponents and Logarithms in Calculus . . . . . . . . . . 53
7 Introductory Problems . . . . . . . . . . 65
8 Advanced Problems . . . . . . . . . . 71
9 Solutions to Introductory Problems . . . . . . . . . . 77
10 Solutions to Advanced Problems . . . . . . . . . . 103
Other Books from XYZ Press . . . . . . . . . . 141

# 1 Basics of Exponents and Logarithms

We begin with some basic properties of exponentiation. Recall that for any positive integer $n$ and real number $a$, we define

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}.$$

For any positive integers $p, q$ we can define

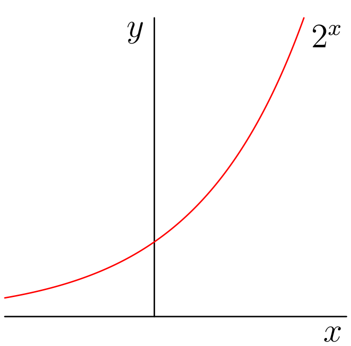
$$a^{\frac{p}{q}} = (\sqrt[q]{a})^p,$$

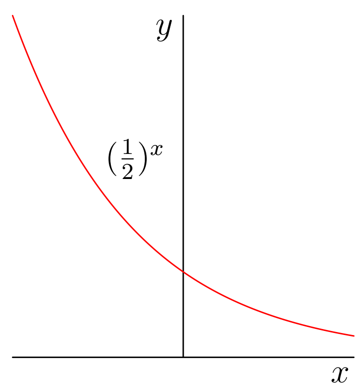
provided $\sqrt[q]{a}$ is defined. We can extend this to negative exponents as follows: if $a^{\frac{p}{q}} \neq 0$, then

$$a^{-\frac{p}{q}} = \frac{1}{a^{\frac{p}{q}}}.$$

We can even extend our definition to allow the exponent to be any real number. Doing so rigorously requires calculus, but for our purposes, we will assume that it can be done so in a way that preserves the properties below.

Now we can graph exponential functions of the form $f(x) = a^x$, where $a$ is a positive real number different from 1 and $x$ is any real number. When $a > 1$, we have a graph that visually looks like the graph of $f(x) = 2^x$ on the left, whereas when $0 < a < 1$, we have a graph that visually looks like the graph of $f(x) = (\frac{1}{2})^x$ on the right (particularly, in these cases the function will be increasing or decreasing, respectively).




**Properties of exponents.** We recall the following properties of exponents. If $a, b$ are positive real numbers, then for all real numbers $x$ and $y$, we have

1. $a^0 = 1$
2. $a^{-x} = \frac{1}{a^x}$
3. $a^x \cdot a^y = a^{x+y}$

4. $\dfrac{a^x}{a^y} = a^{x-y}$

5. $(a^x)^y = a^{xy}$

6. $(ab)^x = a^x b^x$

7. $(\frac{a}{b})^x = \frac{a^x}{b^x}$

We can now use our definition of the exponential function to define the logarithm. If $a \neq 1$ and $x$ are positive real numbers, then

$$y = \log_a x \text{ if and only if } x = a^y.$$

In other words, the logarithm function is the inverse of the exponential function. If the base of a logarithm is unspecified, it is assumed by convention to be the base-10 logarithm.

The notation $\ln x$ refers to the logarithm with base $e$, where $e \approx 2.718\ldots$ is Euler's number. We will see why this number is special in Chapter 6, Exponents and Logarithms in Calculus.

**Properties of logarithms.** Let $a \neq 1$, $x$, and $y$ be positive real numbers, and let $r$ be any real number. Then, from the properties of the exponential function, a few basic properties of the logarithm function immediately follow.

1. $\log_a 1 = 0$

2. $\log_a a = 1$

3. $a^{\log_a x} = x$

4. $\log_a(xy) = \log_a x + \log_a y$

5. $\log_a(\frac{x}{y}) = \log_a x - \log_a y$

6. $\log_a x^r = r \log_a x$

Using these results, we can prove the so-called change of base formula, which allows us to easily modify the base of any logarithm by multiplying by a constant factor.

