Preface

Trigonometry is an area of mathematics which deals with geometrical reasoning and studies the relationship between the lengths and angles of triangles. This study is conducted through the trigonometric functions which are also a key element in the foundation of other mathematical areas, such as Fourier Analysis or Differential equations. In the real life applications, trigonometric functions play a crucial role in navigation, astronomy, architecture, cartography or digital imaging, to mention just a few examples. It is therefore no surprise that Trigonometry is a very popular topic when it comes to mathematical contests. The present book is intended to offer a rich analysis of this field, as well as illustrating through examples and proposed problems the main themes that occur in Olympiad problems involving Trigonometry. The book is structured in three major parts as follows:

The first chapter, entitled *Theory and Examples* offers a comprehensive overview of the trigonometric functions. We start from the elementary definitions and basic properties and then study in depth the properties of the trigonometric functions as real-valued functions. We conclude the chapter with two sections dedicated to applications in Geometry and Inequalities and an Appendix which presents the details of the more involved theorems that rely on Calculus. Each section is accompanied by a series of examples and guided exercises designed to facilitate the learning process and outline the most useful tips and tricks which occur in Olympiad problems.

The second and third chapters offer a selection of 115 carefully selected introductory and advanced problems in Trigonometry. The list comprises questions from world-wide renowned Olympiads and mathematical magazines, as well as problems designed by the authors of this book and their collaborators. In the last part of the book we present the solutions to these questions and outline the key ideas behind them, in a manner which is intended to enhance the problem solving skills of the reader.

We hope that the theory and the problem selection offered by this book will serve as very good resource and teaching material for anybody who wants to explore the beauty of Trigonometry.

Contents

Chapter 1

Theory and Examples

1 Definition of Trigonometric Functions

In this section we will discuss the definition and basic properties of the trigonometric functions. We recall the following definition of a function.

Definition 1.1. For two sets A and B, a **function** (also called a **mapping** or a **map**) f from A to B (written $f : A \to B$) is a correspondence which associates to each element $a \in A$ precisely one element $b \in B$ (written $f(a) = b$). Given the sets A, B, C and two functions $f : A \to B$ and $g : B \to C$, we can form the **composition** of g and f (denoted $g \circ f$), which is a function $g \circ f : A \to C$ defined by the rule $g \circ f(a) = g(f(a))$, for any $a \in A$:

$$A \longrightarrow B \longrightarrow C$$

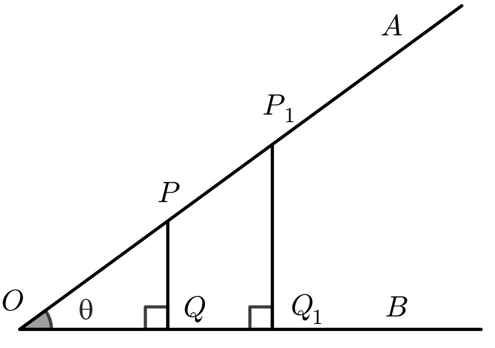
$$a \xrightarrow{f} b = f(a) \xrightarrow{g} c = g(b) = g(f(a)).$$

The **trigonometric functions** are functions of a measure of an angle, which among other applications relate the angles of a triangle to the lengths of its sides.

Given an angle θ (Greek "theta") between $0°$ and $90°$, we define trigonometric functions to describe the size of the angle as follows: let rays OA and OB form an angle θ (see the figure below). Choose a point P on the ray OA and let Q be the foot of the perpendicular line from P to the ray OB. For two points X and Y, we denote by $|XY|$ the length of the line segment between X and Y. We define the sine (sin), cosine (cos), tangent (tan) and cotangent (cot), cosecant (csc) and secant (sec) functions as follows:

$$\sin\theta = \frac{|PQ|}{|OP|}, \quad \csc\theta = \frac{|OP|}{|PQ|},$$
$$\cos\theta = \frac{|OQ|}{|OP|}, \quad \sec\theta = \frac{|OP|}{|OQ|},$$
$$\tan\theta = \frac{|PQ|}{|OQ|}, \quad \cot\theta = \frac{|OQ|}{|PQ|}.$$

One can check that these functions are well-defined, i.e. they only depend on the size of θ, but not on the choice of P: If P_1 is another point on the ray OA and Q_1 is the foot of perpendicular from P_1 to ray OB, then the right triangles OPQ and OP_1Q_1 are similar, hence pairs of corresponding ratios, like $\frac{|PQ|}{|OP|}$ and $\frac{|P_1Q_1|}{|OP_1|}$ are all equal.



