Preface

Polynomials form the cornerstone of modern mathematics and other discrete fields. Apart from the direct link to Algebra, and subsequently, Calculus, they also appear frequently in many branches of sciences. The ubiquity of polynomials and their ability to characterize complex patterns let us better understand generalizations, theorems, and elegant paths to solutions that they provide. We strive to showcase the true beauty of polynomials through a well-thought collection of problems from mathematics competitions and intuitive lectures that follow the sub-topics. Thus, we present a view of polynomials that incorporates various techniques paired with the favorite themes that show up in math contests.

First, the two introductory chapters detail the machinery and notation used to characterize polynomials. Factorization identities, the notion of GCD, composition, types of roots, and the Intermediate Value Theorem, to name some, lay the foundational groundwork that the reader will use to internalize the contents of the later chapters. To give the student a sense of direction or relation to real world problems, polynomials and their properties are discussed based on the number of variables, starting with second degree and up to fourth degree representations. Two additional chapters that deal with key theorems and some applications of polynomials in Number Theory follow. The book concludes with the Introductory and Advanced problem sections, along with their respective solution segments. There are significantly more than 117 problems that are selected with strict considerations, with an abundance of great problems in the first seven chapters. It actually features more than 180 problems in the theory component as well. Most of the harder and classic problems have more than one solution to familiarize the reader with a variety of approaches. We have also added problems proposed recently in journals and competitions for the student to better consolidate and assimilate these techniques.

Since mathematics competitions almost always include algebraic problems, polynomials can form a large subset of the necessary material one should be acquainted with in order to succeed. The authors consider polynomials a dear and classic art. Through the problems, lecture, and theory, we do our best to transfer most of the knowledge, strategies, and tricks explored to our readers. This book is best suited for AMC 10/12, AIME, and USAMO/JMO competitors, and is the first volume in a three-book series.

Many thanks to Richard Stong and Adrian Andreescu for significant improvement of the text. We would also like to thank Alexandr Khrabrov, Peter Boyvalenkov, Gregor Dolinar, and Bayasagalan Banzarach for their contribution.

Enjoy the problems!

The authors

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1 Basic properties of polynomials - Part I

A *polynomial* is an expression of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = \sum_{i=0}^{n} a_i x^i.$$

The numbers a_i are said the *coefficients* of the polynomial $P(x)$. Usually, we consider a_i in $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and we say that the polynomial has integer, rational, real or complex coefficients, respectively. We denote by $\mathbb{Z}[x]$, $\mathbb{Q}[x], \mathbb{R}[x], \mathbb{C}[x]$ the set of polynomials with integer, rational, real or complex coefficients, respectively.

The coefficient a_0 is said the *constant term*.

From the definition, it follows that two polynomials

$$P(x) = \sum_{i=0}^{n} a_i x^i \text{ and } Q(x) = \sum_{j=0}^{m} b_j x^j$$

are equal if and only if $a_i = b_i$ for all i (if $m > n$, then $b_{n+1} = \ldots = b_m = 0$).

We define the *degree* of polynomial $P(x) = \sum\limits_{i=0}^{n} a_i x^i$ as the greatest integer i such that $a_i \neq 0$ and we denote the degree by $\deg P(x)$. If i is the greatest integer i such that $a_i \neq 0$, we say that a_i is the *leading coefficient* of $P(x)$. If the leading coefficient is equal to 1, we say that the polynomial is *monic*. Notice that the degree of a constant polynomial $P(x) = a_0 \neq 0$ is zero. We don't give any degree to the *zero polynomial* $P(x) \equiv 0$ (i.e. the polynomial whose coefficients are all zeros)[1]. Usually, we omit the terms having zero as a coefficient. For example, we write the polynomial $0x^3 + 1x^2 + 2x + 0$ as $x^2 + 2x$. This is clearly an example of a monic polynomial of degree 2.

We can perform some operations on polynomials.

For example, if $P(x) = \sum\limits_{i=0}^{n} a_i x^i$ and $Q(x) = \sum\limits_{j=0}^{m} b_j x^j$ are two polynomials and $m \geq n$, then the sum of $P(x)$ and $Q(x)$ is defined by

$$P(x) + Q(x) = \sum_{h=0}^{m} (a_h + b_h) x^h$$

and the product of $P(x)$ and $Q(x)$ is defined by

$$P(x)Q(x) = \sum_{h=0}^{n+m} \left(\sum_{i+j=h} a_i b_j \right) x^h.$$

[1]By convention, we can also assign to the zero polynomial the degree $-\infty$.

