

Preface

Even though there are plenty of Number Theory books for students preparing for mathematics competitions, many are either outdated or actually addressed to experts in the field. Therefore, *121 Number Theory Problems for Mathematics Competitions* is a new book aimed at those who have little or moderate knowledge of the subject.

The book follows almost the same line of the lectures of the level one Number Theory course taught by the second author at the AwesomeMath Summer Program (AMSP), combined with the vast experience and problem-solving style of the first author. The book contains many examples, proposed problems followed by solutions, and an ample selection from many recent mathematical contests and Olympiads, journals, as well as tests given to the students during recent years at AMSP.

The AwesomeMath Summer Program is an intensive three-week program for highly gifted mathematics students from around the globe. It is designed for bright middle and high school students who wish to sharpen their problem-solving skills and further their mathematics education. Many of the participants seek to improve their performance on contests such as AMC10/12, AIME, or USA(J)MO.

The book is structured in 11 chapters, each containing 11 examples and 11 proposed problems, for a total of 121 examples plus 121 problems. Chapter 12 of the book is a collection of 30 bonus problems. The first chapter contains essential techniques such as proof by contradiction, invariants, double counting and the pigeonhole principle. This material has been included because we regard it as important in mastering the theoretical part of number theory.

The second chapter introduces remarkable factorizations that each serious student should know in order to grasp the material presented later. Here are some of the concepts that readers will encounter in the next few chapters: divisibility and its properties, the division algorithm, infinitude of primes, Euclid's lemma and the fundamental theorem of arithmetic, gcd and the lcm of two positive integers and their properties, congruences and its applications.

In the seventh chapter we discuss perfect squares and perfect cubes which provide a good chance to apply the knowledge and techniques learned in the previous chapters. Chapters 8 through 10 involve divisors, especially the number of positive divisors and the sum of divisors of a positive integer, number bases, factorials, and a brief initiation in the p -adic valuation of the factorial of a positive integer. The eleventh chapter is an introduction to games and tables, a pleasant opportunity to see in a different light what has been discussed throughout the book.

Solutions to all proposed problems are provided in the second part of the book. There is a corresponding solution chapter for each of the twelve chapters in the first part. We hope that students who are training for mathematics contests or wish to improve their Number Theory skills will benefit tremendously from this book. We strongly believe that this book will also be a great resource for teachers and coaches who train students for competitions or organize mathematics circles.

Finally, we would like to thank Richard Stong for his precious contribution and suggestions that enhanced the quality of this work.

We trust that you will enjoy the book!

Titu Andreescu
Alessandro Ventullo

Contents

Preface	vii
1 Essential Techniques	1
1.1 Proposed problems	8
2 Factorizations	11
2.1 Proposed problems	17
3 Divisibility	19
3.1 Proposed problems	27
4 Prime Numbers	29
4.1 Proposed problems	36
5 GCD and LCM	39
5.1 Proposed problems	45
6 Congruences	47
6.1 Proposed problems	56
7 Perfect Squares and Perfect Cubes	59
7.1 Proposed problems	65
8 Divisors	67
8.1 Proposed problems	74
9 Number Bases	77
9.1 Proposed problems	85
10 Factorials	87
10.1 Proposed problems	93

11 Tables and Games	95
11.1 Proposed problems	104
12 Bonus Problems	107
 I Solutions	 113
13 Essential Techniques	115
14 Factorizations	123
15 Divisibility	129
16 Prime Numbers	137
17 GCD and LCM	145
18 Congruences	153
19 Perfect Squares and Perfect Cubes	161
20 Divisors	169
21 Number Bases	177
22 Factorials	187
23 Tables and Games	195
24 Bonus Problems	203
Other Books from XYZ Press	235

