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Preface

This book is intended for mathematical Olympiad students who wish to further their study of inequalities, a favorite topic at many mathematics competitions. We selected the most interesting and beautiful inequalities that appeared in Mathematical Reflections from 2021 to 2024. We also chose new, refined problems from other important online math resources, recent math competitions, and the Romanian journal *Gazeta Matematică*.

The problems are divided into two categories: introductory and advanced. The inequalities from each section are ordered increasingly by the number of variables involved and their degree of difficulty. Each problem has at least one complete solution and many have several solutions or remarks. As a bonus, we have dedicated a section to a collection of 25 elegant inequalities along with their solutions.

The readers will encounter classical theorems such as the Power Mean and AM-GM inequalities, the Cauchy-Schwarz and Hölder's inequalities, Schur's inequality, Jensen's inequality, Muirhead and Maclaurin's inequalities, the mixing variable method, the tangent line method, SOS (sum of squares), SOS-Schur method, and so on. There are also numerous connections with 116 Algebraic Inequalities from the AwesomeMath Year-round Program, 118 Inequalities for Mathematics Competitions, as well as with New, Newer, and Newest Inequalities.

For each of the three sections we present the most ingenious and concise solutions with the goal of stimulating readers to develop their problem-solving skills and passion for the subject. Some of the problems included in this book require laborious algebraic calculations, which may discourage the reader at first, but we wanted to demonstrate that the techniques showcased are very useful for solving strong inequalities.

Many thanks to Richard Stong for significant improvement of the text. We would also like to thank all the contributors to Mathematical Reflections for their beautiful problems that we have selected in this book.

Enjoy the problems!

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Chapter 1

Problems

1.1 Introductory Problems

1. Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$a + b + c + 3abc \geq \frac{4}{a + b + c}.$$

When does the equality occur?

2. Prove that the following inequality holds for all positive real numbers a, b, c :

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \sqrt{\frac{b+c}{2a}} + \sqrt{\frac{c+a}{2b}} + \sqrt{\frac{a+b}{2c}}.$$

3. Let $x, y, z > 0$ such that $x^2 + y^2 + z^2 = 3$. Prove that

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + xyz \geq 4.$$

4. Prove that for any positive real numbers a, b, c

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{81(a+b)(b+c)(c+a)}{16(a+b+c)^3} \geq 3.$$

5. Let a, b, c, x, y, z be positive real numbers such that $x + y + z = 3$. Prove that

$$\frac{a}{a+2bx} + \frac{b}{b+2cy} + \frac{c}{c+2az} \geq 1.$$

6. Let a, b, c be positive real numbers. Prove that

$$\frac{b+c}{a^2} + \frac{c+a}{b^2} + \frac{a+b}{c^2} - \frac{9}{a+b+c} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

7. Let ABC be a triangle. Prove that

$$\frac{\cos A}{\sin^2 A} + \frac{\cos B}{\sin^2 B} + \frac{\cos C}{\sin^2 C} \geq \frac{R}{r}.$$

8. Let x, y, z be positive real numbers with $xy + yz + zx + xyz = 4$. Prove that

$$2 \left(\sqrt{x+1} + \sqrt{y+1} + \sqrt{z+1} \right) \leq 3\sqrt{(x+1)(y+1)(z+1)}.$$

9. Let x, y, z be positive real numbers such that $x + y + z = 1$. Prove that

$$\frac{1}{x^3 + y^3 + z^3} + \frac{24}{xy + yz + zx} \geq 81.$$

10. Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$\frac{1}{1+a^2+b^2} + \frac{1}{1+b^2+c^2} + \frac{1}{1+c^2+a^2} \leq \frac{9}{5}.$$

11. Let a, b, c be positive real numbers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$. Prove that

$$3\sqrt[6]{8abc} \leq \sqrt{a + \frac{b}{c}} + \sqrt{b + \frac{c}{a}} + \sqrt{c + \frac{a}{b}} \leq 3\sqrt{2abc}.$$

12. Let a, b, c be positive real numbers such that $a + b + c + abc = 4$. Prove that

$$a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} + 3 \leq \sqrt{1+4a-a^2} + \sqrt{1+4b-b^2} + \sqrt{1+4c-c^2} \\ \leq ab + bc + ca + 3.$$

13. Let x, y, z be positive real numbers such that $x + y + z = xyz$. Prove that

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2} \geq \frac{1}{1+xy} + \frac{1}{1+yz} + \frac{1}{1+zx}.$$

14. Let a, b, c be real numbers such that

$$\frac{1}{a+2+\sqrt{a^2+8}} + \frac{1}{b+2+\sqrt{b^2+8}} + \frac{1}{c+2+\sqrt{c^2+8}} \leq \frac{1}{2}.$$

Prove that $a + b + c \geq 3$. When does the equality occur?

15. Let $a, b, c \geq 0$ be real numbers such that $ab + bc + ca = a + b + c > 0$.
Prove that

$$1 \leq \frac{1}{1+2a} + \frac{1}{1+2b} + \frac{1}{1+2c} \leq \frac{7}{5}.$$

16. Let a, b, c be positive real numbers. Prove that

$$\sqrt[4]{\frac{2ab}{a^2+b^2}} + \sqrt[4]{\frac{2bc}{b^2+c^2}} + \sqrt[4]{\frac{2ca}{c^2+a^2}} + \frac{(a+b)(b+c)(c+a)}{8abc} \geq 4.$$

17. Let ABC be a triangle. Prove that

$$\frac{a^2}{1 + \cos^2 B + \cos^2 C} + \frac{b^2}{1 + \cos^2 C + \cos^2 A} + \frac{c^2}{1 + \cos^2 A + \cos^2 B} \leq \frac{2}{3} (a^2 + b^2 + c^2).$$

18. Let a, b, c, λ be positive real numbers. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{abc(\lambda+1)^3}{(a+\lambda b)(b+\lambda c)(c+\lambda a)} \geq 4.$$

19. Let a, b, c be positive real numbers such that $(a+b)(b+c)(c+a) = 9abc$.
Prove that

$$\sqrt[3]{2abc} \leq \max\{a, b, c\} \leq \sqrt[3]{4abc}.$$

20. Prove that in any triangle ABC ,

$$\frac{9}{4} \sqrt{\frac{r}{2R}} \leq \sqrt{3} \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \leq 1 + \frac{r}{4R}.$$

21. Let $a, b, c > 1$ be real numbers such that

$$\frac{1}{a-1} + \frac{1}{b-1} + \frac{1}{c-1} = 1.$$

Prove that

$$abc + 44 \geq 9(a + b + c).$$

22. If a, b, c are positive real numbers, then

$$\sqrt{\frac{a}{4b + \frac{bc}{a} + 4c}} + \sqrt{\frac{b}{4c + \frac{ca}{b} + 4a}} + \sqrt{\frac{c}{4a + \frac{ab}{c} + 4b}} \geq 1.$$