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Preface

The problems in this book were selected from the Romanian Team Selection Tests from 1975 to 1986. They are presented chronologically within each chapter.

1975 was the first author's last year to take team selection tests, as he was a high school senior. The second author, much younger, took these tests at the end of the period covered in the book.

We learned a lot of mathematics from the problems encountered in various Romanian national contests (some of which are included) and this is something that stayed with us for our entire careers. The problems are old, some of them more than forty years, but still of use to today's students, especially if they are involved in mathematical competitions. We felt that we had to do a lot more than just present a list of problems accompanied by their solutions. Thus we gave more than one solution when possible, commented on the solutions in order to make them clearer or to get a connection to other problems, and even digressed to other relevant results. We trust that all of this made the exposition more interesting, at least for the beginners in Olympiad mathematics. Here and there we left unanswered questions related to some of the problems presented; we hope the reader will accept and even enjoy these challenges. Of course, nowadays,"all answers are on the internet", but we feel the urge to repeat the good old advice: try first to solve the problem on your own, and only after that seek the answers.

You will definitely see the different proportion between algebra and geometry on one side, and combinatorics and number theory on the other, if one compares to the problems from present days contests. This was not our choice, but rather the reality of those times. And you will find that the two authors of this book kindly invite you to read and learn something from it, so that you will do well in mathematical contests, and, more importantly, in your future STEM journeys. Suggestions, comments, and remarks are highly welcomed.

We cannot end without thanking the authors of these problems (and apologize for not knowing all their names). Many thanks to Richard Stong for significant improvements of the text.

A booklet containing Romanian Team Selection Test problems from 1975 to 1983 along with sketches of their solutions was distributed at the final round of the 1984 competition. We used parts of this booklet (and some other materials, mentioned in the text) in developing the present book, and are grateful to all of those who contributed to these materials.

Titu Andreescu, Marian Tetiva

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Chapter 1

Algebra

1. Find the maximum value of

$$S_n(a_1, \ldots, a_n) = \sum_{k=1}^{n} a_k(1 - a_{k+1}),$$

where $a_1, \ldots, a_n$ are real numbers such that $a_k \geq \frac{1}{2}$ for $k = 1, \ldots, n$, and $a_{n+1} = a_1$.

2. Consider the sets of real numbers

$$A = \{a_1, a_2, \ldots\} \text{ and } B = \{b_1, b_2, \ldots\}$$

such that $a_1 = b_1$ and $b_{n+1} = b_n(1 - a_{n+1}) + (1 - b_n)a_{n+1}$, for any positive integer n. Prove that $\frac{1}{2} \in A$ if and only if $\frac{1}{2} \in B$.

3. Determine all functions $f : \mathbb{Q} \to \mathbb{Q}$ which satisfy the equation

$$f(x + y) + f(x - y) = 2(f(x) + f(y) + 1) \text{ for every } x, y \in \mathbb{Q}.$$

4. Solve in complex numbers the system of equations

$$\begin{aligned}(x_1 + x_2 + x_3)x_4 = (x_1 + x_2 + x_4)x_3 &= (x_1 + x_3 + x_4)x_2 \\ &= (x_2 + x_3 + x_4)x_1 = a,\end{aligned}$$

where a is a given complex number.

5. Determine the real numbers x_1, x_2, x_3, x_4 for which

$$\begin{aligned}\max\{x_1, x_2+x_3+x_4\} &= \max\{x_2, x_1+x_3+x_4\} = \max\{x_3, x_1+x_2+x_4\}\\ &= \max\{x_4, x_1+x_2+x_3\} = 1,\end{aligned}$$

and then formulate and solve the analogous problem for n numbers.

6. If a, b, c are the lengths of the sides of a triangle with perimeter 6 and A, B, C the measures, in radians, of its angles, then prove that

$$2\pi \leq aA + bB + cC < 3\pi.$$

7. Let n be an integer greater than 1 and let $a > 1$ be an irrational number. Prove that

$$\sqrt[n]{a+\sqrt{a^2-1}} + \sqrt[n]{a-\sqrt{a^2-1}}$$

is also irrational.

8. Let a, b, c, d be non-negative numbers such that

$$a \leq 1,\ a+b \leq 5,\ a+b+c \leq 14,\ \text{and } a+b+c+d \leq 30.$$

Prove that

$$\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d} \leq 10.$$

9. Prove that any polynomial function with real coefficients can be written as the difference of two increasing polynomial (real) functions.

