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Foreword

Marston Morse once said, “Mathematics are the result of mysterious powers which no one understands, and which the unconscious recognition of beauty must play an important part. Out of an infinity of designs a mathematician chooses one pattern for beauty’s sake and pulls it down to earth.” To many, mathematics is yet another domain of study grouped among the sciences. There is no debate about this; however, many also consider such scientific paradigms to be “beautiful”, a term that typically resonates more with the study of the arts. I, among many, share the opinion that mathematics, despite being a precise language to specifically express numerical concepts, is beautiful. It seems like every artist chooses a certain set of themes that they expound upon in their works, perhaps reflecting their personal vision or interpretation of the world. These elements are chosen for good reason, and I think that this process is no different for mathematics.

I am honored to introduce this book which is not an ordinary problem-solving or typical XYZ Press publication. More importantly, the problems contained within these pages tell a story.

My father, Titu Andreescu, was born in Romania. From a young age, he showed a keen interest in mathematics and began focusing his time and effort on the study of problem-solving. He went on to achieve respectable results in the competitive field winning the Romanian National Contest organized by *Gazeta Matematică* in 1973, 1974, and 1975. He eventually graduated from the University of Timișoara and became a mathematics professor at the same high school he once attended, Constantin Diaconovici Loga School of Mathematics and Physics. Later on, my father continued spreading the gift of problem-solving teaching in the U.S. and through his many publications.

Ever since I was young, I recall my father constantly scribbling down equations on any surface he could find, whether it was on a piece of paper or even a napkin in the absence of anything else. I came to recognize these

moments as a sign of his mind at work, deeply engaged in the creative process. This book is a collection of those problems, some of which might have started their lives on the back of an envelope, or in thin air, until they of course took on their ultimate polished form. However, there is also something unique about them, something that is not shared with his problems from other publications. Many of these are his *first* problems.

The problems presented are exclusively my father's, most of which have rarely, if ever, been seen before. From the age of fifteen, he began creating them, continuing throughout his younger adult years. Georg Cantor once said, "In mathematics, the art of proposing a question must be held of higher value than solving it." Students of mathematics can be seen solving problems, not usually writing them. What caused this sudden problem-writing metamorphosis? Well, it was a way for him to reach the world and an outlet for creative expression. He began submitting his problems to the popular journal *Revista Matematică* which would start publishing more and more of them, cementing his reputation as not only a problem-solver but more importantly, a problem-writer. This breakthrough caused him to gain a sort of notoriety in the local mathematical community, and in turn, encouraged him to continue in his endeavors. I also think that this was a start of a life-long journey unbeknownst to him at the time, and signaled an important shift in the way he would approach mathematics. These problems paint a picture of his own personal journey in the world of problem-solving, and his intellectual and mathematical evolution is evident in the progression of the problems from the first to the last page.

A short story: when my father immigrated to the U.S. settling near Chicago, he was advised to apply to the best school in the state at the time, Illinois Mathematics and Science Academy. During his interview, he was asked to present a full lesson to a panel of 12 math teachers from the department. They were impressed by his problems and a senior member asked, "Do you know more problems like these?", my father answered, "I know at least three-thousand of them." A few months later, that older colleague of his told him, "Titu, I really thought you were bluffing that day but you were certainly not."

I think it is safe to say that my father is not only a respected mathematician, but also a walking problem bank. It is still quite remarkable, for me, when I sit and think that from the time he was a teen up until his early mathematics career, he had already created hundreds of problems. This dedication and steadfastness to his craft is something that has left

a profound mark on me and served as an important source of motivation in my own life. My father always told me that you don't learn math by reading solutions or solving easy problems, but rather by challenging yourself and struggling with difficult ones. I think there is something to say about the lessons one learns through struggle and how it not only sharpens your intellect but also prepares you for other life challenges.

Several times I was a teaching assistant and then teacher at the summer program he founded, AwesomeMath, which was in-person at the time. There I had the pleasure of interacting with many of his colleagues who also taught there and we frequently shared interesting math problems we had stumbled across. I, too, would share some of my father's, but without revealing the author. They would usually stare at the problem for a couple of minutes, crack a smile, and then say something along the lines of "This seems like the kind of problem your father would write."

What makes his problems special is his style. My father has a propensity for creating problems with relatively simple statements and compact solutions. Don't let this fool you though - his problems are dense and pack a punch.