By the above definitions, we see that $\sin\theta$, $\cos\theta$ and $\tan\theta$ are the reciprocals of $\csc\theta$, $\sec\theta$ and $\cot\theta$, respectively. Also, if we look at the complementary angle $\angle OPQ$ of θ in the triangle OPQ, it has the value $90^\circ - \theta$. If we use our definitions of trigonometric functions for this angle instead, we have that

$$\sin(90^\circ - \theta) = \frac{|OQ|}{|OP|} = \cos\theta,$$

and

$$\tan(90^\circ - \theta) = \frac{|OQ|}{|QP|} = \cot\theta,$$

hence $\cos\theta$ is the same as the sine evaluated at the complementary angle to θ, and similarly, $\cot\theta$ is just $\tan(90^\circ - \theta)$. This shows that the "co" trigonometric functions (i.e. those whose names start with "co") are just the trigonometric functions of the complementary angles.

Example 1.2. *Prove that* $\sin(30^\circ) = \frac{1}{2}$.

Solution. It is well-known that in any right triangle, the side which is opposite to an angle equal to 30° is half of the hypotenuse. Therefore, by the definition of the sine function, we obtain

$$\sin(30^\circ) = \frac{1}{2},$$

as we wanted. □

Exercise 1.3. *Use the properties of the right triangle to deduce the following values for the trigonometric functions*

θ	30°	45°	60°
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Show also that $\sin(0^\circ) = 0$ *and* $\cos(0^\circ) = 1$, *while* $\sin(90^\circ) = 1$ *and* $\cos(90^\circ) = 0$ *(notice that some of the other trigonometric functions are not defined when* $\theta = 0^\circ$ *or* $\theta = 90^\circ$*: for example, when* $\theta = 90^\circ$*, we have that* $|OQ| = 0$*, so we cannot define* $\tan(90^\circ)$*, as we cannot divide by* 0*).*

The above definitions for the trigonometric functions apply whenever we have to deal with angles θ whose values are between 0° and 90°, because then θ can be an acute angle in a right triangle. For the purpose of many applications, including those in Geometry, we would like to be able to evaluate these trigonometric functions even for angles which are outside this range between 0° and 90° (for example obtuse angles). The key thing which will allow us to generalize the above definitions is the following observation: in the above figure, if the length of the hypotenuse OP in the right triangle OPQ is equal to 1, we will have $\sin\theta = |PQ|$ and $\cos\theta = |OQ|$. So if we look at a system of Cartesian coordinates xOy in the plane and we take a point P in the plane such that P lies in the first quadrant (i.e. both its x-coordinate and y-coordinate are non-negative) and $|OP| = 1$, then the angle θ between OP and Ox-axis will have value between 0° and 90° and according to the previous observation, $\sin\theta$ will be just the y-coordinate of P and $\cos\theta$ will be just the x-coordinate of P. This suggests that we could generalize the definitions of sine and cosine for angles which are bigger than 90° in the following way: consider an angle θ between 0° and 360° and let P be a point in the plane such that $|OP| = 1$ (i.e P is on the unit circle centered at the origin) and the angle between the rays Ox and OP (measured from the ray Ox to the ray OP

in the counter-clockwise direction) has value θ. We then define $\sin\theta$ as the y-coordinate of P and $\cos\theta$ as the x-coordinate of P.

In what follows, we will formalize the above described approach to generalizing the definitions of trigonometric functions. Notice that once we generalized the definitions for sine and cosine, we obtained the definitions for all the other trigonometric functions, since we can express all of them just in terms of sine and cosine. It will also be convenient to use radians for measuring our angles, since the radian is just a real number and we have a well-developed theory for studying functions which take as inputs real numbers (Real Analysis). We make this more precise in the following definitions.

