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Preface

Inequalities permeate all fields of mathematics. The aim of *118 Inequalities for Mathematics Competitions* is to present selected techniques in the field. We chose refined problems from Mathematical Reflections, the Art of Problem Solving website, and the Romanian journal Gazeta Matematică. Many of the problems featured in the book were created by the authors.

In the first section, the readers will encounter the Muirhead Inequality, as well as older and newer methods of proving inequalities. Among these we mention the substitution method, where, by suggestive examples, some famous inequalities are explored through homogenization, normalization, and typical substitutions in practical problems. Some of these include substitutions that transform a geometric inequality into an algebraic one and vice versa. Another method presented is the tangent line method, a powerful tool used to ease computations in the case of polynomial or rational functions. Also, to form a thorough intuition, we provide graphical representations for selected examples. Undetermined coefficients and the contradiction method also guarantee the success of solving certain classes of inequalities as shown in examples. In the following two sections we present a set of strong theorems, first for symmetrical inequalities in three variables and then in several variables, some of which concur with other sources such as the *pqr* or *uvw* methods. Finally, we introduce two more recent methods known as SOS (sum of squares), SOS-Schur method, and a multitude of examples to illustrate as many aspects as possible.

All of the material presented throughout the book is intended for a wide audience: high school students, teachers, undergraduates, or and anyone with a passion for mathematics.

The subsequent sections are dedicated to the proposed problems, which are divided into introductory and advanced. The inequalities from each section are ordered increasingly by the number of variables and and the degree of difficulty. Each problem has at least one complete solution, and many problems have multiple solutions, useful in developing the necessary array of mathematical machinery for competitions.

This book would certainly help Olympiad students who wish to prepare for the study of inequalities, a topic now of frequent use at various competitive levels. We hope the book will be a source of inspiration for proving algebraic inequalities and some of their newfound applications. Thanks to all Mathematical Reflections contributors and math enthusiasts who post problems on the AoPS website.

Enjoy the problems!

1

Some Classical and Some New Inequalities

1.1 Muirhead's Inequality

This inequality is an important generalization of the AM-GM Inequality. It is a powerful tool for solving inequality problem. First we present some introductory notions and then prove the inequality in the case of three variables.

Definition 1. Let $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$ be two sequences of real numbers. We say that the sequence p majorizes the sequence q and we write $p \succ q$ or $q \prec p$, if, after a possible reordering, the terms of the sequences p and q satisfy the following three conditions:

1° $p_1 \geq p_2 \geq \ldots \geq p_n$ and $q_1 \geq q_2 \geq \ldots \geq q_n$;

2° $p_1 + p_2 + \ldots + p_k \geq q_1 + q_2 + \ldots + q_k$, for each k, $1 \leq k \leq n-1$;

3° $p_1 + p_2 + \ldots + p_n = q_1 + q_2 + \ldots + q_n$.

The first condition is obviously no restriction, since we can always rearrange the sequence. The second condition is essential. Clearly, $p \succ p$ holds for an arbitrary sequence p.
Note 1. If $p = (p_i)_{i=1}^n$ is an arbitrary sequence of nonnegative numbers, having the sum equal to 1, then

$$(1, 0, \ldots, 0) \succ (p_1, p_2, \ldots, p_n) \succ \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right).$$

Definition 2. Let $a_1, a_2, \ldots, a_n$ be positive real numbers and

$$p = (p_1, p_2, \ldots, p_n)$$

be a sequence of real numbers. The $p - mean$ of $a_1, a_2, \ldots, a_n$ is defined by

$$[p] = \frac{1}{n!} \sum_{\sigma \in S_n} a_{\sigma(1)}^{p_1} a_{\sigma(2)}^{p_2} \cdots a_{\sigma(n)}^{p_n},$$

where S_n is the set of all permutations of $\{1, 2, \ldots, n\}$.

Note 2. We have

$$[(1, 0, \ldots, 0)] = \frac{1}{n}\sum_{i=1}^{n} a_i,$$

which is the arithmetic mean of $a_1, a_2, \ldots, a_n$, and

$$\left[\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)\right] = \sqrt[n]{a_1 a_2 \ldots a_n}$$

is their geometric mean.