**Theorem 1.1** *(Change of base formula)* For any positive real numbers $a, b$ different from 1 and any positive real number $x$, we have

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

*Proof.* Take the base-$b$ logarithm of both sides of the equation $x = a^{\log_a x}$, obtaining

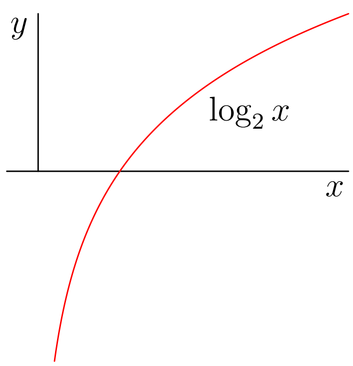
$$\log_b x = \log_b a^{\log_a x} = (\log_a x) \cdot (\log_b a),$$

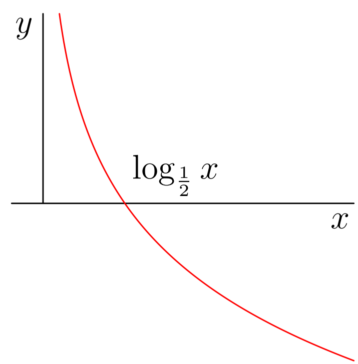
which is equivalent to the desired result.

*Remark.* Setting $x = b$ in the change of base formula gives the following useful result. For any positive real numbers $a, b$ with $a \neq 1$ and $b \neq 1$, we have

$$\log_a b = \frac{1}{\log_b a}.$$

We can also analyze the graphs of the function $f(x) = \log_a x$. When $a > 1$, the graph will visually look like the function $f(x) = \log_2 x$ on the left, and when $0 < a < 1$, the graph will look like the function $f(x) = \log_{\frac{1}{2}} x$ on the right (particularly, in these cases the function will be increasing or decreasing, respectively).




**Example 1.1.** Prove that if $a$ and $b$ are positive real numbers with $a \neq 1$, and $r$ is any nonzero real number, then $\log_a b = \log_{a^r} b^r$.

**Solution.** Let $x = \log_a b$. Then we only need to show that $(a^r)^x = b^r$. Indeed,

$$(a^r)^x = (a^x)^r = b^r,$$

where the second equality follows from the definition of $x$.

**Example 1.2.** Let $a, x, b$ be positive real numbers such that $a^x = x = b^3$. Express $x^{x^{-\frac{1}{3}}}$ in terms of $a$ and $b$.

**Solution.** Note that $a = x^{\frac{1}{x}}$ and $b = \sqrt[3]{x}$, so

$$x^{x^{-\frac{1}{3}}} = x^{\frac{1}{b}} = \left(x^{\frac{1}{x}}\right)^{\frac{x}{b}} = a^{\frac{b^3}{b}} = a^{b^2}.$$

The answer is thus $a^{b^2}$.

**Example 1.3.** If $6 \cdot 5^x - 3^{x+1} = 3^x + 5^{x+1}$, find the value of $\left(\frac{3}{5}\right)^x$.

**Solution.** Dividing both sides of the equation by $5^x$, we obtain

$$6 - 3 \cdot \left(\frac{3}{5}\right)^x = \left(\frac{3}{5}\right)^x + 5,$$

so $\left(\frac{3}{5}\right)^x = \frac{1}{4}$.

**Example 1.4.** If $\dfrac{1 + 2^x + 3^x + 6^x}{2^x + 1} = 82$, find the value of $x$.

**Solution.** The numerator is equal to $(1+2^x)(1+3^x)$. Therefore, the equation becomes $1 + 3^x = 82$, so $x = 4$.

**Example 1.5.** If $a$ is a nonzero integer and $b$ is a positive real number such that $ab^2 = \log b$, what is the median of the set $\left\{0, 1, a, b, \frac{1}{b}\right\}$?

**Solution.** Note that $\log b < b$ for all $b > 0$. If $b > 1$, then $0 < \dfrac{\log b}{b^2} < 1$, so $a = \dfrac{\log b}{b^2}$ cannot be an integer. Therefore $0 < b < 1$, so $\log b < 0$ and $a < 0$. Thus $a < 0 < b < 1 < \frac{1}{b}$, and the median is $b$.

**Example 1.6.** Simplify

$$\frac{1}{\log_a abc} + \frac{1}{\log_b abc} + \frac{1}{\log_c abc}$$

where $a, b, c$ are positive real numbers.

**Solution.** By the change of base formula, $\dfrac{1}{\log_a abc} = \log_{abc} a$, and similarly for the other two terms. Therefore, the expression becomes

$$\log_{abc} a + \log_{abc} b + \log_{abc} c = \log_{abc} abc = 1.$$

**Example 1.7.** Compute

$$\lfloor \log_3 1 \rfloor + \lfloor \log_3 2 \rfloor + \cdots + \lfloor \log_3 100 \rfloor.$$

**Solution.** To simplify our calculation, we will group together the terms that are equal. We do this by bounding the integers $n \in \{1, 2, \cdots, 100\}$ between consecutive powers of 3.