Definition 1.4. Consider a system of Cartesian coordinates xOy of the plane. An **angle** in the xOy plane is a configuration determined by an ordered pair (l_1, v, l_2), where l_1 and l_2 are two rays in the xOy plane (called the **sides of the angle**) which emerge from a common point v, called the **vertex** of the angle. To each such configuration, we will assign a real number (or more precisely a set of real numbers), called **the measure of the angle**, which uniquely describes the configuration.

Consider the unit circle $\mathcal{C}_v$ centered at v and, as usual, denote by 2π its length (so that for any circle in the plane the ratio of its length to its diameter is π). Let A_1 and A_2 be the intersections of l_1 and l_2 with the circle $\mathcal{C}_v$, respectively. We would like somehow to relate the angle configuration (l_1, v, l_2) to the counter-clockwise arc $\widehat{A_1A_2}$ on $\mathcal{C}_v$ from A_1 to A_2. It might be tempting to define the angle measure of the angle configuration (l_1, v, l_2) to be the length of the arc $\widehat{A_1A_2}$. The disadvantage in doing that is that if we look as the two rays in the angle configuration move smoothly past each other, the angle measure has a sudden drop from 2π to 0. The most natural way to overcome this difficulty is to make the numbers 2π and 0 "equal". More precisely, we let $\theta \in [0, 2\pi)$ be the unique real number with the property that the length of the counter-clockwise arc $\widehat{A_1A_2}$ is θ. Then any number in the set $\{\theta + 2k\pi : k \in \mathbf{Z}\}$ can be used to represent **the measure** of the angle (l_1, v, l_2), though in practice one uses θ as the representative for the measure. For example, one can say that the angle (Ox, O, Oy) has measure $\frac{5\pi}{2}$, but it is more common to choose the number in the interval $[0, 2\pi)$ as the measure for this angle, which is $\frac{\pi}{2}$. Notice also that two distinct real numbers x and y define two equal angle measures if and only if there is an integer k such that $x - y = 2\pi \cdot k$.

Remark 1.5. The above defined measure for an angle is also called the **measure in radians**. In particular, the radian is a scalar (i.e. a real number). For the readers who were used to having angles measured in degrees, it is easy to see from the definition that one has the conversion formula

$$x \quad \text{radians} = \frac{180x}{\pi} \quad \text{degrees,}$$

for a real number x with $0 \leq x < 2\pi$. The following table provides the conversion for some of the most used angle measures.

degrees	$0°$	$1°$	$30°$	$45°$	$60°$	$90°$	$180°$
radians	0	$\frac{\pi}{180}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π

□

By a slight abuse of language, from now on we will use the word "angle" as referring to the "measure of an angle", whenever no confusion can occur. In geometry, it is common to use the notation $\angle AOB$ to denote the angle which has measure in the interval $[0, \pi]$ between the angles (OA, O, OB) and (OB, O, OA). We also agree on the convention that if no unit is specified for an angle, the angle is measured in radians.

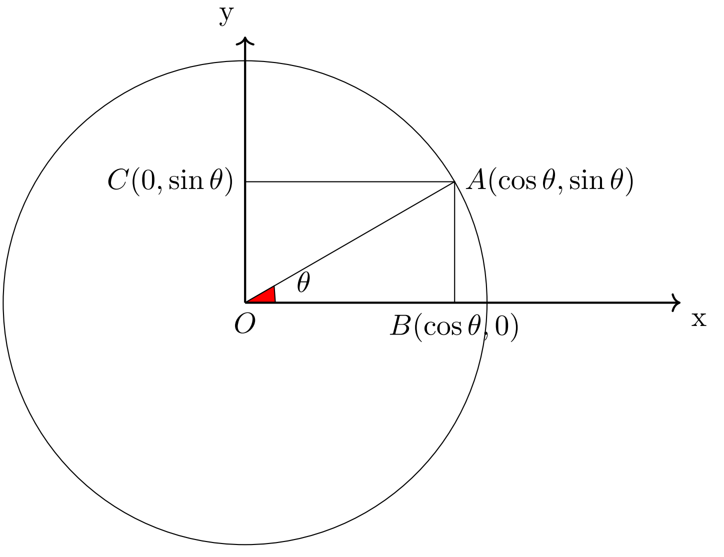
Remark 1.6. Since we use real numbers for measuring angles, the trigonometric functions will be real-valued functions. Moreover, from the definition of the measure of an angle, the trigonometric functions must be constant on sets of the form $\{x + 2k\pi : k \in \mathbf{Z}\}$, so they will be periodic functions of period 2π.

