

124 Challenging Algebra Problems

+ 30 Bonus Problems

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Preface

There are countless algebra books available today for students preparing for mathematics competitions. However, many of these books assume a level of knowledge that can be overwhelming for those just beginning their mathematical Olympiad journey.

We wrote *124 Challenging Algebra Problems + 30 Bonus Problems* to help fill this gap: a new book designed for students who are interested in proof-based problem solving and want to sharpen their skills. This book is suitable for students with a basic understanding of algebra who are ready to tackle problems that range from intermediate to advanced difficulty. But, it is more than just a series of problems. With rich theory and worked examples, students will have all of the necessary formative materials at their fingertips.

Together, the chapters provide a complete view of algebra as its own domain, specializing on the subjects most accessible through contest math. But, the accompanying examples and theoretical presentations expound upon the relevant mathematical machinery used to solve a wide array of problems in general. When relevant, some problems bridge the gap to other related branches of mathematics. While each section provides a comprehensive analysis of topics from the ground up, the concepts and problems eventually increase in difficulty and can become quite challenging.

The book comprises of twelve chapters. Each of the first eleven chapters contains theoretical discussions and examples that solidify the key concepts appearing in at least eleven problems that follow for the reader to solve. The last chapter is a collection of 30 bonus problems serving as an extra training set. The problems come from numerous sources around the world, including journals, contests, and Olympiads. Furthermore, virtually all of them are from within the last three years. It is safe to say that the application of the illustrated techniques are cutting edge and will help students preparing for any kind of mathematics contest.

The topics covered include factorizations, quadratic trinomials, absolute value, rational and irrational numbers, polynomials, functions, inequalities, sequences and sums, equations, systems of equations, integer part and fractional part — all of which are essential for success in competition math and beyond.

Solutions to all proposed problems are provided in the second part of the book. For roughly half of all problems, a second solution is also provided to augment the reader's understanding and make connections to other adjacent concepts. All aspects discussed focus on rigorous reasoning, and we hope that the beautiful solutions will also serve as a primer on how to write solid proofs.

The intended audience of this book are students preparing for mathematics competitions such as the USA(J)MO and other national or even international Olympiads, as well as teachers and math enthusiasts.

We would like to express our gratitude to Richard Stong for his precious contribution and suggestions that have elevated the quality of the book.

As always, enjoy the problems!

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Chapter 1

Factorizations

Factorization is one of the most important tools in algebra. It allows us to rewrite complex expressions as products of simpler factors, making equations easier to solve and expressions simpler to analyze. In this chapter, we will see various factorization techniques that will help you solve challenging problems throughout the book.

Remarkable Identities

$$(x \pm y)^2 = x^2 \pm 2xy + y^2$$

$$(x \pm y)^3 = x^3 \pm y^3 \pm 3xy(x \pm y)$$

$$x^2 - y^2 = (x - y)(x + y)$$

$$x^3 \pm y^3 = (x \pm y)(x^2 \mp xy + y^2)$$

If n is a positive integer, then

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}).$$

If n is a positive odd integer, then

$$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + \cdots + y^{n-1}).$$

Beautiful Cubes Identity

$$\begin{aligned} a^3 + b^3 + c^3 - 3abc &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= \frac{1}{2}(a + b + c) \left((a - b)^2 + (b - c)^2 + (c - a)^2 \right). \end{aligned}$$

Sophie Germain's Trick

$$\begin{aligned} a^4 + 4b^4 &= a^4 + 4a^2b^2 + 4b^4 - 4a^2b^2 \\ &= (a^2 + 2b^2)^2 - (2ab)^2 \\ &= (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2) \\ &= \left((a - b)^2 + b^2 \right) \left((a + b)^2 + b^2 \right). \end{aligned}$$

Example 1. The sum of two numbers is 6. If the sum of their cubes is 90, what is the sum of their squares?

Croatian Mathematical Olympiad 2022

First solution. Let the required numbers be x and y . According to the conditions of the problem, the following equalities hold

$$x + y = 6, \quad x^3 + y^3 = 90. \quad (1)$$

Applying the formula for the cube of the binomial $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$ and using (1) we get

$$216 = 90 + 3xy \cdot 6,$$

which gives $xy = 7$.