First we have, for $1 \le n < 3$, $\lfloor \log_3 n \rfloor = 0$. Similarly, for $3 \le n < 9$, $\lfloor \log_3 n \rfloor = 1$. For $9 \le n < 27$, $\lfloor \log_3 n \rfloor = 2$. For $27 \le n < 81$, $\lfloor \log_3 n \rfloor = 3$; and for $81 \le n \le 100$, $\lfloor \log_3 n \rfloor = 4$. So the sum is equal to

$$0 \cdot (3-1) + 1 \cdot (9-3) + 2 \cdot (27-9) + 3 \cdot (81-27) + 4 \cdot (100-81+1) = 284.$$

**Example 1.8.** Evaluate the expression

$$3^{\log_5 7} - 7^{\log_5 3}.$$

**Solution.** Let $\log_5 7 = x$. Then $5^x = 7$ and

$$3^{\log_5 7} - 7^{\log_5 3} = 3^x - (5^x)^{\log_5 3} = 3^x - 5^{x \log_5 3} = 3^x - 5^{\log_5 3^x} = 3^x - 3^x = 0.$$

Therefore,

$$3^{\log_5 7} - 7^{\log_5 3} = 0.$$

**Example 1.9.** For an integer $n > 1$ let $f(n) = \dfrac{1}{\log_n (10!)}$. Find

$$f(3!) + f(5!) + f(7!) + f(10!).$$

**Solution.** From the change of base formula,

$$f(n) = \frac{1}{\log_n (10!)} = \log_{10!} n.$$

Thus,

$$\begin{aligned} f(3!) + f(5!) + f(7!) + f(10!) &= \log_{10!} 3! + \log_{10!} 5! + \log_{10!} 7! + \log_{10!} 10! \\ &= \log_{10!} (3! \cdot 5! \cdot 7!) + 1 \\ &= \log_{10!} (3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 7!) + 1 \\ &= \log_{10!} ((5 \cdot 2)(3 \cdot 3)(4 \cdot 2)7!) + 1 \\ &= \log_{10!}(10!) + 1 = 2. \end{aligned}$$

**Example 1.10.** If $1 < a < b$ and $1 < c$, show that

$$\log_a b > \log_{ac} bc.$$

**Solution.** By the change of base formula, the left-hand side is equal to

$$\log_a b = \frac{\log b}{\log a},$$

while the right-hand side is equal to

$$\log_{ac} bc = \frac{\log bc}{\log ac} = \frac{\log b + \log c}{\log a + \log c}.$$

Letting $x = \log a, y = \log b$, and $z = \log c$, the desired inequality becomes

$$\frac{y}{x} > \frac{y+z}{x+z}.$$

From the given conditions on $x, y$, and $z$ we know $0 < x < y$ and $z > 0$, implying that

$$yx + yz > yx + xz,$$

which is equivalent to the desired inequality upon dividing by the positive value $x(x+z)$.

**Example 1.11.** Let $f : \mathbb{R} \to (1, \infty)$ be defined by $f(x) = e^{2x} + e^x + 1$. Find the inverse of $f$.

**Solution.** Suppose that

$$y = e^{2x} + e^x + 1 = \left(e^x + \frac{1}{2}\right)^2 + \frac{3}{4}.$$

Then

$$y - \frac{3}{4} = \left(e^x + \frac{1}{2}\right)^2 \Rightarrow \pm\sqrt{y - \frac{3}{4}} = e^x + \frac{1}{2}.$$

Since $e^x + \dfrac{1}{2}$ is positive, we take the positive square root, obtaining

$$e^x = \sqrt{y - \frac{3}{4}} - \frac{1}{2}.$$

Taking natural logarithms of both sides, we obtain

$$x = f^{-1}(y) = \ln\left(\sqrt{y - \frac{3}{4}} - \frac{1}{2}\right).$$

**Example 1.12.** Trying to solve a problem, Jimmy used the following "formula":

$$\log_{ab} x = (\log_a x)(\log_b x),$$

where $a, b, x$ are positive real numbers different from 1. Prove that this is correct only if $x$ is a solution to the equation $\log_a x + \log_b x = 1$.

**Solution.** Let $u = \log_{ab} x, v = \log_a x$, and $w = \log_b x$. If $u = vw$, then we know

$$x = (ab)^u = a^u b^u = a^{vw} b^{vw} = (a^v)^w (b^w)^v = x^w x^v = x^{v+w}.$$

This implies that $v + w = 1$, or equivalently $\log_a x + \log_b x = 1$.