In Section 7.2 we give a brief exposition of how one can define the trigonometric functions on the whole complex plane. □

Definition 1.7. Let ω denote the unit circle centered at the origin O of the coordinate system xOy. We define two functions $\sin : \mathbf{R} \to [-1, 1]$ (called the **sine function**) and $\cos : \mathbf{R} \to [-1, 1]$ (called the **cosine function**) as follows: for θ an arbitrary real number, we let A be the unique point on ω such that the measure of the angle xOA is θ. We also let x_A and y_A denote the standard Cartesian coordinates of the point A (see the figure below). Then we define $\sin\theta$ and $\cos\theta$ by

$$\cos\theta = x_A, \quad \text{and} \quad \sin\theta = y_A.$$

Since the point A is uniquely determined by θ, both functions are well-defined.



Let us now present a list of properties for sine and cosine which follow immediately from their definition. To ease the subsequent exposition, for a given angle θ, we will call the unique point on $A \in \omega$ with the property that the angle (Ox, O, OA) equals θ the *point corresponding to* θ.

1) Since for any $k \in \mathbf{Z}$ the angles θ and $\theta + 2k\pi$ define the same point on ω, it follows that

$$\sin(\theta + 2k\pi) = \sin\theta, \quad \cos(\theta + 2k\pi) = \cos\theta.$$

In particular, both sine and cosine are periodic with period 2π.

2) For any $\theta \in \mathbf{R}$, the points on ω corresponding to θ and $\theta+\pi$, respectively, are symmetric with respect to the origin. Hence

$$\sin(\theta + \pi) = -\sin\theta, \quad \cos(\theta + \pi) = -\cos\theta.$$

For any $\theta \in \mathbf{R}$, the points corresponding to θ and $\pi - \theta$ are symmetric with respect to the Oy-axis, hence

$$\sin(\pi - \theta) = \sin\theta, \quad \cos(\pi - \theta) = -\cos\theta.$$

Similarly, the points corresponding to θ and $-\theta$ are symmetric with respect to the Ox-axis, hence

$$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta.$$

Finally, if $A \in \omega$ corresponds to θ and $B \in \omega$ corresponds to $\frac{\pi}{2} - \theta$, then $x_A = y_B$ and $y_A = x_B$, hence

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \quad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta.$$

3) Whenever x and y are real numbers satisfying $x^2 + y^2 = 1$, there exists a unique number $0 \leq \alpha < 2\pi$ such that $x = \cos\alpha$ and $y = \sin\alpha$; this is because (x, y) determines a point on the unit circle which will correspond to an angle $0 \leq \alpha < 2\pi$. (Alternately, we could view α as an arbitrary real number which is unique up to adding integer multiples of 2π.) Conversely, for any $\theta \in \mathbf{R}$, if $A \in \omega$ corresponds to θ, then $x_A = \cos\theta$ and $y_A = \sin\theta$. Since $|OA| = 1$, by the Pythagorean theorem, we always have $x_A^2 + y_A^2 = 1$, hence

$$\sin^2\theta + \cos^2\theta = 1,$$

for any real number θ.

Remark 1.8. Let $\theta \in [0, \frac{\pi}{2}]$ be an angle measured in radians, let A be the point on ω corresponding to θ and let B denote the foot of the perpendicular from A onto the Ox-axis. Then AOB is a right triangle and from the above definitions we have $\sin\theta = |AB|$ and $\cos\theta = |OB|$. Since $|OA| = 1$, we can write in fact

$$\sin\theta = \frac{|AB|}{|OA|}, \quad \cos\theta = \frac{|OB|}{|OA|}. \tag{*}$$

If we are given another right triangle MNP with the property that $\angle MNP = \frac{\pi}{2}$ and $\angle PMN = \theta$, then the triangles MNP and OBA are similar, hence

$$\frac{|MN|}{|OB|} = \frac{|NP|}{|AB|} = \frac{|MP|}{|OA|}.$$

It follows that

$$\frac{|AB|}{|OA|} = \frac{|NP|}{|MP|} \quad \text{and} \quad \frac{|OB|}{|OA|} = \frac{|MN|}{|MP|}.$$