We have the following.

Theorem

Let $P(x)$ and $Q(x)$ be two polynomials and let k be a positive integer. Then,

1. $\deg(P(x)Q(x)) = \deg P(x) + \deg Q(x)$
2. $\deg(P(x) + Q(x)) \leq \max(\deg P(x), \deg Q(x))$
3. $\deg[(P(x))^k] = k \cdot \deg P(x)$.

1.1 Identities

The term *identity* designates an equality that holds for all allowed values of the unknowns it contains (usually all real numbers or all complex neumbers). When it is clear from context one often omits explicitly specifying the allowed range for the unknowns. For example, the following equations are identities:

$$a^2 - b^2 = (a-b)(a+b),$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2),$$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc),$$

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a} = \frac{b^2}{a+b} + \frac{c^2}{b+c} + \frac{a^2}{a+c}.$$

The first three of these hold for all real (or all complex) values of a, b, c, whereas the last only holds if none of $a+b$, $b+c$, and $c+a$ vanishes. However, the equalities below do not meet this criterion since they do not hold universally:

$$2x + 1 = 5,$$

$$\frac{1}{x-2} + \frac{1}{x} = 3,$$

$$a^3 + b^3 + c^3 = 3abc.$$

Identities are the pillars of our mathematical computations. They are commonly encountered in mathematics competitions, where many problems require such knowledge.

Here we record some of the most important identities. It is important that, in order to enhance your strength in working with algebraic expressions, you force yourself to learn these identities.

Useful Identities

Conjugate Identity:

$$a^2 - b^2 = (a-b)(a+b)$$

Square Identity I:

$$(a+b)^2 = a^2 + 2ab + b^2$$

Square Identity II:

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+bc+ca)$$

Power Difference:

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$$

Power Sum:

$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \cdots + b^{n-1}) \qquad \text{if } n \text{ is odd.}$$

Euler's Identity:

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

Binomial Identity:

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b + \cdots + \binom{n}{k} a^{n-k}b^k + \cdots + b^n.$$

Comment I. *Euler's Identity* implies that if $a+b+c=0$, then

$$a^3 + b^3 + c^3 = 3abc.$$

Comment II. The general case of the *Square Identity* is of special interest. That is, let $x_1, \ldots, x_n$ be n numbers. Then,

$$(x_1 + \cdots + x_n)^2 = x_1^2 + \cdots + x_n^2 + 2 \sum_{1 \le i < j \le n} x_i x_j$$

For example, when $n = 4$, we have,

$$(a+b+c+d)^2 = a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad + bc + bd + cd).$$

Comment III. The general case of *Binomial Identity* is of special interest.

Let $x_1, \ldots, x_n$ be n numbers and let m be a positive integer. Then,

$$(x_1 + \cdots + x_n)^m = \sum_{\substack{0 \le i_1, i_2, \ldots, i_n \le m \\ i_1 + i_2 + \ldots + i_n = m}} \binom{m}{i_1, i_2, \ldots, i_n} x_1^{i_1} x_2^{i_2} \cdot \ldots \cdot x_n^{i_n}.$$

The above identity is called the *Multinomial Identity.* For example, when $n = m = 3$ we have

$$(a + b + c)^3 = a^3 + b^3 + c^3 + 3\left(a^2 b + b^2 c + c^2 a + b^2 a + c^2 b + a^2 c\right) + 6abc.$$

Many elementary algebra problems are greatly simplified by the above formulas. Nevertheless, we will also see some of their creative applications in more complex solutions.

Example 1.1. (Saint-Petersburg Mathematical Olympiad 1999) Let $n \ge 3$ be a positive integer. Prove that n^{12} can be represented as a sum of cubes of three natural numbers.
Solution. Note that

$$(a - 3b)^3 = a^3 - 9a^2 b + 27ab^2 - 27b^3.$$

Multiplying both sides by a, we have

$$a(a - 3b)^3 = a^4 - 9b(a - b)^3 - 9b^4.$$

Now, set $a = n^3$ and $b = 3m^3$. Then,

$$n^{12} = n^3(n^3 - 9m^3)^3 + 27m^3(n^3 - 3m^3)^3 + 27m^3(3m^3)^3.$$

This is equivalent to

$$n^{12} = (n^4 - 9nm^3)^3 + (3mn^3 - 9m^4)^3 + (9m^4)^3.$$

Example 1.2. (Serbian Mathematical Olympiad 2012) If $x+y+z = 0$ and $x^2 + y^2 + z^2 = 6$ what is the maximal value of