Chapter 1

Essential Techniques

Proof by contradiction

Proof by contradiction and *reductio ad absurdum* are special methods for proving statements in mathematics. Both methods rely on one of the fundamental laws of classical logic: “Between any statement and its negation, one is true and the other is false and there is no third choice”. A proof by contradiction begins by assuming that the negation of the statement that we want to prove is true. Based on this assumption, we derive some consequences leading to a contradiction: they contradict either some hypothesis or a statement whose truth has already been proved. We then reason like this: if our assumption were true, then any conclusion we derive from it by sound logic must be a true conclusion. But since we have obtained a false conclusion, the assumption we have made must be false. This leads us to conclude that the thesis we want to prove is true. Reducto ad absurdum is a variation on this idea, where we basically combine proof by contradiction with induction. Suppose we want to prove that no instance of a particular object exists. In a proof by contradiction, we would assume that on the contrary there is an instance. In a reducto ad absurdum argument, we go one step further and assume we have instance which has the smallest value of some non-negative integer parameter. This uses the basic fact that any nonempty subset of the non-negative integers has a smallest element, a statement that is logically equivalent to induction. If we can build another instance with a smaller value for this parameter, then we will have a contradiction, and hence we may conclude that no instance exists.

Example 1. Prove that the numbers from 1 to 15 cannot be split into two disjoint subsets A and B with A containing two numbers and B containing 13 numbers so that the sum of the numbers in B is equal to the product of the numbers in A .

N. Agakhanov - All Russian Mathematical Olympiad 1999

Solution. Assume that such a partition is possible and let A contain numbers x and y , with $1 \leq x < y \leq 15$. Then the sum of the numbers in B is

$$1 + 2 + \dots + 15 - x - y = 120 - x - y,$$

so $xy = 120 - x - y$. Rewriting this equality as

$$(x + 1)(y + 1) = 121,$$

and noting that since $x < y$ we cannot have $x + 1 = y + 1 = 11$, we get that $x + 1 = 1$, $y + 1 = 121$, i.e., $x = 0$, $y = 120$, a contradiction. ■

Example 2. Is it possible to color all integers greater than 1 in three colors (each integer in one color, all three colors must be used) so that the color of the product of any two differently colored numbers is different from the color of each of the factors?

Mikhail Evdokimov - Tournament of Towns 2023

Solution. The answer is no. Assume that we have such a coloring, and call the colors blue, red and green. By symmetry, we can assume that the number 2 is blue, and 4 is not red. Take any red number k , which must exist by the hypothesis that all three colors are used. Then since 2 is blue and k is red, the number $2k$ must be green. Since 2 is blue and $2k$ is green, $2 \cdot 2k$ must be red. On the other hand, since 4 is not red and k is red, $4 \cdot k$ must be whichever of blue and green is not used by 4. Hence $4 \cdot k$ is not red, a contradiction. ■

Example 3. Masha took four different positive numbers and wrote down six of their pairwise products in a row in ascending order. Can all five differences between adjacent numbers of this sequence be the same?

I. Rubanov - Leonhard Euler Competition 2023

Solution. The answer is no. Assume Masha wrote down four numbers a, b, c, d with

$$0 < a < b < c < d,$$

and that on the contrary the five differences are all the same. Then, the first and the second number in the sequence of products are ab and ac , and the fifth and the sixth are bd and cd . Hence, in particular, we have $ac - ab = cd - bd$, which gives $a(c - b) = d(c - b)$, i.e., $(a - d)(c - b) = 0$. Therefore, either $b = c$ or $a = d$, a contradiction. ■

Invariants

A common situation in mathematics is that we are given an initial state of an object and a collection of transformations we can apply to it in sequence. We are then asked to prove that these transformations cannot lead to a certain final configuration. The notion of an invariant lets us solve such a problem. An invariant is a quantity, a relation or a property that remains unchanged after the application of a transformation. If the starting position and the requested ending position have different values for this invariant, then it is impossible to get from one to the other by any sequence of the transformations.

Example 4. There are four numbers on the blackboard: 1, 3, 6, 10. Each time we can erase any two numbers a and b written on the blackboard, we write the numbers $a + b$ and ab instead. Can we obtain the numbers 2015, 2016, 2017, 2018 after several moves?

Ukrainian Mathematical Olympiad 2016

Solution. The answer is no. Observe that the count of numbers divisible by 3 cannot decrease after each operation. Indeed, if both a and b are divisible by 3, then both $a + b$ and ab are divisible by 3 and if one of a and b is divisible by 3, then ab is divisible by 3. The starting state has the property that at least two of its numbers are divisible by 3, and the reasoning above shows that this property is invariant under the transformations. But the desired ending state has only one number divisible by 3, hence it does not have this property, which is impossible. ■

Example 5. Starting with n 1s, we keep picking two of the numbers, say a and b , and replace them by $\sqrt[3]{\frac{a^2b^2}{a+b}}$. We continue this operation until there is only one number left. Prove that this number is not less than $\frac{1}{\sqrt[3]{n}}$.