Therefore, from (*) we deduce that

$$\sin\theta = \frac{|NP|}{|MP|} \quad \text{and} \quad \cos\theta = \frac{|MN|}{|MP|}.$$

This shows that in any right triangle where one of the acute angles has value θ, $\sin\theta$ defines the ratio between the length of the side opposite to θ and the length of the hypotenuse, while $\cos\theta$ defines the ratio between the side opposite to the other acute angle and the hypotenuse. We have therefore recovered the first definition for sine and cosine which we gave before. □

Exercise 1.9. *Each point A in the plane is uniquely determined by the distance $r = |OA|$ it has from the origin (sometimes called the* **norm***) and the measure θ of the angle (Ox, O, OA) it determines with respect to the Ox-axis (sometimes called the* **argument***). These coordinates are called the* **polar coordinates** *of a A. Show that if a point A has polar coordinates (r, θ), then A has Cartesian coordinates $(r\cos\theta, r\sin\theta)$.*

Exercise 1.10. *Using suitable properties of the right triangle together with the symmetry properties listed above, deduce the following values for sine and cosine:*

θ	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$\sin\theta$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$

It is an immediate consequence of the definition of the cosine function that one has $\cos\theta = 0$ precisely when $\theta = \frac{\pi}{2} + k\pi$, for some $k \in \mathbf{Z}$. This allows us to introduce the following definition:

Definition 1.11. The function $\tan : \mathbf{R} \setminus \{\frac{\pi}{2} + k\pi : k \in \mathbf{Z}\} \to \mathbf{R}$ (called the **tangent function** and written tan or sometimes tg) is defined as

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

A series of properties for the tangent function can now be easily deduced from those for sine and cosine listed above. We assume that in each case, θ is a real number at which all the functions that occur are defined.

1) Since $\sin(\theta + \pi) = -\sin\theta$ and $\cos(\theta + \pi) = -\cos\theta$, we have that

$$\tan(\theta + \pi) = \tan\theta.$$

Thus tan is periodic with period π.

2) From $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$, it follows that

$$\tan(-\theta) = -\tan\theta.$$

From $\sin(\pi - \theta) = \sin\theta$, and $\cos(\pi - \theta) = -\cos\theta$ we deduce

$$\tan(\pi - \theta) = -\tan\theta.$$

3) From the identities

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \quad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta,$$

we deduce that

$$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\tan\theta}.$$

Remark 1.12. If $\theta \in [0, \frac{\pi}{2})$ and we consider as before a right triangle MNP with $\angle MNP = \frac{\pi}{2}$ and $\angle PMN = \theta$, then by definition we have

$$\tan\theta = \frac{|NP|}{|MN|}.$$

Hence the tangent function relates the lengths of the two smaller sides in a right triangle. □

Particular values for the tangent function can now be deduced from those that we already know for sine and cosine.

Exercise 1.13. *Use the values that you worked out in Exercise* 1.10 *to deduce the following particular values of the tangent function:*

θ	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$
$\tan\theta$	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$-\sqrt{3}$

Example 1.14. *Find all pairs of positive integers* (n, k) *such that for all real numbers* x *one has*

$$\sin^n x + \cos^k x = \sin^k x + \cos^n x.$$

Solution. First take $x = \frac{\pi}{3}$. This gives

$$\left(\frac{\sqrt{3}}{2}\right)^n + \left(\frac{1}{2}\right)^k = \left(\frac{\sqrt{3}}{2}\right)^k + \left(\frac{1}{2}\right)^n.$$

This shows that n and k must have the same parity. If $k = 2m + 1$ and $n = 2l + 1$, then the above equation can be rewritten as

$$\frac{\sqrt{3}}{2}\left(\left(\frac{3}{4}\right)^m - \left(\frac{3}{4}\right)^l\right) = \frac{1}{2}\left(\frac{1}{4^m} - \frac{1}{4^l}\right).$$

This forces $m = l$, hence $n = k$.