$$|(x - y)(y - z)(z - x)|?$$

Solution. Set $z = -x - y$. We find that $2x^2 + 2xy + 2y^2 = 6$, i.e., $x^2 + xy + y^2 = 3$. Likewise, $y^2 + yz + z^2 = z^2 + zx + x^2 = 3$, thus $xy + yz + zx = -3$. Now,

$$(x - y)^2 = x^2 + xy + y^2 - 3xy = 3 - 3xy,$$

thus

$$((x - y)(y - z)(x - z))^2 = (3 - 3xy)(3 - 3yz)(3 - 3zx)$$

The right-hand side is equal to

$$\begin{aligned} 27\left(1-xy-yz-zx+xyz(x+y+z)-x^2y^2z^2\right) &= 27\left(4-x^2y^2z^2\right) \\ &\leq 27\cdot 4. \end{aligned}$$

Therefore, $|(x-y)(y-z)(z-x)| \leq 6\sqrt{3}$. Since this value is attained when $\{x,y,z\}=\{0,\sqrt{3},-\sqrt{3}\}$, this is the maximum value.

Example 1.3. (Mathematics and Youth Journal 2003) Solve the system of equations

$$\begin{aligned} x^2(y+z)^2 &= \left(3x^2+x+1\right)y^2z^2 \\ y^2(z+x)^2 &= \left(4y^2+y+1\right)z^2x^2 \\ z^2(x+y)^2 &= \left(5z^2+z+1\right)x^2y^2. \end{aligned}$$

Solution. If $x=0$, then all three equations reduce to $y^2z^2=0$ and we find the solutions $(x,y,z)=(0,t,0)$ or $(0,0,t)$ for any real t. Similarly, the cases $y=0$ or $z=0$ add one more family of solutions $(x,y,z)=(t,0,0)$.
Now assume $xyz\neq 0$. Setting, $x=\frac{1}{a}$, $y=\frac{1}{b}$, $z=\frac{1}{c}$, we obtain the following system of equations

$$\begin{aligned} (b+c)^2 &= 3+a+a^2 \\ (c+a)^2 &= 4+b+b^2 \\ (a+b)^2 &= 5+c+c^2 \end{aligned}$$

By summing them, we find that $(a+b+c)^2-(a+b+c)-12=0$. We substitute $t=a+b+c$, and simplify to $t^2-t-12=0$, which gives $t=4$ or $t=-3$. If $t=4$, then $a+b+c=4$, and by substituting into the above equations, we find $(4-a)^2=3+a+a^2$, i.e., $9a=13$, so $x=9/13$. By the same argument $y=3/4$ and $z=9/11$. If $t=3$, then $a+b+c=-3$, and by the same argument we find the solution $(x,y,z)=(-5/6,-1,-5/4)$.

Example 1.4. (Ivan Tonov - Bulgarian Mathematical Olympiad 2008)
If the following equation is an identity

$$(x+y)^{2n+1}-x^{2n+1}-y^{2n+1}=(2n+1)xy(x+y)(x^2+xy+y^2)^{n-1},$$

find the value of n.
Solution. Let $x=y=1$. Then $2^{2n+1}-2=2(2n+1)3^{n-1}$, thus

$$2^{2n}-1=(2n+1)3^{n-1}.$$

We prove that the equality does not occur for $n>3$. Write the equality as

$$\left(\frac{4}{3}\right)^n=\frac{2n+1}{3}+\frac{1}{3^n}.$$

Then, for $n > 3$ we have

$$\begin{aligned}\frac{2n+1}{3}+\frac{1}{3^n} &= \left(\frac{4}{3}\right)^n \\ &= \left(1+\frac{1}{3}\right)^n \\ &= 1+\frac{n}{3}+\frac{n(n-1)}{2\cdot 3^2}+\ldots+\frac{1}{3^n} \\ &> 1+\frac{n}{3}+\frac{n(n-1)}{2\cdot 3^2}+\frac{1}{3^n}.\end{aligned}$$

Now

$$\frac{2n+1}{3} > 1+\frac{n}{3}+\frac{n(n-1)}{2\cdot 3^2},$$

which leads to the inequality $n^2-7n+12=(n-3)(n-4)<0$, which is false for $n \geq 4$. If $n \in \{1,2,3\}$, the equation is indeed an identity.
Note. We can also use a number theory argument to refute the identity case: $2^{2n}-1=(2n+1)3^{n-1}$. Let $v_3(N)$ denote the exact number of times the prime 3 divides N. Since $v_3(2^{2n}-1)=v_3(4^n-1)\geq n-1$, we find that $3^{n-2} \mid n$, which is false for $n > 3$.