Liubomir Chiriac - Mathematical Reflections, Problem J22

Solution. The key point is finding an invariant property of the list of written numbers. We claim that if numbers $a_1, \dots, a_k$ are written on the blackboard, then the quantity

$$E(a_1, \dots, a_k) = \frac{1}{a_1^3} + \frac{1}{a_2^3} + \dots + \frac{1}{a_k^3}$$

does not increase. Assuming this, we start with $E(1, 1, \dots, 1) = n$ and the property that $E \leq n$ is an invariant. Thus when we have just one number x

written on the board, we must have $n \geq E(x) = \frac{1}{x^3}$, implying $x \geq \frac{1}{\sqrt[3]{n}}$.

To see that E cannot increase, we must prove that

$$\frac{1}{a^3} + \frac{1}{b^3} \geq \frac{1}{\frac{a^2b^2}{a+b}} = \frac{a+b}{a^2b^2}$$

or $a^3 + b^3 \geq ab(a+b)$, which is equivalent to $(a+b)(a-b)^2 \geq 0$. Hence it is true. ■

Example 6. Eight numbers, all zero, are written on a blackboard. An *operation* consists in randomly selecting four of them, a, b, c, d , and replacing them by $a+3, b+3, c+2$ and $d+1$, respectively.

- (a) What is the minimum number of operations we need to perform to get eight consecutive integers on the blackboard?
- (b) Is it possible that, after a certain number of operations, all the numbers written on the blackboard are equal to 2015?
- (c) Is it possible that, after a certain number of operations, the product of the numbers on the blackboard is equal to 2145?

Gheorghe Rotariu - Romanian Mathematical Olympiad 2015

Solution. (a) After each operation, the sum of the numbers increases by 9. Since the sum of the smallest 8 consecutive natural numbers is

$$0 + 1 + 2 + \dots + 7 = 28,$$

at least four operations are required. On the other hand, four operations are sufficient, as shown below:

Start	0	0	0	0	0	0	0	0
1st Operation	1	2	0	0	0	0	3	3
2nd Operation	1	2	2	1	0	0	6	6
3rd Operation	1	2	2	1	3	3	7	8
4th Operation	1	2	3	4	5	6	7	8

- (b) The answer is no. After k operations, the sum of the eight numbers is $9k$. If after a certain number of operations all the eight numbers are equal to 2015, then their sum is $8 \cdot 2015$, which is not divisible by 9, a contradiction.

- (c) The answer is no. Assume on the contrary, that this is possible. Since $2145 = 3 \cdot 5 \cdot 11 \cdot 13$, at least four numbers from the final eight are equal to 1. If there are exactly four, then the numbers are 3, 5, 11, 13, 1, 1, 1, 1 in some order. Their sum equals $3 + 5 + 11 + 13 + 1 + 1 + 1 + 1 = 36$, so four operations were performed to obtain these numbers. This cannot happen, as four operations cannot give the number 13. Notice that among the final eight numbers one cannot have six or more numbers equal to 1, since each operation increases three numbers by more than 1. Suppose now that five numbers among the final eight are equal to 1 and let a , b and c be the remaining three numbers. In each operation, one of the four numbers chosen must be one of the places that ended with a 1, and hence the operation must have increased that place from 0 to 1. Thus exactly five operations were performed in order to obtain five numbers equal to 1, so $a + b + c + 1 + 1 + 1 + 1 + 1 = 45$, i.e.,

$$a + b + c = 40.$$

If multiply two factors from the set $\{3, 5, 11, 13\}$ and take the sum of the three resulting numbers, then the smallest sum we can get is

$$11 + 13 + 3 \cdot 5 = 39.$$

Multiplying any other pair will give a sum of at least $5 + 13 + 3 \cdot 11 = 51$. Thus we cannot get a sum of 40, a contradiction. ■

Double Counting

Double counting is a technique for showing that two expressions are equal by observing that there are two ways of counting the elements of a finite set. The argument is that since both expressions are equal to the cardinality of the same set, they are equal.