If $n = 2m$ and $k = 2l$, then we get

$$\left(\frac{3}{4}\right)^m - \left(\frac{3}{4}\right)^l = \frac{1}{4^m} - \frac{1}{4^l}.$$

This gives

$$\frac{3^m - 1}{4^m} = \frac{3^l - 1}{4^l}.$$

If one defines the sequence $(a_j)_{j\geq 1}$ by $a_j = \frac{3^j-1}{4^j}$, then for $j \geq 2$ we have

$$a_j - a_{j+1} = \frac{3^j - 3}{4^{j+1}} > 0,$$

so the sequence is decreasing. This leaves only the cases $m = l$ and $\{l, m\} = \{1, 2\}$. The case $\{n, k\} = \{2, 4\}$ satisfies the initial conditions, hence the solutions are $n = k$ or $\{n, k\} = \{2, 4\}$. □

Each of the sine, cosine and tangent functions has a well-defined reciprocal function. These are called cosecant (csc), secant (sec) and cotangent (cot or ctg), respectively and are defined by

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta},$$

for every real number θ for which the corresponding fractions are well-defined. The properties of these functions are an immediate consequence of those we deduced from sin, cos and tan.

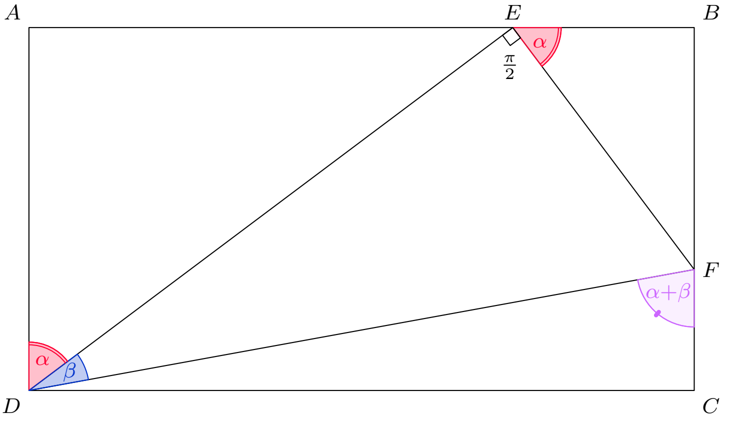
2 Trigonometric Identities

It is natural to ask how the functions which we introduced behave under angle addition, for example whether there is a formula for $\sin(\alpha+\beta)$ which relates it to $\sin\alpha$ and $\sin\beta$. It is not hard to see that we **cannot** have

$$\sin(\alpha+\beta) = \sin\alpha + \sin\beta,$$

as for example taking $\alpha = \beta = \frac{\pi}{6}$ would give $\frac{\sqrt{3}}{2} = 1$, which is absurd. It turns out that the formula for $\sin(\alpha+\beta)$ is more elaborate than this and we shall use a geometric approach to deduce it. An alternative approach which uses 2×2 matrices and rotations of the plane is provided in guided steps in Exercise 2.1.

We first treat the situation when α and β are such that $\alpha, \beta, \alpha+\beta \in [0, \frac{\pi}{2}]$. Consider a right triangle DEF with $\measuredangle DEF = \frac{\pi}{2}$, $\measuredangle FDE = \beta$ and $|DF| = 1$. Construct a line l_1 which passes through D and does not intersect the interior of the triangle DEF and such that l_1 and DE form an acute angle congruent to α. Now we construct a line l_2 which passes through D and is perpendicular to l_1. Let A be the foot of the perpendicular from E to l_1, C the foot of the perpendicular from F to l_2 and B the intersection of the lines AE and CF. In this way, we obtained a rectangle $ABCD$ with the triangle DEF inscribed inside it. The resulting diagram is illustrated in the figure below.



We now compute the lengths of the segments inside this rectangle. In triangle DEF, we have $|DE| = |DF|\cos\beta = \cos\beta$ and $|EF| = |DF|\sin\beta = \sin\beta$. Now in triangle ADE, one has

$$|AD| = |DE|\cos\alpha = \cos\alpha\cos\beta \quad \text{and} \quad |AE| = |DE|\sin\alpha = \sin\alpha\cos\beta.$$