Side note – I think this is probably why he favored the following statement:

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \cdots + \frac{x_n^2}{y_n} \geq \frac{(x_1 + x_2 + \cdots + x_n)^2}{y_1 + y_2 + \cdots + y_n}$$

for $x_i \in \mathbb{R}$, $y_i \in \mathbb{R}^+$. This inequality, a variation of the Cauchy-Schwarz inequality, is known as Titu's lemma, colloquially abbreviated as T_2 's lemma. It can be applied in some ingenious ways and frequently shortens the solutions to many challenging inequalities. I encourage you to prove this statement. It is no surprise why he popularized the above inequality as he was never one who was interested in lengthy computations. For him, the true priority was highlighting the beauty of whatever aspect of mathematics was at hand.

His style employs concepts that aren't too distant for a wide audience. Nevertheless, he extends them in creative and non-obvious ways, pairing up the most seemingly difficult of problems with clear and accessible solutions. You might think to yourself, "That's it?", or, "I can't believe I didn't see that!" I still to this day have these moments when he shows me some of his latest creations. I guarantee that some of the smartest people you know, even seasoned professors, might not be able to solve some of the more challenging problems at first glance. The problems

that you will see are not the menial and repetitive ones encountered in standardized courses where you may be asked for the solution to a bland integral that consumes both you and two whole pages. Don't worry, there isn't a quiz at the end of this book.

As a testament to the kind of problem-writer he is, my father has had at least one problem selected to be on either the USAJMO or USAMO contests every year from 1994 to 2020, and again in 2023 and 2024. If I didn't make it clear already, I, among many, consider his problems to be works of art – not just because the statements are visually appealing or they have concise, elegant solutions. More importantly, they teach you how to reason and expand your thought processes to eventually tackle bigger challenges. And you don't need to be a mathematician to appreciate them.

When he mentioned his desire to publish this book in our conversations, he told me, “These are not my best problems.” He did, in fact, create a lot of them when he was young and still learning. However, I must advise you to not be swayed by his humble words. It could very well be that these are not his “best problems”, but they are damn good ones. I have worked through these problems myself and I must say that they are special. They present a view of mathematics that will ring familiar with readers in various stages of their problem-solving journeys and lives. They personify the individual's initial discovery of the beauty of mathematics, a universal, shared experience among problem-solvers.

There comes a point when a student has progressed significantly in their own craft, following in the footsteps of their teachers. Countless depictions across the arts explore the journey of the student who ultimately becomes the teacher, continuing the age-old cycle of learning. This is the story that these problems tell of my father's journey from student to teacher.

Adrian Andreescu
Dallas, Texas
February 2025

Preface

In contrast with the foreword written by my son, I choose to use fewer words. This book is, in a sense, the culmination of some of my life's work. It contains problems that are very special to me. I started writing math problems when I was 15. At first, it was something that paralleled my study of math. I didn't think it would become what it is today.

I was often asked, "How do you keep creating all these problems?" The answer to this question is not simple. Some just came to me, but for most, I pulled together various notions and combined them in original ways. There were also times when I needed to work hard to solve my own "conjectures". And there were instances when, after years, I had to sit down again and solve my own problems whose solutions I had forgotten. Most of the time, I succeeded!

This book is a collection of problems and solutions across four important domains of competitive mathematics: algebra, number theory, geometry, and trigonometry. The core concepts from all of them that are essential for success in competitions are individually showcased in great detail through the accompanying solutions. You will find roughly 30 instructive problems in each of these sections. There isn't a unifying theme across the presentation of these four domains, however, there are numerous ideas and techniques used in this kind of problem-solving that are shared between them. Given the breadth of topics presented in this book, I think that it will serve well as a training resource for Olympiads, namely the level of USA(J)MO or similar contests. But, you don't have to be involved in contests to appreciate the problems. Students, teachers, and math enthusiasts alike will all find their place in this book.

As always, I hope you enjoy them.