Now, from the formula for the square of the binomial, the required sum of squares is expressed as $x^2 + y^2 = (x + y)^2 - 2xy$, so

$$x^2 + y^2 = 36 - 2 \cdot 7 = 22. \quad \blacksquare$$

Second solution. As before, we have $x + y = 6$ and $x^3 + y^3 = 90$. Applying the formula for the sum of cubes $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ we have

$$90 = 6(x^2 - xy + y^2) \implies x^2 - xy + y^2 = 15,$$

which gives $(x + y)^2 - 3xy = 15$, thus $36 - 3xy = 15$, and so $xy = 7$. Then arguing as in the first solution, we get $x^2 + y^2 = 22$. ■

Example 2. If for real numbers x, y we have $\frac{1}{x} - \frac{1}{y} = 4$ and $xy = \frac{1}{4}$, what is $\frac{1}{x^4} + \frac{1}{y^4}$?

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Solution. Squaring both sides of the equality $\frac{1}{x} - \frac{1}{y} = 4$, we get

$$\frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2} = 16.$$

Since $\frac{1}{xy} = 4$, we get

$$\frac{1}{x^2} + \frac{1}{y^2} = 16 + 2 \cdot 4 = 24.$$

Squaring both sides of this equality, it follows that

$$\frac{1}{x^4} + \frac{2}{x^2y^2} + \frac{1}{y^4} = 576.$$

Since $\frac{1}{x^2y^2} = 16$, we get

$$\frac{1}{x^4} + \frac{1}{y^4} = 576 - 2 \cdot 16 = 544. \quad \blacksquare$$

Example 3. Let a, b, c be numbers such that $a + b + c = 0$ and $a^2 + b^2 + c^2 = 1$. Find the value of the expression $a^4 + b^4 + c^4$.

Moscow Mathematical Regata 2024

Solution. Since

$$(a + b + c)^2 = (a + b)^2 + c^2 + 2(a + b)c = a^2 + b^2 + c^2 + 2(ab + bc + ca),$$

substituting the values from the condition, we get $0 = 1 + 2(ab + bc + ca)$, and thus

$$ab + bc + ca = -\frac{1}{2}.$$

Squaring both sides, we get $a^2b^2 + b^2c^2 + c^2a^2 + 2abc(a + b + c) = \frac{1}{4}$ and since $a + b + c = 0$, we get

$$a^2b^2 + b^2c^2 + c^2a^2 = \frac{1}{4}.$$

Since $(a^2 + b^2 + c^2)^2 = a^4 + b^4 + c^4 + 2(a^2b^2 + b^2c^2 + c^2a^2)$, we get

$$1 = a^4 + b^4 + c^4 + 2 \cdot \frac{1}{4} \implies a^4 + b^4 + c^4 = \frac{1}{2}.$$

Example 4. Factorize into factors, each of which has a degree not greater than 2:

$$x^4 + y^4 + (x + y)^4.$$

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First solution. Let $z = x + y$. Squaring both sides and reordering, we get $x^2 + y^2 = z^2 - 2xy$. Squaring again we get

$$x^4 + y^4 = (z^2 - 2xy)^2 - 2x^2y^2 = z^4 - 4xyz^2 + 2x^2y^2.$$

Therefore

$$\begin{aligned} x^4 + y^4 + (x + y)^4 &= x^4 + y^4 + z^4 \\ &= 2z^4 - 4xyz^2 + 2x^2y^2 \\ &= 2(z^2 - xy)^2 \\ &= 2(x^2 + xy + y^2)^2 \end{aligned} \quad \blacksquare$$

Second solution. Expanding the term in brackets, we have

$$\begin{aligned} x^4 + y^4 + (x + y)^4 &= x^4 + y^4 + (x^2 + 2xy + y^2)^2 \\ &= 2x^4 + 2y^4 + 4x^2y^2 + 4x^3y + 4xy^3 + 2x^2y^2 \\ &= 2(x^4 + y^4 + x^2y^2 + 2x^2y^2 + 2x^3y + 2xy^3) \\ &= 2(x^2 + xy + y^2)^2. \end{aligned} \quad \blacksquare$$

Example 5. Given real numbers x, y, z such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x + y + z}$, prove that

$$\frac{1}{x^{2023}} + \frac{1}{y^{2023}} + \frac{1}{z^{2023}} = \frac{1}{x^{2023} + y^{2023} + z^{2023}}.$$

