Contents

Part I

Problems

Chapter 1

Quadratics

1. Find all pairs (a, b) of real numbers such that the roots of the equation $x^2 + ax + b = 0$ are $3a$ and $3b$.

2. Let p and q be real numbers such that one of the roots of the quadratic equation $x^2 + px + q = 0$ is the square of the other. Prove that $p \leq \frac{1}{4}$ and

$$p^3 - 3pq + q^2 + q = 0.$$

3. Let a and b be real numbers. Prove that at least one of the equations

$$x^2 + ax + b = 0, \quad x^2 + bx + a = 0, \quad x^2 + (ab + 8)x + a^3 + b^3 + 16 = 0$$

has real roots.

4. **(a)** Prove that for any real numbers a, b, c,

$$a^2 + \left(2 - \sqrt{2}\right) b^2 + c^2 \geq \sqrt{2}(ab - bc + ca).$$

(b) Find the smallest constant k such that for all real numbers a, b, c,

$$a^2 + kb^2 + c^2 \geq \sqrt{2}(ab + bc + ca).$$

5. Let a and b be postive real numbers less than 2 such that $ab = 2$. Solve in real numbers the equation

$$4(x^2 + ax + b)(x^2 + bx + a) + a^3 + b^3 = 9.$$

6. Let $f(x) = ax^2 + bx + c$, where a, b, c are real numbers such that $a > 0$ and $ab \geq \frac{1}{8}$. Prove that $f(b^2 - 4ac) \geq 0$.

7. Let $f(x) = ax^2 + bx + c$, where $a < 0 < b$ and $b\sqrt[3]{c} \geq \dfrac{3}{8}$. Prove that

$$f\left(\frac{1}{\Delta^2}\right) \geq 0,$$

where $\Delta = b^2 - 4ac$.

8. Let a, b, c, d be positive real numbers. Prove that

$$3(a^2 - ab + b^2)(c^2 - cd + d^2) \geq 2(a^2c^2 - abcd + b^2d^2).$$

9. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that for any positive real number t,

$$(at^2 + bt + c)(bt^2 + ct + a)(ct^2 + at + b) \geq t^3.$$

10. Prove that for all real numbers a, b, c the following inequality holds

$$3(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) \geq a^3b^3 + b^3c^3 + c^3a^3.$$

Chapter 2

Equations in Real Numbers

11. Solve in real numbers the equation

$$(3x+1)(4x+1)(6x+1)(12x+1)=5.$$

12. Solve in real numbers the equation

$$\left(x^2-2\sqrt{2}x\right)\left(x^2-2\right)=2021.$$

13. Solve in real numbers the equation

$$x^4+2x^3-1850x^2-2x+1=0.$$

14. Find all real numbers a for which the equation

$$\left(\frac{x}{x-1}\right)^2+\left(\frac{x}{x+1}\right)^2=a$$

has four distinct real roots.

15. Prove that the equation $x^3-6x^2+3x-2=0$ has a root of the form $\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}$ for some positive integers a, b, c.

16. The equation

$$x^3-\frac{1}{x}=4$$

has two real roots, x_1 and x_2. Evaluate $x_1^2+x_2^2$.

17. Solve in real numbers the equation

$$4x^3+\frac{127}{x}=2016.$$

18. Solve in real numbers the equation

$$\left(\frac{2}{x^2+1}\right)^2 - \left(\frac{1}{x^2-1}\right)^2 = \left(\frac{1}{2x}\right)^2.$$

19. Solve in real numbers the equation

$$(x^3-3x)^2 + (x^2-2)^2 = 4.$$

20. Solve in positive real numbers the equation

$$\frac{1}{2}\left(\frac{x^3}{y} + \frac{y^3}{x}\right) = 2 - \frac{1}{xy}.$$