**Example 1.13.** Let $c \neq 1$ be a positive real number, and define the function $f : (1, \infty) \to \mathbb{R}$ by $f(x) = \log_x c$. For what values of $c$ is $f$ increasing?

**Solution.** By the change of base formula, $f(x) = \dfrac{\log c}{\log x}$. Recall that $\log x$ is increasing, so $\dfrac{1}{\log x}$ is decreasing. Hence, $f$ is increasing if and only if $\log c < 0$, or $c < 1$. By the same reasoning, if $c > 1$ then $f$ is decreasing.

**Example 1.14.** Given are $n$ positive real numbers $x_1, x_2, \ldots, x_n$ such that

$$\begin{aligned} x_1 &= \log_{x_{n-1}} x_n \\ x_2 &= \log_{x_n} x_1 \\ &\vdots \\ x_n &= \log_{x_{n-2}} x_{n-1}. \end{aligned}$$

Prove that $\displaystyle\prod_{k=1}^{n} x_k = 1$.

**Solution.** By the change of base formula,

$$x_k = \log_{x_{k-2}} x_{k-1} = \frac{\log x_{k-1}}{\log x_{k-2}},$$

where $x_{-1}$ and $x_0$ refer to $x_{n-1}$ and $x_n$, respectively. Therefore,

$$\prod_{k=1}^{n} x_k = \prod_{k=1}^{n} \frac{\log x_{k-1}}{\log x_{k-2}} = \frac{\prod_{k=1}^{n} \log x_{k-1}}{\prod_{k=1}^{n} \log x_{k-2}} = 1.$$

**Example 1.15.** If $a, b, c$ are positive real numbers different from 1 and

$$x = \log_a \frac{b}{c}, \quad y = \log_b \frac{c}{a}, \quad z = \log_c \frac{a}{b},$$

show that $xyz + x + y + z = 0$.

**Solution.** Note that

$$a^{xyz} = (a^x)^{yz} = \left(\frac{b}{c}\right)^{yz}$$

by the definition of $x$. But using the definitions of $y$ and $z$ gives

$$\left(\frac{b}{c}\right)^{yz} = \frac{(b^y)^z}{(c^z)^y} = \left(\frac{c}{a}\right)^z \left(\frac{a}{b}\right)^{-y} = \frac{c^z b^y}{a^{z+y}}.$$

Using the definitions of $x, y, z$ once more yields

$$\frac{c^z b^y}{a^{z+y}} = \frac{\frac{a}{b}\cdot\frac{c}{a}}{a^{y+z}} = \frac{\frac{c}{b}}{a^{y+z}} = \frac{1}{a^{x+y+z}}.$$

Therefore $a^{xyz+x+y+z} = 1$, so $xyz + x + y + z = 0$, since $a \neq 1$.

**Example 1.16.** Prove that $(\log_{24} 48)^2 + (\log_{12} 54)^2 > 4$.

**Solution.** To obtain a lower bound for the first term on the left-hand side, we wish to find an inequality of the form $48^a > 24^b$ (as this would imply that $\log_{24} 48 > \frac{b}{a}$). We can rewrite such an inequality as $2^a > 24^{b-a}$; the simplest such inequality is when $b - a = 1$, in which case, we note that $2^5 = 32 > 24^1$, so

$$48^5 > 24^6 \Rightarrow \log_{24} 48 > \frac{6}{5}.$$

Similarly, for the second term, we need an inequality of the form $54^a > 12^b$, or $3^{3a-b} > 2^{2b-a}$. After some experimentation, we note that $3^7$ and $2^{11}$ are reasonably close: $3^7 = 2187 > 2048 = 2^{11}$, so

$$54^5 > 12^8 \Rightarrow \log_{12} 54 > \frac{8}{5}.$$

Combining our two inequalities, we obtain

$$(\log_{24} 48)^2 + (\log_{12} 54)^2 > \frac{36}{25} + \frac{64}{25} = 4.$$

**Example 1.17.** Let $n$ be an integer greater than 1. Show that

$$\prod_{k=2}^{n} \log_k(n-k+2) = 1.$$

**Solution.** Let the left-hand side be $S$. By the change of base formula,

$$S^2 = \prod_{k=2}^{n} \log_k(n-k+2) \prod_{k=2}^{n} \frac{1}{\log_{n-k+2} k}.$$