Example 7. Raffaella painted over several cells in a 10×10 square. After that, it turned out that all the 2×2 squares have the same number of painted cells and all the 1×5 strips have the same number of painted cells. Prove that Raffaella painted over all the cells.

Alessandro Ventullo

Solution. Let m cells be filled in each 2×2 square, and let n cells be filled in each 1×5 strip. Since a 10×10 square can be divided into 25 2×2 squares, $25m$ cells are filled in total. Similarly, we get that $20n$ cells are filled in total.

Hence, $25m = 20n$, thus $5m = 4n$. But $0 < m \leq 4$ and $0 < n \leq 5$, so $m = 4$ and $n = 5$, as desired. ■

Example 8. Find the largest integer d for which a 43×47 table can be filled with the numbers 1 and 2 so that the sum of the numbers in each row and in each column is divisible by d .

Tomáš Bárta - Czech-Slovak Mathematical Olympiad 2022

Solution. First, we prove that for $d = 47$ the required filling of the table is indeed possible. Let's arrange that in each row the 47 numbers are all identical, with 1 in the first 39 rows and 2 in the remaining 4 rows. Then the sum of the numbers in each row is equal to either 47 or $2 \cdot 47$, and the sum of the numbers in each column is equal to $39 \cdot 1 + 4 \cdot 2 = 47$, which are all divisible by 47 as we wanted.

Now, we prove by contradiction that the required filling does not exist for any $d > 47$. On the contrary, suppose that we have a 43×47 table filled with 1 and 2 such that for some $d > 47$ the sum of the numbers in each row and in each column is divisible by d . Since each of these sums is positive, at most equal to $2 \cdot 47 < 2d$, and is also divisible by d , each sum is equal to d . By adding all the numbers of the table row by row, we get that the sum of all the numbers in the table is $43d$, while adding by column we get that the sum of all the numbers in the table is $47d$. So, $43d = 47d$, which is a contradiction. ■

Remark. The given problem can be generalized in an obvious way to every table $m \times n$, where $m < n \leq 2m$, and the largest d is equal to the number n . ■

Pigeonhole Principle

The result called *Pigeonhole Principle* or *Dirichlet's Principle* can be stated in the following way.

Pigeonhole Principle

Let m, n be positive integers with $m \geq n$. If $m + 1$ pigeons fly to n pigeonholes, then at least one pigeonhole contains at least $\left\lfloor \frac{m}{n} \right\rfloor + 1$ pigeons.

The Pigeonhole Principle is very easy to understand. The Pigeonhole Principle is useful in proving *existence theorems* non-constructively, that is, we prove that something exists without identifying it concretely. The difficulty in its application lies in identifying the “pigeons” and the “pigeonholes”.

Example 9. At the AwesomeMath Summer Program there are 10 classes in the morning and 10 classes in the evening. There are 101 students and each of them attends at least one class in the morning and at least one class in the evening. Prove that there are two students who attend a common class both in the morning and in the evening.

Alessandro Ventullo

Solution. By the Pigeonhole Principle, there is some morning class that is attended by at least $\left\lfloor \frac{100}{10} \right\rfloor + 1 = 11$ students. Fix such a set of 11 students. Now, by the Pigeonhole Principle once again, there is some evening class attended by at least $\left\lfloor \frac{10}{10} \right\rfloor + 1 = 2$ of these 11 students. ■

Example 10. Is it possible to fill an $n \times n$ table with numbers $-1, 0, 1$ so that all the sums of the numbers along the rows, along the columns and along the two main diagonals are different?

Solution. The answer is no. Since we are filling the table with numbers $-1, 0, 1$, every sum can assume only integer values from $-n$ to n , inclusive. Thus these are only $2n + 1$ possible values for the sums. However, we have $2n + 2$ sums (n rows, n columns and two diagonals), therefore by the Pigeonhole Principle there are two sums that are equal. ■

Example 11. Given $n + 1$ distinct positive integers less than $2n$ ($n > 1$). Prove that among them there are three numbers such that the sum of two of them is equal to the third.

