Contents

Preface

Inequalities abound all fields of mathematics. The aim of *116 Algebraic Inequalities from the AwesomeMath Year-round Program* is to present elementary techniques in the theory of inequalities. We selected refined problems from Mathematical Reflections, the Art of Problem Solving website, and Gazeta Matematică. Many of the problems featured in the book were created by the authors.

In the first section, the readers will meet classical theorems including the Power Mean and AM-GM inequalities, the Cauchy-Schwarz Inequality, Hölder's theorem, the Rearrangement and Chebyshev's inequalities, Schur's inequality, Jensen's inequality, and so on. For each of the theorems, we included proofs and one or more examples with interesting and accessible solutions. All of them are intended for a wide audience spectrum: high school students and teachers, undergraduates, and anyone with a passion for mathematics.

The following sections are dedicated to the proposed problems, which are divided into introductory and advanced. The inequalities from each section are ordered increasingly by the number of variables: one, two, three, four, and multi-variables. Each problem has at least one complete solution, and many problems have multiple solutions, useful in developing the necessary array of mathematical machinery for competitions.

This book would certainly help Olympiad students who wish to prepare for the study of inequalities, a topic now of frequent use at various competitive levels. We hope the book will be a source of inspiration for proving algebraic inequalities and some of their newfound discoveries. Thanks to all Mathematical Reflections contributors and math enthusiasts who post problems on the AoPS website.

Enjoy the problems!

1

Some Classical and Some New Inequalities

1.1 Squares Are Nonnegative

One of the simplest inequalities is: $x^2 \geq 0$ for any real number x. Hence, when trying to prove that a expression is nonnegative, we try to write the expression as the square, or the sum of a number of squares. But writing an expression as a square, or as a sum of squares, is usually far from obvious. It requires a certain level of intuition and creativity, but perhaps more importantly, experience. We thus start with some introductory examples.

Example 1. Let a, b, c be real numbers such that $a + b + c \geq 0$. Prove that

$$a^3 + b^3 + c^3 \geq 3abc.$$

Solution. We have

$$\begin{aligned} a^3 + b^3 + c^3 - 3abc &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= \frac{1}{2}(a + b + c)\left[(a - b)^2 + (b - c)^2 + (c - a)^2\right] \geq 0 \end{aligned}$$

using the hypothesis. □

Note 1. This result is one way of writing the three variable case of the AM-GM Inequality. The analogous form of the two variable case is the even easier fact that $a^2 + b^2 \geq 2ab$, which simply rearranges to $(a - b)^2 \geq 0$. The solution above also established the inequality $a^2 + b^2 + c^2 \geq ab + bc + ca$, which rearranges to $(a + b + c)^2 \geq 3(ab + bc + ca)$. These four inequalities will be used very often in the rest of the text.

Example 2. (Titu Andreescu) Let m and n be integers greater than 1. Prove that

$$(m^3 - 1)(n^3 - 1) \geq 3m^2n^2 + 1.$$

Solution. The inequality rewrites

$$(mn)^3 + (-m)^3 + (-n)^3 - 3(mn)(-m)(-n) \geq 0,$$

which is equivalent to

$$\frac{1}{2}(mn - m - n)\left[(mn + m)^2 + (mn + n)^2 + (m - n)^2\right] \geq 0.$$

Clearly, all factors are nonnegative, as $mn - m - n = (m-1)(n-1) - 1 \geq 0$. The equality holds if and only if $m = n = 2$. □