1.2 The coefficients of x^d in polynomial products

Suppose that we want to compute the coefficient of x^{50} in the following product

$$\left(1+2x+3x^2+\cdots+101x^{100}\right)\left(1+x+x^2+\cdots+x^{25}\right).$$

For sake of this, we need to study in which ways a monomial from first factor and a monomial from second factor generate the term x^{50}. That is,

$$x^{50}=x^{50}\cdot 1=x^{49}\cdot x=\cdots=x^{25}\cdot x^{25}.$$

Hence, the coefficient of x^{50} is the sum of coefficients of the constructed monomials and is equal to

$$51+50+\ldots+26=1001.$$

Example 1.5. (Navid Safaei) Let k be a positive integer such that

$$1+x^k+x^{2k}=(1+a_1x+x^2)(1+a_2x+x^2)\cdot\ldots\cdot(1+a_kx+x^2).$$

Find the value of $a_1^2+\ldots+a_k^2$.
Solution. If $k=1$, then $a_1=1$ and the desired value is 1. If $k=2$, then

$$1+x^2+x^4=(1+a_1x+x^2)(1+a_2x+x^2).$$

Comparing the coefficients of x^2 and x in both sides, we find that

$$a_1 + a_2 = 0, \quad a_1 a_2 = -1$$

i.e.

$$a_1^2 + a_2^2 = (a_1 + a_2)^2 - 2a_1 a_2 = 2.$$

Moreover, one can find that $\{a_1, a_2\} = \{1, -1\}$. Hence,

$$1 + x^2 + x^4 = (1 - x + x^2)(1 + x + x^2).$$

Assume now $k \geq 3$. Then, the coefficients of x and x^2 in the product

$$(1 + a_1 x + x^2)(1 + a_2 x + x^2) \cdot \ldots \cdot (1 + a_k x + x^2)$$

must be zero. Examining the aforementioned coefficients, one can find that

$$a_1 + \ldots + a_k = 0$$

and

$$k + \sum_{1 \leq i < j \leq k} a_i a_j = 0,$$

i.e., $\sum\limits_{1 \leq i < j \leq k} a_i a_j = -k$. Hence, by the generalization of the square identity, we find that

$$a_1^2 + \ldots + a_k^2 = (a_1 + \ldots + a_k)^2 - 2 \sum_{1 \leq i < j \leq k} a_i a_j = 2k.$$

Hence, the answer is k if $k = 1, 2$ and $2k$ if $k \geq 3$.

Example 1.6. Find the coefficient of x^{100} in the expression:

$$(1 + x + x^2 + \ldots + x^{100})^3.$$

Solution. Note that

$$(1+x+x^2+\ldots+x^{100})^3 = (1+x+\ldots+x^{100})(1+x+\ldots+x^{100})(1+x+\ldots+x^{100}).$$

Thus, a term x^{100} can arise from a product of the form $x^a x^b x^c$ from the respective factors where $a + b + c = 100$ and $a, b, c \geq 0$. Let $a = 0$. Then $b + c = 100$ and there are 101 cases. If $a = 1$, then $b + c = 99$ and there are 100 cases. Similarly when $a = 100$, then $b + c = 0$, and there is only one case. Hence the total number of cases is

$$1 + 2 + \ldots + 101 = 5151.$$

Example 1.7. (Federico Poloni - Italian Mathematical Olympiad 2013, Local Round) Let $P(x)$ and $Q(x)$ be two trinomials. How many non-zero monomials does their product $P(x)Q(x)$ have at least?
Solution. Assume that

$$P(x) = Ax^R + Bx^S + Cx^T, \quad Q(x) = ax^r + bx^s + cx^t$$

where, $A, B, C, a, b, c \neq 0$ and $R, S, T, r, s, t \geq 0$ are integers, $R \neq S \neq T$ and $r \neq s \neq t$. Without loss of generality assume that $R > S > T$ and $r > s > t$. Then,

$$P(x)Q(x) = AaAx^{R+r} + \ldots + Ccx^{T+t},$$

a product with 9 terms. It is clear that the monomials x^{R+r} and x^{T+t} cannot cancel out because of minimality and maximality. Thus, the product has at least two terms. Now we provide an example with exactly two terms, that is, consider the polynomial $x^4 + 4$. If we factor it, we find that

$$x^4 + 4 = (x^2 - 2x + 2)(x^2 + 2x + 2).$$

Thus, the answer is 2.