Example 1. Let $a_1, a_2, \ldots, a_n$ be positive real numbers and

$$p = (p_1, p_2, \ldots, p_n),\ q = (q_1, q_2, \ldots, q_n)$$

be two sequences of real numbers. We have

$$\frac{[p] + [q]}{2} \geq \left[\frac{p+q}{2}\right].$$

Solution. This is because by the AM-GM Inequality,

$$\frac{a_{\sigma(1)}^{p_1} \cdots a_{\sigma(n)}^{p_n} + a_{\sigma(1)}^{q_1} \cdots a_{\sigma(n)}^{q_n}}{2} \geq a_{\sigma(1)}^{(p_1+q_1)/2} \cdots a_{\sigma(n)}^{(p_n+q_n)/2}.$$

Summing over $\sigma \in S_n$ and dividing by $n!$, we get the inequality. □

Example 2. (Schur's Inequality) Let x, y, z be nonnegative real numbers and let $a \in \mathbb{R}$, $b > 0$. Then we have

$$[(a + 2b, 0, 0)] + [(a, b, b)] \geq 2[(a + b, b, 0)].$$

Solution. By definition, we get

$$3![(a + 2b, 0, 0)] = 2(x^{a+2b} + y^{a+2b} + z^{a+2b}),$$
$$3![(a, b, b)] = 2(x^a y^b z^b + x^b y^a z^b + x^b y^b z^a),$$
$$3![(a + b, b, 0)] = x^{a+b}(y^b + z^b) + y^{a+b}(z^b + x^b) + z^{a+b}(x^b + y^b).$$

With elementary algebraic transformations, we have

$$\frac{1}{2}[(a + 2b, 0, 0)] + \frac{1}{2}[(a, b, b)] - [(a + b, b, 0)]$$
$$= x^a(x^b - y^b)(x^b - z^b) + y^a(y^b - x^b)(y^b - z^b) + z^a(z^b - y^b)(z^b - x^b).$$

Thus the given inequality is equivalent to

$$x^a(x^b - y^b)(x^b - z^b) + y^a(y^b - x^b)(y^b - z^b) + z^a(z^b - y^b)(z^b - x^b) \geq 0.$$

Assume, without loss of generality, that $x \geq y \geq z$.
If $a \geq 0$ then

$$\begin{aligned} x^a(x^b - y^b)(x^b - z^b) &\geq x^a(x^b - y^b)(y^b - z^b) \\ &\geq y^a(x^b - y^b)(y^b - z^b) \\ &= -y^a(y^b - x^b)(y^b - z^b), \end{aligned}$$

which means that

$$x^a(x^b - y^b)(x^b - z^b) + y^a(y^b - x^b)(y^b - z^b) \geq 0,$$

and since $z^a(z^b - y^b)(z^b - x^b) \geq 0$ we get the required result.
Similarly when $a < 0$, we assume without loss of generality, that $x \leq y \leq z$ and the proof is essentially the same. □

Note 3. Replacing $x \to yz$, $y \to zx$, $z \to xy$ we get the following equivalent form of Schur's Inequality:

$$[(a + 2b, a + 2b, 0)] + [(a + b, a + b, 2b)] \geq 2[(a + 2b, a + b, b)].$$

Theorem 1 (Muirhead's Inequality). Let $a_1, a_2, \ldots, a_n$ be positive real numbers and $p, q \in \mathbb{R}^n$ be two sequences of real numbers. If $p \succ q$, then $[p] \geq [q]$. Furthermore, for $p \neq q$, equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

Proof. We will prove this theorem for the case n = 3. Let us first notice that for $n = 2$ the inequality follows easily. Indeed, it's easy to check

$$\begin{aligned} &a_1^{p_1}a_2^{p_2} + a_1^{p_2}a_2^{p_1} - a_1^{q_1}a_2^{q_2} - a_1^{q_2}a_2^{q_1} \\ &= a_1^{p_2}a_2^{p_2}\left(a_1^{p_1-p_2} + a_2^{p_1-p_2} - a_1^{q_1-p_2}a_2^{q_2-p_2} - a_1^{q_2-p_2}a_2^{q_1-p_2}\right) \\ &= a_1^{p_2}a_2^{p_2}\left(a_1^{q_1-p_2} - a_2^{q_1-p_2}\right)\left(a_1^{q_2-p_2} - a_2^{q_2-p_2}\right) \geq 0, \end{aligned}$$