Example 3. (Adrian Andreescu) Prove that for any real numbers a, b, c, d,

$$a^4 + b^4 + c^4 + d^4 - 4abcd \geq 2|(a^2 - b^2 + c^2 - d^2)(ab - cd)|.$$

Solution. We have

$$\begin{aligned} a^4 + b^4 + c^4 + d^4 - 4abcd &= (a^2 - b^2)^2 + (c^2 - d^2)^2 + 2(ab - cd)^2 \\ &\geq \frac{1}{2}\left[(a^2 - b^2) + (c^2 - d^2)\right]^2 + 2(ab - cd)^2 \\ &\geq 2|(a^2 - b^2 + c^2 - d^2)(ab - cd)|, \end{aligned}$$

where we have used the inequalities $x^2 + y^2 \geq \dfrac{1}{2}(x+y)^2$ and $x^2 + y^2 \geq 2xy$, both equivalent to $(x - y)^2 \geq 0$. □

Example 4. (Mathlinks) Let a, b be real numbers such that

$$ab(a^2 - b^2) = a^2 + b^2 + 1.$$

Find the minimum value of $a^2 + b^2$.

Solution. Using polar coordinates, i.e.

$$a = r\cos\alpha,\ \ b = r\sin\alpha,\ \ r > 0,\ \ \alpha \in [0, 2\pi),$$

we see that the given condition becomes

$$1 + r^2 = r^4 \sin\alpha\cos\alpha(\cos^2\alpha - \sin^2\alpha) = r^4 \cdot \frac{\sin 4\alpha}{4} \leq \frac{r^4}{4},$$

which means that

$$r^4 - 4r^2 - 4 \geq 0 \iff (r^2 - 2)^2 - 8 \geq 0$$

$$\iff (r^2 - 2 - 2\sqrt{2})\left(r^2 - 2 + 2\sqrt{2}\right) \geq 0.$$

Therefore

$$a^2 + b^2 = r^2 \geq 2\left(1 + \sqrt{2}\right).$$

The equality holds when $r = \sqrt{2\left(1 + \sqrt{2}\right)}$, and $\alpha \in \left\{\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}\right\}$. $\square$

Example 5. Let a, b be nonnegative real numbers such that $a + b \leq 2$. Prove that

$$(1 + a^2)(1 + b^2) \geq \left[1 + \left(\frac{a+b}{2}\right)^2\right]^2.$$

Solution. Expanding and regrouping the terms according to their degree as polynomials in a and b gives the equivalent inequality

$$a^2 + b^2 - 2\left(\frac{a+b}{2}\right)^2 \geq \left(\frac{a+b}{2}\right)^4 - a^2b^2.$$

Factoring $(a-b)^2$ out of both sides, this becomes

$$\frac{1}{2}(a-b)^2 \geq \frac{1}{16}(a-b)^2(a^2 + 6ab + b^2).$$

Hence it is enough to show that $8 \geq a^2 + 6ab + b^2$, but this follows easily by noting that $a^2 + 6ab + b^2 = 2(a+b)^2 - (a-b)^2 \leq 2(a+b)^2 \leq 8$. $\square$

Example 6. (An Zhenping) Let $a, b > 0$ such that $ab = 1$. Prove that

$$\frac{2}{a^2 + b^2 + 1} \leq \frac{1}{a^2 + b + 1} + \frac{1}{a + b^2 + 1} \leq \frac{2}{a + b + 1}.$$

Solution. We have $b = \dfrac{1}{a}$, so the inequality can be rewritten as

$$\frac{2a^2}{a^4 + a^2 + 1} \leq \frac{a}{a^3 + a + 1} + \frac{a^2}{a^3 + a^2 + 1} \leq \frac{2a}{a^2 + a + 1}.$$

First, we will prove the right inequality. We need to prove that

$$\frac{a^2}{a^4 + a^2 + a} + \frac{a^2}{a^3 + a^2 + 1} \leq \frac{2a^2}{a^3 + a^2 + a}.$$