Example 1.8. Define a family of polynomials recursively by

$$\begin{aligned} P_0(x) &= x - 2, \\ P_k(x) &= P_{k-1}^2(x) - 2 \quad \text{if } k \geq 1. \end{aligned}$$

Find the coefficient of x^2 in $P_k(x)$ in terms of k.
Solution. Note that for all $k \geq 1$ we have $P_k(0) = 2$, thus

$$P_k(x) = 2 + a_k x + b_k x^2 + \ldots,$$

then $P_{k+1}(x) = 2 + a_{k+1}x + b_{k+1}x^2 + \ldots = (2 + a_k x + b_k x^2 + \ldots)^2 - 2$.
An easy calculation shows that $a_{k+1} = 4a_k$ and $b_{k+1} = a_k^2 + 4b_k$.
Since $a_1 = -4$, we find that $a_k = -4^k$, so

$$b_k = 4^{2k-2} + \ldots + 4^{k-1} = 4^{k-1}(1 + 4 + \ldots + 4^{k-1}) = \frac{4^{2k-1} - 4^{k-1}}{3}.$$

Example 1.9. (AIME 2016) Let $P(x) = 1 - \dfrac{x}{3} + \dfrac{x^2}{6}$ and define

$$Q(x) = P(x)\,P\left(x^3\right)P\left(x^5\right)P\left(x^7\right)P\left(x^9\right) = \sum_{i=0}^{50} a_i x^i.$$

Find $\displaystyle\sum_{i=0}^{50} |a_i|$.

Solution. Note that all the coefficients of the polynomial $P(-x)$ are nonnegative, indeed its coefficients are the absolute values of the coefficients of the polynomial $P(x)$. Thus all the coefficients of the polynomial

$$Q(-x) = P(-x)P(-x^3)P(-x^7)P(-x^9)$$

are nonnegative and are the absolute values of the coefficients of the polynomial $Q(x)$. Hence

$$\sum_{i=0}^{50} |a_i| = Q(-1) = P(-1)^5 = \left(\frac{3}{2}\right)^5.$$

Example 1.10. (V.A. Senderov - Russian Mathematical Olympiad 2008) Find all positive integers n such that there exist nonzero a, b, c, d such that $(ax+b)^{1000} - (cx+d)^{1000}$ has exactly n nonzero coefficients.

Solution. The answer is $n \in \{500, 1000, 1001\}$. Indeed for $n = 1001$ consider the expression $(2x+2)^{1000} - (x+1)^{1000}$ and for $n = 1000$ consider the expression $(2x+1)^{1000} - (x+1)^{1000}$. If we have more than one zero coefficient then there exists two coefficients, say the coefficients of x^r and x^t such that

$$a^r b^{1000-r} = c^r d^{1000-r}, \quad a^t b^{1000-t} = c^t d^{1000-t}.$$

Thus

$$\left(\frac{ad}{bc}\right)^r = \left(\frac{d}{b}\right)^{1000} = \left(\frac{ad}{bc}\right)^t,$$

so $\left|\frac{ad}{bc}\right| = \left|\frac{d}{b}\right| = 1$, which gives also $\left|\frac{a}{c}\right| = 1$. Now it is clear that if we replace the polynomial $ax+b$ with $-ax-b$, the condition does not change. Thus without loss of generality we assume that $a/c = 1$ and we have two cases. If $d/b = 1$, then we have $(ax+b)^{1000} - (ax+b)^{1000}$ which has only zero coefficients, a contradiction. If $d/b = -1$, then we have $(ax+b)^{1000} - (ax-b)^{1000}$, which has exactly 500 nonzero coefficients.

Example 1.11. (Tournament of Towns 2012) Let $P(0) = 1$ and

$$P(x)^2 = 1 + x + x^{100}Q(x).$$

Find the coefficients of x^{99} in the expansion of $(1 + P(x))^{100}$.