All Russian Mathematical Olympiad 2013

Solution. Let $a_1, a_2, \dots, a_{n+1}$ be distinct positive integers in the interval $[1, 2n - 1]$. Let a_1 be the smallest of these numbers. Consider the list

$$A = \{a_2, \dots, a_{n+1}, a_2 + a_1, a_3 + a_1, \dots, a_{n+1} + a_1\}.$$

There are $2n$ integers in A and all of them are in the interval $[a_1 + 1, a_1 + 2n - 1]$, which consists of $2n - 1$ integers. By the Pigeonhole Principle there are two integers in A that are equal, and since $a_i \neq a_j$ for any $i \neq j$, we must have $a_i = a_j + a_1$ for some distinct i and j , as desired. ■

1.1 Proposed problems

Problem 1. The numbers $1, 2, \dots, 9$ are randomly arranged on a circle. Prove that there are three adjacent numbers whose sum is at least 16.

Ivan Borsenco - Mathematical Reflections, Problem J103

Problem 2. Let $a_1, a_2, \dots, a_{15}$ be positive integers such that

$$(a_1 + 1)(a_2 + 1) \cdot \dots \cdot (a_{15} + 1) = 2015a_1a_2 \cdot \dots \cdot a_{15}.$$

Prove that there are at least six and at most ten numbers among $a_1, a_2, \dots, a_{15}$ that are equal to 1.

Titu Zvonaru, Neculai Stanciu - Mathematical Reflections, Problem S350

Problem 3. Is it possible to arrange 36 distinct numbers in the cells of a 6×6 table, so that in each 1×5 rectangle (both vertical and horizontal) the sum of the numbers equals 2022 or 2023?

Egor Bakaev - Tournament of Towns 2022

Problem 4. There are 99 positive numbers in a circle. It turns out that for any four consecutive numbers, the sum of the first two of them clockwise is equal to the product of the last two of them clockwise. What is the sum of all 99 numbers?

S. Berlov - Leonhard Euler Competition 2023

Problem 5. Several positive integers are written on a board. At each step, we can pick any two numbers u and v , where $u \geq v$, and replace them with $u + v$ and $u - v$. Prove that after a finite number of steps we can never obtain the initial set of numbers.

Marius Cavachi - Mathematical Reflections, Problem J296

Problem 6. Numbers 1 through 24 are written on a board. At any time, numbers a, b, c may be replaced by

$$\frac{2b + 2c - a}{3}, \frac{2c + 2a - b}{3}, \frac{2a + 2b - c}{3}.$$

Can a number greater than 70 eventually appear on the board?

Titu Andreescu - Mathematical Reflections, Problem O115

Problem 7. The numbers $2^3 - 2, 3^3 - 3, 4^3 - 4, \dots, (2n + 1)^3 - (2n + 1)$, where $n \geq 2$ is a natural number, are written on a blackboard. An *operation* consists in deleting three arbitrary numbers a, b, c from the blackboard and replacing them with the number $\frac{abc}{ab + bc + ca}$. Prove that the sum of the last two numbers left on the blackboard is greater than 16.

Gheorghe Rotariu - Romanian JBMO Team Selection Test 2023

Problem 8. There are n boys and n girls in a class ($n \geq 3$). They sit down at a round table so that no two boys and no two girls sit side by side. The teacher has $2n$ cards, the numbers $1, 2, 3, \dots, 2n$ are written on them, each one time. He gives each student one card in such a way that the number of any girl is greater than the number of any boy. Then each girl writes on a piece of paper the sum of the numbers on three cards: her own card and the cards of the boys sitting next to her. For what n could all the resulting n numbers be equal?

Saint Petersburg Mathematical Olympiad 2023

Problem 9. A class of 37 students wrote a paper. No matter how we select 7 students from the class, there will always be at least 2 students among those selected who got the same grade. Prove that there are at least 4 boys or 4 girls in the class who got the same grade.

Hungarian Mathematical Olympiad 2022

Problem 10. One of the numbers 1 or 2 is written in each square of a 4×4 table. For each row, the sum of the numbers written in it was calculated, and for each column, the product of the numbers written in it was calculated. Prove that at least two of these eight results are equal.

Polish Junior Mathematical Olympiad 2023

Problem 11. Let $a_1, a_2, \dots, a_{2n+1}$ be distinct positive integers less than $3n + 2$. Prove that for each $m \in \{1, 2, \dots, n\}$ there are a_i and a_j such that

$$a_i - a_j = m.$$

Ivan Borsenco - Mathematical Reflections, Problem J37