This inequality is equivalent to each of the following

$$\frac{1}{a^4 + a^2 + a} + \frac{1}{a^3 + a^2 + 1} \leq \frac{2}{a^3 + a^2 + a},$$

$$\frac{1}{a^4+a^2+a}-\frac{1}{a^3+a^2+a}\le\frac{1}{a^3+a^2+a}-\frac{1}{a^3+a^2+1},$$
$$\frac{a^3-a^4}{(a^4+a^2+a)(a^3+a^2+a)}\le\frac{1-a}{(a^3+a^2+a)(a^3+a^2+1)},$$
$$\frac{a^2(1-a)}{a^3+a+1}\le\frac{1-a}{a^3+a^2+1},$$
$$(1-a)(a^5+a^4-a^3+a^2-a-1)\le 0,$$
$$(a-1)^2(a^4+2a^3+a^2+2a+1)\ge 0,$$

which is obviously true.
To prove the left inequality, we proceed similarly. We have to show that

$$\frac{2a^2}{a^4+a^2+1}\le\frac{a^2}{a^4+a^2+a}+\frac{a^2}{a^3+a^2+1},$$

which is equivalent to each of the following

$$\frac{2}{a^4+a^2+1}\le\frac{1}{a^4+a^2+a}+\frac{1}{a^3+a^2+1},$$
$$\frac{1}{a^4+a^2+1}-\frac{1}{a^3+a^2+1}\le\frac{1}{a^4+a^2+a}-\frac{1}{a^4+a^2+1},$$
$$\frac{a^3-a^4}{(a^4+a^2+1)(a^3+a^2+1)}\le\frac{1-a}{(a^4+a^2+a)(a^4+a^2+1)},$$
$$\frac{a^3(1-a)}{a^3+a^2+1}\le\frac{1-a}{a^4+a^2+a},$$
$$(1-a)(a^7+a^5+a^4-a^3-a^2-1)\le 0,$$
$$(a-1)^2(a^6+a^5+2a^4+3a^3+2a^2+a+1)\ge 0,$$

obviously true.
The equality holds in both sides of the inequality when $a=b=1$. $\square$

Example 7. (Titu Andreescu) Prove that for all real numbers a,b,c,d,e,

$$2a^2+b^2+3c^2+d^2+2e^2\ge 2(ab-bc-cd-de+ea).$$

Solution. The inequality is equivalent to

$$2e^2-2e(a-d)+2a^2+b^2+3c^2+d^2-2ab+2bc+2cd\ge 0,$$

or

$$2\left(e-\frac{a-d}{2}\right)^2+2\left(\frac{a+d}{2}\right)^2+2(a+d)c+2c^2+a^2+b^2+c^2-2ab-2ac+2bc\ge 0.$$

This reduces to

$$2\left(e-\frac{a-d}{2}\right)^2+2\left(\frac{a+d}{2}+c\right)^2+(a-b-c)^2\geq 0,$$

which is clear.
Equality holds if and only if

$$\begin{cases} 2e-a+d=0 \\ a+d+2c=0 \\ a-b-c=0 \end{cases}$$

that is, if and only if

$$\begin{cases} a=d+2e \\ b=2d+3e \\ c=-(d+e). \end{cases}$$

□

Example 8. Let a, b, c be nonnegative real numbers such that $a+b+c=\frac{3}{2}$. Prove that

$$(1+a^2)(1+b^2)(1+c^2)\geq\frac{125}{64}.$$

Solution. We can use the identity

$$(a^2+1)(b^2+1)(c^2+1)=(a+b+c-abc)^2+(1-ab-bc-ca)^2.$$

From *Example 1*, we have

$$a+b+c=\left(\sqrt[3]{a}\right)^3+\left(\sqrt[3]{b}\right)^3+\left(\sqrt[3]{c}\right)^3\geq 3\sqrt[3]{abc},$$