because $q_1 - p_2 = p_1 - q_2 \geq q_1 - q_2 \geq 0$, $q_2 - p_2 = p_1 - q_1 \geq 0$.
Now, let us consider the case $n = 3$. We assume that $p \neq q$ and not all the a_i are equal. Let $p = (p_1, p_2, p_3)$, $q = (q_1, q_2, q_3)$ and let us consider the following cases:
1. $q_1 \geq p_2$. Since

$$(p_1, p_2) \succ (p_1 + p_2 - q_1, q_1) \text{ or } (p_1, p_2) \succ (q_1, p_1 + p_2 - q_1)$$

and

$$(p_1 + p_2 - q_1, p_3) \succ (q_2, q_3),$$

using Muirhead's Inequality twice for the case $n = 2$, proven before, it follows that

$$\begin{aligned}
6[p] &= \sum_{cyc} (a_1^{p_1} a_2^{p_2} + a_1^{p_2} a_2^{p_1})\, a_3^{p_3} \\
&\geq \sum_{cyc} \left(a_1^{p_1+p_2-q_1} a_2^{q_1} + a_1^{q_1} a_2^{p_1+p_2-q_1}\right) a_3^{p_3} \\
&= \sum_{cyc} a_1^{q_1} \left(a_2^{p_1+p_2-q_1} a_3^{p_3} + a_2^{p_3} a_3^{p_1+p_2-q_1}\right) \\
&\geq \sum_{cyc} a_1^{q_1} \,(a_2^{q_2} a_3^{q_3} + a_2^{q_3} a_3^{q_2}) \\
&= 6[q].
\end{aligned}$$

2. $q_1 \leq p_2$. It follows from $3q_1 \geq q_1 + q_2 + q_3 = p_1 + p_2 + p_3 \geq q_1 + p_2 + p_3$ that

$$(p_2, p_3) \succ (q_1, p_2 + p_3 - q_1)$$

and since $p_1 \geq q_1 \geq q_2$, $p_1 \geq q_1 = 2q_1 - q_1 \geq p_2 + p_3 - q_1$, we get

$$(p_1, p_2 + p_3 - q_1) \succ (q_2, q_3).$$

Therefore, applying Muirhead's Inequality twice for the case $n = 2$, it follows that

$$\begin{aligned}
6[p] &= \sum_{cyc} a_1^{p_1} \,(a_2^{p_2} a_3^{p_3} + a_2^{p_3} a_3^{p_2}) \\
&\geq \sum_{cyc} a_1^{p_1} \left(a_2^{q_1} a_3^{p_2+p_3-q_1} + a_2^{p_2+p_3-q_1} a_3^{q_1}\right) \\
&= \sum_{cyc} a_2^{q_1} \left(a_1^{p_1} a_3^{p_2+p_3-q_1} + a_1^{p_2+p_3-q_1} a_3^{p_1}\right) \\
&\geq \sum_{cyc} a_2^{q_1} \,(a_1^{q_2} a_3^{q_3} + a_1^{q_3} a_3^{q_2}) \\
&= 6[q].
\end{aligned}$$

Equality holds if and only if $a_1 = a_2 = a_3$. □

Note 4. Since $(1, 0, \ldots, 0) \succ (1/n, 1/n, \ldots, 1/n)$, the AM-GM Inequality is a consequence.

Example 3. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2(b+c)}{b^2+c^2} + \frac{b^2(c+a)}{c^2+a^2} + \frac{c^2(a+b)}{a^2+b^2} \geq a+b+c.$$

Solution. Clearing denominators, the inequality becomes

$$\sum_{cyc} a^2(b+c)(c^2+a^2)(a^2+b^2) \geq (a+b+c)(a^2+b^2)(b^2+c^2)(c^2+a^2).$$

Expanding and canceling terms, we reach

$$a^6(b+c) + b^6(c+a) + c^6(a+b) \geq a^5(b^2+c^2) + b^5(c^2+a^2) + c^5(a^2+b^2),$$

or

$$[(6,1,0)] \geq [(5,2,0)],$$

which follows by Muirhead's Inequality.
Equality holds if and only if $a = b = c$. $\square$

Example 4. (Nguyen Viet Hung, Mathematical Reflections) Prove that for any positive real numbers a, b, c the following inequality holds:

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq \frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2}.$$