Titu Andreescu

Part I

Problems

Chapter 1

Algebra

1. Prove that if x, y, z are nonzero real numbers with $x + y + z = 0$, then

$$\frac{x^2 + y^2}{x + y} + \frac{y^2 + z^2}{y + z} + \frac{z^2 + x^2}{z + x} = \frac{x^3}{yz} + \frac{y^3}{zx} + \frac{z^3}{xy}.$$

2. Let a, b, c be nonzero real numbers such that

$$a + b + c = 0 \quad \text{and} \quad a^3 + b^3 + c^3 = a^5 + b^5 + c^5.$$

Prove that

$$a^2 + b^2 + c^2 = \frac{6}{5}.$$

3. Prove that if $a \geq \frac{1 + \sqrt{5}}{2}$, then

$$\left[\frac{1 + \left[\frac{1 + na^2}{a} \right]}{a} \right] = n, \quad n = 0, 1, 2, \dots$$

4. Prove that if x, y, z are real numbers such that $x^3 + y^3 + z^3 \neq 0$, then the ratio

$$\frac{2xyz - (x + y + z)}{x^3 + y^3 + z^3}$$

equals $\frac{2}{3}$ if and only if $x + y + z = 0$.

5. Find all n -tuples $(x_1, x_2, \dots, x_n)$ of real numbers in the interval $(0, 2)$ such that

$$2x_1 - \frac{x_2}{2 - x_2} = 2x_2 - \frac{x_3}{2 - x_3} = \dots = 2x_n - \frac{x_1}{2 - x_1} = 1.$$

6. Let a, b, c be real numbers such that the sum of any two of them is not equal to zero. Prove that

$$\frac{a^5 + b^5 + c^5 - (a + b + c)^5}{a^3 + b^3 + c^3 - (a + b + c)^3} \geq \frac{10}{9}(a + b + c)^2.$$

7. Let a, b, c be real numbers such that $abc = 1$. Prove that at most two of the numbers

$$2a - \frac{1}{b}, \quad 2b - \frac{1}{c}, \quad 2c - \frac{1}{a}$$

are greater than 1.

8. Let m and n be positive integers. Prove that

$$\frac{x^{mn} - 1}{m} \geq \frac{x^n - 1}{x}$$

for any positive real number x .

9. Prove that between n and $3n$ there is at least one perfect cube for any integer $n \geq 10$.
10. Let $k \geq 2$ be an integer. Find in closed form the general term a_n of the sequence defined by $a_0 = 0$ and $a_n - a_{\lfloor \frac{n}{k} \rfloor} = 1$ for all $n > 0$.
11. Let $a, b, c, d, e \in [-1, 1]$ such that $a > b > c > d > e$. Prove that

$$\frac{1}{a - b} + \frac{1}{b - c} + \frac{1}{c - d} + \frac{1}{d - e} \geq 8.$$

12. Let $a_1, a_2, \dots, a_n$ be positive real numbers and let $k \geq 0$. Prove that

$$k + \sqrt[n]{\prod_{i=1}^n a_i} \leq \sqrt[n]{\prod_{i=1}^n (k + a_i)} \leq k + \frac{1}{n} \sum_{i=1}^n a_i.$$

13. Determine all polynomials P with real coefficients such that

$$(P(x))^n = P(x^n)$$

for all real numbers x , where $n > 1$ is a given integer.

14. Let

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n, \quad a_n \neq 0,$$

be a polynomial with complex coefficients such that there is a positive integer m with

$$\left| \frac{a_m}{a_n} \right| > \binom{n}{m}.$$

Prove that the polynomial P has at least one zero with absolute value less than 1.

15. Let $a_1, a_2, \dots, a_n, b_1 \leq b_2 \leq \cdots \leq b_n$ be positive numbers such that
- $$a_1 \leq b_1, \quad a_1 + a_2 \leq b_1 + b_2, \dots, \quad a_1 + a_2 + \cdots + a_n \leq b_1 + b_2 + \cdots + b_n.$$

Prove that

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n} \leq \sqrt{b_1} + \sqrt{b_2} + \cdots + \sqrt{b_n}.$$

16. Let a, b, c be positive real numbers such that $ab + bc + ca = \frac{1}{3}$.
Prove that

$$\frac{a}{a^2 - bc + 1} + \frac{b}{b^2 - ca + 1} + \frac{c}{c^2 - ab + 1} \geq \frac{1}{a + b + c}.$$

17. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right) \left(b - 1 + \frac{1}{c}\right) \left(c - 1 + \frac{1}{a}\right) \leq 1.$$

18. Let p be a positive integer and let x_1 be a positive real number. Define the sequence $(x_n)_{n \geq 1}$ by

$$x_{n+1} = \sqrt{p^2 + 1}x_n + p\sqrt{x_n^2 + 1}$$

for all positive integers n . Prove that among the first m terms of the sequence there are at least $\left\lceil \frac{m}{3} \right\rceil$ irrational numbers.