Solution. Note that in the expansion of $\big(1 + P(x)\big)^{100} + \big(1 - P(x)\big)^{100}$ we have only even powers of $P(x)$. Indeed, the above expression is a polynomial in $P(x)^2 = 1 + x + x^{100}Q(x)$ of degree 50. Taking the equation modulo x^{100}, the highest coefficient of the expression is $2(1+x)^{50}$; thus there is no x^{99} term in the expansion. Moreover, since $P(0) = 1$, one can find that the polynomial $P(x) - 1$ is divisible by x. Hence, $(1 - P(x))^{100} = x^{100}R(x)$, for some polynomial $R(x)$, which implies that there is no x^{99} in the expansion. Thus there doesn't exist any x^{99} in the expansion of $(1+P(x))^{100}$ too.

Example 1.12. (Moscow Mathematical Olympiad 1997) Let

$$1+x+x^2+\ldots+x^{n-1}=F(x)G(x),$$

where $n>1$ and where F and G are polynomials, whose coefficients are zeros and ones. Prove that one of the polynomials F and G can be represented in the form $(1+x+x^2+\ldots+x^{k-1})T(x)$, where $k>1$ and where T is also a polynomial whose coefficients are zeros and ones.

Solution. Set $F(x)=a_0+a_1x+\cdots$ and $G(x)=b_0+b_1x+\cdots$. From the constant term we get $a_0b_0=1$, which gives $a_0=b_0=1$. From the x coefficient we therefore get $a_1+b_1=1$. Without loss of generality, assume that $a_1=1$ and $b_1=0$. If $G(x)=1$, then we are done, so we may assume there is some least non-zero monomial in $G(x)$, say x^k, so that $G(x)=1+x^k+\cdots$. Looking at the coefficients of x^i for $i=2,\ldots,k$, we conclude that $a_0=a_1=a_2=\cdots=a_{k-1}=1$ and $a_k=0$.

Now we will show that every monomial in G is of the form x^{kr} for some r (or in terms of polynomials $G(x)=Q(x^k)$ for some polynomial Q with coefficients zero and one) and that every non-zero monomial in F occurs in a run of nonzero monomials of the form $x^{kr}+x^{kr+1}+x^{kr+2}+\cdots+x^{kr+(k-1)}$ (or in terms of polynomials $F(x)=(1+x+x^2+\cdots+x^{k-1})P(x^k)$ for some polynomial $P(x)$ with coefficients zero and one). Note that this will solve the problem by setting $T(x)=P(x^k)$.

Suppose on the contrary that this is not the case. Then there is some lowest degree monomial where it fails. There are two ways this can happen.

If the first bad monomial is in G, then there is some monomial x^{kr+s} in G where $0<s<k$. Since the product $F(x)G(x)$ contains the monomial x^{kr} there must be some monomial x^a in F and some monomial x^b in G with $a+b=kr$. Since $a,b<kr+s$ and the monomial x^{kr+s} is the first deviation from our proposed pattern, it follows that $b=kr'$ for some $r'\le r$ and hence $a=k(r-r')$. Again since this is x^{kr+s} was the first bad monomial, this must begin a run of monomials $x^{k(r-r')}+x^{k(r-r')+1}+\cdots+x^{k(r-r')+(k-1)}$ in F. But then the coefficient of x^{kr+s} in $F(x)G(x)$ gets a contribution of 1 from $x^{k(r-r')+s}\cdot x^{kr'}$ and another contribution of 1 form $x^0\cdot x^{kr+s}$. Since any additional contributions would only be positive, the coefficient of x^{kr+s} in the product will be at least 2, a contradiction.

If the first bad monomial is in F, then there is some r such that F contains only a proper subset of the monomials $x^{kr},x^{kr+1},\ldots,x^{kr+k-1}$. Say it contains x^{kr+i} but not x^{kr+j} for some $0\le i,j<k$. The monomial x^{kr+j} must occur in $F(x)G(x)$, say as $x^a\cdot x^b$. Since the run $x^{kr},\ldots,x^{rk+k-1}$ contains the smallest degree bad monomial, we must have $b=kr'$ for some $1\le r'\le r$ and hence

$a = k(r-r') + j$. Since this is smaller, F must contain the entire run

$$x^{k(r-r')} + x^{k(r-r')+1} + \cdots + x^{k(r-r')+k-1}.$$

Now look at how many times the monomial x^{kr+i} occurs in $F(x)G(x)$. It occurs once as $x^{k(r-r')+i} \cdot x^{kr'}$ and once as $x^{kr+i} \cdot x^0$. Thus it occurs at least twice, a contradiction.