Solution. The inequality can be rewritten as

$$(a^4+b^4+c^4)(a^2+b^2+c^2) \geq 3abc(a^3+b^3+c^3),$$

or

$$a^6+b^6+c^6+a^4(b^2+c^2)+b^4(c^2+a^2)+c^4(a^2+b^2) \geq 3abc(a^3+b^3+c^3),$$

or

$$[(6,0,0)] + 2[(4,2,0)] \geq 3[(4,1,1)],$$

which follows from Muirhead's Inequality,

$$[(6,0,0)] \geq [(4,1,1)],$$

$$[(4,2,0)] \geq [(4,1,1)].$$

Equality holds if and only if $a = b = c$. $\square$

Example 5. Let a, b, c be positive real numbers. Prove that

$$(a^2+bc)(b^2+ca)(c^2+ab) \geq (a^2+ab)(b^2+bc)(c^2+ca).$$

Solution. Expanding, we obtain the following equivalent forms

$$\begin{aligned} &a^4bc+b^4ca+c^4ab+a^3b^3+b^3c^3+c^3a^3+2a^2b^2c^2 \\ \geq\ &a^3b^2c+a^3bc^2+b^2c^2a+b^3ca^2+c^3a^2b+c^3ab^2+2a^2b^2c^2, \\ &a^4bc+b^4ca+c^4ab+a^3b^3+b^3c^3+c^3a^3 \\ \geq\ &a^3b^2c+a^3bc^2+b^2c^2a+b^3ca^2+c^3a^2b+c^3ab^2, \end{aligned}$$

$$[(4,1,1)]+[(3,3,3)] \geq 2[(3,2,1)],$$

which follows from Muirhead's Inequality,

$$[(4,1,1)] \geq [(3,2,1)],$$

$$[(3,3,3)] \geq [(3,2,1)].$$

Equality holds if and only if $a=b=c$. □

Example 6. Let a,b,c be positive real numbers such that $abc=1$. Prove that

$$a^2+b^2+c^2+3 \geq 2(ab+bc+ca).$$

Solution. Using the substitutions

$$a=\frac{x^2}{yz},\ b=\frac{y^2}{zx},\ c=\frac{z^2}{xy},\ x,y,z>0,$$

the inequality becomes

$$\frac{x^4}{y^2z^2}+\frac{y^4}{z^2x^2}+\frac{z^4}{x^2y^2}+3 \geq 2\left(\frac{xy}{z^2}+\frac{yz}{x^2}+\frac{zx}{y^2}\right),$$

or

$$x^6+y^6+z^6+3x^2y^2z^2 \geq 2(x^3y^3+y^3z^3+z^3x^3),$$

or

$$[(6,0,0)]+[(2,2,2)] \geq 2[(3,3,0)].$$

But this follows from the following inequality

$$[(6,0,0)]+[(2,2,2)] \geq 2[(4,2,0)],$$

which is Schur's Inequality and

$$[(4,2,0)] \geq [(3,3,0)],$$

which is Muirhead's Inequality.
Equality holds if and only if $a=b=c$. □

Example 7. If a, b, c are nonnegative real numbers prove that

$$a^2 + b^2 + c^2 \leq \sqrt{abc}\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) + (a-b)^2 + (b-c)^2 + (c-a)^2.$$

Solution. Note that rearranging the requested inequality gives

$$a^2 + b^2 + c^2 + \sqrt{abc}\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) \geq 2(ab + bc + ca).$$

Thus substituting $a = x^2$, $b = y^2$, $c = z^2$ the inequality becomes

$$x^4 + y^4 + z^4 + xyz(x + y + z) \geq 2(x^2y^2 + y^2z^2 + z^2x^2),$$

or

$$[(4,0,0)] + [(2,1,1)] \geq 2[(2,2,0)].$$

But this follows from Schur's Inequality i.e.

$$[(4,0,0)] + [(2,1,1)] \geq 2[(3,1,0)]$$

and

$$[(3,1,0)] \geq [(2,2,0)]$$

which is Muirhead's Inequality.
Equality holds if and only if $a = b = c$. □

Example 8. (An Zhenping, Mathematical Reflections) Let a, b, c be nonnegative real numbers such that $a^2 + b^2 + c^2 \geq a^3 + b^3 + c^3$. Prove that

$$a^3b^3 + b^3c^3 + c^3a^3 \leq a^2b^2 + b^2c^2 + c^2a^2.$$

Solution. It is sufficient to prove that

$$a^3b^3 + b^3c^3 + c^3a^3 \leq \left(\frac{a^3 + b^3 + c^3}{a^2 + b^2 + c^2}\right)^2 \left(a^2b^2 + b^2c^2 + c^2a^2\right).$$