Estonian Mathematical Olympiad 2023

Solution. We will prove the thesis for any odd positive integer n and the conclusion will follow by taking $n = 2023$. The given condition can be rewritten as

$$(yz + zx + xy)(x + y + z) = xyz.$$

By viewing this as an equation in x , we get

$$x^2(y + z) + x(y + z)^2 + yz(y + z) = 0,$$

and factoring we get

$$(y + z)(x^2 + x(y + z) + yz) = 0 \implies (x + y)(y + z)(z + x) = 0.$$

Thus $x = -y$ or $y = -z$ or $z = -x$. Assuming without loss of generality that $x = -y$, we get $x^n = -y^n$, and hence for all odd positive integers n , we have

$$\frac{1}{x^n} + \frac{1}{y^n} + \frac{1}{z^n} = \frac{1}{x^n + y^n + z^n} = \frac{1}{z^n}. \quad \blacksquare$$

Example 6. Let a, b, c be integers such that $a + b + c = 0$. Let $S = ab + bc + ca$, $A = a^2 + a + 1$, $B = b^2 + b + 1$, $C = c^2 + c + 1$. Prove that the number $(S + A)(S + B)(S + C)$ is a perfect square.

Belarusian Mathematical Olympiad 2022

Solution. From the equality $a + b + c = 0$, it follows that $b + c = -a$. Therefore

$$S + A = bc + a(b + c) + A = bc + a + 1 = bc - (b + c) + 1 = (b - 1)(c - 1).$$

Similarly, $S + B = (c - 1)(a - 1)$ and $S + C = (a - 1)(b - 1)$. Hence

$$(S + A)(S + B)(S + C) = ((a - 1)(b - 1)(c - 1))^2,$$

which is a perfect square. ■

Example 7. Calculate

$$\frac{x^2}{x^4 - 3x^2 + 25}$$

knowing that $\frac{x}{x^2 + 7x + 5} = a$, where a is a given real number.

Write the answer as a ratio of two polynomials $\frac{p(a)}{q(a)}$.

Lithuanian Team Mathematical Olympiad 2022

Solution. If $a \neq 0$, then $x \neq 0$ and

$$x + \frac{5}{x} = \frac{x^2 + 7x + 5}{x} - 7 = \frac{1}{a} - 7 \implies \left(\frac{1}{a} - 7\right)^2 = x^2 + 10 + \frac{25}{x^2} = 13 + \frac{x^4 - 3x^2 + 25}{x^2}.$$

It follows that

$$\frac{x^2}{x^4 - 3x^2 + 25} = \frac{1}{\left(\frac{1}{a} - 7\right)^2 - 13} = \frac{a^2}{36a^2 - 14a + 1}.$$

If $a = 0$, then $x = 0$ and so $\frac{x^2}{x^4 - 3x^2 + 25} = \frac{a^2}{36a^2 - 14a + 1}$. ■

Example 8. The integers a, b, c satisfy the relations $a + b \neq c$, $b + c \neq a$, $c + a \neq b$, $a + b + c \neq 0$ and

$$\frac{a}{b + c - a} = \frac{b}{c + a - b} = \frac{c}{a + b - c}.$$

Determine the value of the expression

$$(2a - b - c - 1)^{2023} + (2b - c - a - 1)^{2023} + (2c - a - b - 1)^{2023}.$$

Lucian Petrescu - Laurențiu Panaitopol Competition 2023

Solution. Taking the reciprocal of these fractions and adding 2 to all,

$$\begin{aligned} \frac{b + c - a}{a} + 2 &= \frac{c + a - b}{b} + 2 = \frac{a + b - c}{c} + 2 \implies \\ \frac{a + b + c}{a} &= \frac{a + b + c}{b} = \frac{a + b + c}{c}. \end{aligned}$$

Since $a + b + c \neq 0$, we get $\frac{1}{a} = \frac{1}{b} = \frac{1}{c}$, which yields $a = b = c$. Thus

$$(2a - b - c - 1)^{2023} + (2b - c - a - 1)^{2023} + (2c - a - b - 1)^{2023} = -3. \quad \blacksquare$$

Example 9. Let x, y, z be nonzero real numbers with

$$\frac{x + y}{z} = \frac{y + z}{x} = \frac{z + x}{y}.$$

Determine all possible values of $\frac{(x + y)(y + z)(z + x)}{xyz}$.

Walther Janous - Austrian Mathematical Olympiad 2023