Example 1.13. (Moscow Mathematical Olympiad 1994) Is there a polynomial $P(x)$ with a negative coefficient while all the coefficients of any power $(P(x))^n$ are positive for $n > 1$?

Solution. The answer is yes. Let $P(x) = a_d x^d + \ldots + a_0$ have positive coefficients. Then all of its powers have positive coefficients.

Assume $f(x) = x^4 + x^3 + x + 1$ and set $g(x) = f(x) - \varepsilon x^2$ for some $\varepsilon > 0$ sufficiently small. Then all the coefficients of $(g(x))^2$ and $(g(x))^3$ are close to the coefficients of

$$(f(x))^2 = x^8 + 2x^7 + x^6 + 2x^5 + 4x^4 + 2x^3 + x^2 + 2x + 1$$

and

$$(f(x))^3 = x^{12}+3x^{11}+3x^{10}+4x^9+9x^8+9x^7+6x^6+9x^5+9x^4+4x^3+3x^2+3x+1.$$

The coefficients of $(f(x))^2$ and $(f(x))^3$ are all positive and thus the coefficients of $(g(x))^2$ and $(g(x))^3$ must be positive. Then, since all positive integers n can be written as $n = 2a + 3b$ for some nonnegative integers a, b, all the powers $(g(x))^n$ have only positive coefficients.

1.3 Factoring and its implications

Factoring of an algebraic expression means the decomposition of the original expression into a product of expressions with smaller degrees. We have two main tools for this: grouping and using identities. Applying the former tool includes dividing the expression into groups with a common factors.

For example,

$$a^2 + ab + bc + ca = (a^2 + ab) + (ac + bc) = a(a+b) + c(a+b) = (a+c)(a+b).$$

Example 1.14. Factor the following expression:

$$xyz + 3xy + 2xz - yz + 6x - 3y - 2z - 6.$$

Solution. We group the above expression as

$$(xyz + 3xy) + (2xz + 6x) - (yz + 3y) - (2z + 6).$$

Now we can factor out a $z+3$ from every group,

$$(z+3)(xy+2x-y-2).$$

Again, we can group the expression $xy+2x-y-2$ as

$$x(y+2)-(y+2)=(x-1)(y+2).$$

Therefore, our expression can be factored as

$$(x-1)(y+2)(z+3).$$

The latter case refers to identities for factoring. For example, we can factor the expression

$$\left(a^2-b^2\right)^3+\left(b^2-c^2\right)^3+\left(c^2-a^2\right)^3.$$

First, note that $a^2-b^2+b^2-c^2+c^2-a^2=0$. Hence by Euler's identity,

$$\left(a^2-b^2\right)^3+\left(b^2-c^2\right)^3+\left(c^2-a^2\right)^3=3\left(a^2-b^2\right)\left(b^2-c^2\right)\left(c^2-a^2\right).$$

Now by the conjugate identity, we find that

$$3\left(a^2-b^2\right)\left(b^2-c^2\right)\left(c^2-a^2\right)=3\left(a-b\right)\left(b-c\right)\left(c-a\right)\left(a+b\right)\left(b+c\right)\left(c+a\right).$$

Example 1.15. (Mathematics and Youth Journal 2004) Solve the system of equations

$$\begin{aligned} x^3+y^3 &= 4y^2-5y+3x+4 \\ 2y^3+z^3 &= 4z^2-5z+6y+6 \\ 3z^3+x^3 &= 4x^2-5x+9z+8. \end{aligned}$$

Solution. Write the system as:

$$\begin{aligned} x^3-3x-2 &= -y^3+4y^2-5y+2 \\ 2y^3-6y-4 &= -z^3+4z^2-5z+2 \\ 3z^3-9z-6 &= -x^3+4x^2-5x+2. \end{aligned}$$

Both sides can now be factored as

$$\begin{aligned} (x-2)\left(x+1\right)^2 &= (2-y)(y-1)^2 \\ 2\left(y-2\right)\left(y+1\right)^2 &= (2-z)(z-1)^2 \\ 3\left(z-2\right)\left(z+1\right)^2 &= (2-x)(x-1)^2 \end{aligned}$$

Now if $x=2$, then $y=2$ or $y=1$. If $y=2$, then from the second and the third equation, we get $z=2$. If $y=1$, then comparing the second and the