After clearing denominators this is equivalent to

$$\sum_{cyc}(a^7b^3 + a^7c^3) + \sum_{cyc} a^4b^3c^3 \leq \sum_{cyc}(a^8b^2 + a^8c^2) + \sum_{cyc} a^6b^2c^2,$$

or

$$2[7,3,0)] + [(4,3,3)] \leq 2[(8,2,0)] + [(6,2,2)].$$

However this inequality is true, since by Muirhead's Inequality,

$$[7,3,0)] \leq [(8,2,0)],$$

and

$$[(4,3,3)] \leq [(6,2,2)].$$

Equality holds if and only if $a = b = c$. □

Example 9. Let a, b, c be positive real numbers such that $a+b+c = a^2+b^2+c^2$. Prove that

$$ab + bc + ca \leq abc + 2.$$

Solution. First, we homogenize the inequality and we get the following inequality

$$(a + b + c)^2(a^2 + b^2 + c^2)(ab + bc + ca) \leq abc(a + b + c)^3 + 2(a^2 + b^2 + c^2)^3.$$

By some easy computations, the inequality is reduced to

$$2(a^2 + b^2 + c^2)^3 \geq (a + b + c)^2 \sum_{cyc}(a^3b + a^3c),$$

or

$$2\sum_{cyc}(a^6 + 2a^4b^2 + 2a^4c^2 + 2a^2b^2c^2)$$
$$\geq \sum_{cyc}(a^5b + a^5c + 4a^4bc + 2a^3b^3 + 3a^3b^2c + 3a^3bc^2),$$

that is

$$[(6, 0, 0)]+4[(4, 2, 0)]+2[(2, 2, 2)] \geq [(5, 1, 0)]+2[(4, 1, 1)]+[(3, 3, 0)]+3[(3, 2, 1)].$$

Applying Schur's Inequality, we get

$$[(6, 0, 0)] + [(4, 1, 1)] \geq 2[(5, 1, 0)],$$
$$[(6, 0, 0)] + [(2, 2, 2)] \geq 2[(4, 2, 0)].$$

Also, by the AM-GM Inequality, we have

$$3[(4, 2, 0)] + 3[(2, 2, 2)] \geq 6[(3, 2, 1)].$$

On other hand, by the Muirhead's Inequality, it follows that

$$5[(4, 2, 0)] \geq 5[(4, 1, 1)],$$
$$2[(4, 2, 0)] \geq 2[(3, 3, 0)].$$

Summing up the above inequalities, dividing by 2, we get the desired result. Equality holds if and only if $a = b = c$. □

Example 10. Let x, y, z be positive real numbers such that $x + y + z = 2$. Prove that

$$\frac{x^2\sqrt{y}}{\sqrt{x+z}} + \frac{y^2\sqrt{z}}{\sqrt{y+x}} + \frac{z^2\sqrt{x}}{\sqrt{z+y}} \leq \sqrt{x^3 + y^3 + z^3}.$$

Solution. By the Cauchy-Schwarz Inequality

$$\left(\sum_{cyc} \frac{x^2\sqrt{y}}{\sqrt{x+z}}\right)^2 \le \sum_{cyc} \frac{xy}{(x+z)(y+z)} \sum_{cyc} x^3(y+z).$$

Hence, it remains to prove that

$$(x+y+z)(x^3+y^3+z^3) \ge \sum_{cyc} \frac{2xy}{(x+z)(y+z)} \sum_{cyc}(x^3y+x^3z),$$

or

$$(x+y)(y+z)(z+x)(x+y+z)(x^3+y^3+z^3) \ge \sum_{cyc} 2xy(x+y) \sum_{cyc}(x^3y+x^3z).$$