which means

$$abc\leq\frac{1}{8},$$

and

$$(a+b+c)^2\geq 3(ab+bc+ca),$$

which implies

$$ab+bc+ca\leq\frac{3}{4}.$$

Therefore

$$a+b+c-abc\geq\frac{3}{2}-\frac{1}{8}=\frac{11}{8},$$

and

$$1-ab-bc-ca\geq 1-\frac{3}{4}=\frac{1}{4}.$$

Hence,

$$(a^2+1)(b^2+1)(c^2+1) \geq \frac{121}{64} + \frac{1}{16} = \frac{125}{64}.$$

The equality holds when $a = b = c = \dfrac{1}{2}$. $\square$

Example 9. (Mathlinks) Let x, y, z be nonnegative real numbers. Prove that

$$(x+2y+3z)\left(x^2+y^2+z^2\right) \geq \frac{20-2\sqrt{2}}{27}(x+y+z)^3.$$

Solution. First, we observe that if $z \geq x$ then

$$LHS \geq 2(x+y+z)(x^2+y^2+z^2) \geq \frac{2}{3}(x+y+z)^3 \geq RHS$$

hence, it remains to prove the inequality in the case $z < x$.
Without loss of generality assume that $x+y+z=3$.
Let $x = a+1,\ y = b+1,\ z = c+1$, which means $a+b+c=0$, and $-1 \leq a, b, c \leq 2$.
The our inequality becomes

$$(6+c-a)(2a^2+2ac+2c^2+3) \geq 20-2\sqrt{2},$$

or

$$3(c-a)+12(a^2+ac+c^2)+2(c-a)(a^2+ac+c^2) \geq 2-2\sqrt{2}.$$

Let $u = a-c$, $v = a+c$, so $0 < u \leq 3$ and the last inequality is written as

$$-3u+9v^2+3u^2-2u\left(\frac{3v^2}{4}+\frac{u^2}{4}\right) \geq 2-2\sqrt{2},$$

or

$$(2+2\sqrt{2}-u)(u-2+\sqrt{2})^2+3v^2(6-u) \geq 0,$$

which is obviously true.
The equality holds when $u = 2-\sqrt{2}, v = 0$. This translates into

$$x = 2-\frac{1}{\sqrt{2}},\ y = 1,\ z = \frac{1}{\sqrt{2}},$$

and restoring generality by dropping the assumption $x+y+z=3$, we see that equality holds for any multiple of this example. $\square$

∇

1.2 Inequalities Between Means

One of the most important inequalities, which has enormously many applications is the inequalities between means:

Theorem 1. Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Then,

$$Q_n \geq A_n \geq G_n \geq H_n,$$

where

$Q_n = \sqrt{\dfrac{a_1^2 + a_2^2 + \ldots + a_n^2}{n}}$ is the root-mean-square,

$A_n = \dfrac{a_1 + a_2 + \ldots + a_n}{n}$ is the arithmetic mean,

$G_n = \sqrt[n]{a_1 a_2 \ldots a_n}$ is the geometric mean,

$H_n = \dfrac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}$ is the harmonic mean of the numbers $a_1, a_2, \ldots, a_n$.

Solution. First, we prove the inequality (AM-GM Inequality)

$$A_n \geq G_n. \tag{1}$$

For $n = 2$, it is easy to see that

$$\frac{a_1 + a_2}{2} - \sqrt{a_1 a_2} = \frac{\left(\sqrt{a_1} - \sqrt{a_2}\right)^2}{2} \geq 0.$$

Assume that it is valid for $n = k$, i.e.,

$$A_k \geq G_k.$$

By induction,

$$A = \frac{a_{k+1} + (k-1)A_{k+1}}{k} \geq \left(a_{k+1} A_{k+1}^{k-1}\right)^{\frac{1}{k}} = G.$$

It follows that

$$A_{k+1} = \frac{A_k + A}{2} \geq \sqrt{A_k A} \geq \sqrt{G_k G} = \sqrt{\left(G_{k+1}^{k+1} A_{k+1}^{k-1}\right)^{\frac{1}{k}}},$$

so, we get

$$A_{k+1}^{2k} \geq G_{k+1}^{k+1} A_{k+1}^{k-1}$$

or

$$A_{k+1} \geq G_{k+1}.$$