Expanding, we obtain the inequality

$$\sum_{cyc} \left[(x^6y+x^6z)-(x^4y^3+x^4z^3)+2(x^4y^2z+x^4yz^2)-2x^3y^3z-2x^3y^2z^2\right] \ge 0,$$

which can be written as

$$[(6,1,0)] - [(4,3,0)] + 2[(4,2,1)] - [(3,3,1)] - [(3,2,2)] \ge 0$$

But this follows by Muirhead's Inequality,

$$[(6,1,0)] \ge [(4,3,0)],$$

$$[(4,2,1)] \ge [(3,3,1)],$$

$$[(4,2,1)] \ge [(3,2,2)].$$

Equality holds when $x = y = z$. □

Example 11. Let x, y, z be positive real numbers such that $xyz = 1$. Prove that

$$x+y+z \ge \frac{3}{x+2} + \frac{3}{y+2} + \frac{3}{z+2}.$$

Solution. Multiplying throughout by the product of the (clearly positive) denominators and rearranging terms, the proposed inequality is equivalent to

$$2\sum_{cyc}(x^2y+x^2z) + 4(x^2+y^2+z^2) + 5(xy+yz+zx) - 3(x+y+z) - 30 \ge 0.$$

Using the substitutions

$$x = \frac{a^2}{bc},\ y = \frac{b^2}{ca},\ z = \frac{c^2}{ab},\ a, b, c > 0,$$

our inequality can be written successively in the following forms

$$2\sum_{cyc}\left(\frac{a^3}{c^3}+\frac{a^3}{b^3}\right)+4\sum_{cyc}\frac{a^4}{b^2c^2}+5\sum_{cyc}\frac{ab}{c^2}-3\sum_{cyc}\frac{a^2}{bc}-30\geq 0,$$

$$2\sum_{cyc}(a^6b^3+a^6c^3)+4\sum_{cyc}a^7bc+5\sum_{cyc}a^4b^4c-3\sum_{cyc}a^5b^2c^2-30a^3b^3c^3\geq 0,$$

$$2[(6,3,0)]+2[(7,1,1)]+\frac{5}{2}[(4,4,1)]-\frac{3}{2}[(5,2,2)]-5[(3,3,3)]\geq 0.$$

Now according to Muirhead's Inequality, we have

$$\frac{3}{2}[(6,3,0)]\geq\frac{3}{2}[(5,2,2)],$$

$$\frac{1}{2}[(6,3,0)]\geq\frac{1}{2}[(3,3,3)],$$

$$2[(7,1,1)]\geq 2[(3,3,3)],$$

$$\frac{5}{2}[(4,4,1)]\geq\frac{5}{2}[(3,3,3)].$$

If we add the last four inequalities we obtain the required result. Equality occurs if and only if $a=b=c=1$. □

Example 12. (Pham Kim Hung) Let a, b, and c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{a^2b}{4-bc}+\frac{b^2c}{4-ca}+\frac{c^2a}{4-ab}\leq 1.$$

Solution. Since for nonnegative real numbers such that $a+b+c=3$ we have the well known inequality

$$a^2b+b^2c+c^2a+abc\leq 4, \tag{1}$$

then

$$\begin{aligned}&4\left(\frac{a^2b}{4-bc}+\frac{b^2c}{4-ca}+\frac{c^2a}{4-ab}-1\right)\\&=a^2b\left(\frac{bc}{4-bc}+1\right)+b^2c\left(\frac{ca}{4-ca}+1\right)+c^2a\left(\frac{ab}{4-ab}+1\right)-4\\&\leq abc\left(\frac{ab}{4-bc}+\frac{bc}{4-ca}+\frac{ca}{4-ab}-1\right).\end{aligned}$$

So it suffices to prove that

$$\frac{ab}{4-bc}+\frac{bc}{4-ca}+\frac{ca}{4-ab}\le 1.$$

Clearing denominators, it becomes

$$\begin{aligned}32(ab+bc+ca)+abc(a^2b+b^2c+c^2a+abc)\\ -64-8abc(a+b+c)-4(a^2b^2+b^2c^2+c^2a^2)\le 0.\end{aligned}$$

After applying inequality (1) and homogenizing, it remains to prove that

$$\begin{aligned}\frac{32}{9}(ab+bc+ca)(a+b+c)^2+\frac{4}{3}abc(a+b+c)\\ -\frac{64}{81}(a+b+c)^4-8abc(a+b+c)-4(a^2b^2+b^2c^2+c^2a^2)\le 0.\end{aligned}$$

Clearing denominators again and expanding, it becomes

$$16\,([(3,1,0)]-[(4,0,0)])+33\,([(2,1,1)]-[(2,2,0)])\le 0,$$

which follows from the Muirhead's Inequality,

$$[(3,1,0)]\le[(4,0,0)],$$

$$[(2,1,1)]\le[(2,2,0)].$$

Equality holds for $a=b=c=1$, or $a=2$, $b=1$, $c=0$, or any permutations of these values. □

∇