10. Consider a polynomial P with integer coefficients and some distinct integers $a_1, \ldots, a_n$. Prove that if there exists a permutation σ of the set $\{1, \ldots, n\}$ such that $P(a_k) = a_{\sigma(k)}$, $k = 1, 2, \ldots, n$, then $\sigma \circ \sigma$ is the identity.

11. Given that the equation $ax^2 + bx + c = 0$, where a, b, c are integers and $a > 0$ has two distinct zeros in $(0, 1)$, prove that $a \geq 5$.

12. Prove that for any $x_k \in [1,2]$, $k = 1, \ldots, n$,

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k}\right)^2 \leq n^3.$$

13. Consider three cubic real polynomials P, Q, R such that

$$\begin{aligned} &P(x) \leq Q(x) \leq R(x) \text{ for every real } x, \text{ and} \\ &P(a) = Q(a) = R(a) \text{ for some real } a. \end{aligned}$$

Show that there exists $\lambda \in [0,1]$ such that

$$Q = (1-\lambda)P + \lambda R.$$

Prove that the same result does not hold for quartic polynomials.

14. Show that there exists a function $f : \mathbb{N}^* \to \mathbb{N}^*$ such that $f(f(n)) = n^2$ for any $n \in \mathbb{N}^*$.

15. Find a second degree polynomial $P \in \mathbb{R}[X]$ such that

$$\left|P(x) + \frac{1}{x-4}\right| \leq 0.01$$

for any $x \in [-1,1]$.
Are there first degree polynomials with this property?

16. Let $n \geq 2$ be an integer. Compute

$$\max_{1 \leq k \leq n} \max_{n_1 + \cdots + n_k = n} \binom{n_1}{2} \cdots \binom{n_k}{2}.$$

17. Let n be a positive integer. Find all polynomials $P \in \mathbb{R}[X]$ such that

$$P\left(x + \frac{1}{n}\right) + P\left(x - \frac{1}{n}\right) = 2P(x)$$

for all $x \in \mathbb{R}$.

18. For a real number x let (x) (this is an ad hoc notation) denote the nearest integer to x, that is, we have $(x) = p$ if and only if p is the only integer satisfying

$$p - \frac{1}{2} \leq x < p + \frac{1}{2}.$$

For instance, $(k) = k$ for any integer k, $(2.3) = 2$, and $(-2.5) = -2$. Find a polynomial P such that for all $n \in \mathbb{N}^*$,

$$\sum_{k=1}^{n^2} \left(\sqrt{k}\right) = P(n).$$

19. Let a, b, c and λ be real numbers with $a^2 + b^2 + c^2 = 1$ and $\lambda > 0$, $\lambda \neq 1$. Show that if $x, y, z \in \mathbb{R}$ and $x - \lambda y = a$, $y - \lambda z = b$, $z - \lambda x = c$ then

$$x^2 + y^2 + z^2 \leq \frac{1}{(\lambda - 1)^2}.$$

20. Let M be the set of all bijections $\phi : \mathbb{N} \to \mathbb{N}$. Show that there is no bijection $F : \mathbb{N} \to M$.

21. Consider the expansion

$$(1 + X + X^2)^n = a_0^{(n)} + a_1^{(n)} X + \cdots + a_{2n}^{(n)} X^{2n}.$$

Show that:

(a) $a_0^{(n)} \leq a_1^{(n)} \leq \cdots \leq a_n^{(n)}$ and $a_n^{(n)} \geq a_{n+1}^{(n)} \geq \cdots \geq a_{2n}^{(n)}$.

(b) Three of the sums

$$S_j^{(n)} = \sum_{k \geq 0} a_{4k+j}^{(n)}, \quad j = 0, 1, 2, 3$$

are equal, and the fourth differs from them by 1.

(Each sum is over all nonnegative indices k satisfying also $4k + j \leq 2n$.)

22. Let $n \geq 2$ be an integer, and let a be a positive real number. Find

$$\max_{0 \leq a_1, \ldots, a_n \leq a} \sum_{k=1}^{n} (a - a_1) \cdots (a - a_{k-1}) a_k (a - a_{k+1}) \cdots (a - a_n).$$

23. Let p be an odd prime, and let n be an even natural number. Denote

$$P(X) = X^{p-1} + X^{p-2} + \cdots + X + 1.$$

Show that the polynomial

$$-1 + \prod_{k=0}^{n-1} P(X^{p^k})$$

is divisible by $X^2 + 1$.

24. (a) Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the identity

$$f(x) + f(\lfloor x \rfloor) f(\{x\}) = x, \quad \forall x \in \mathbb{R}.$$

As usual, by $\lfloor x \rfloor$ and $\{x\}$ we denote the integer part and the fractional part of the real number x respectively.

(b) For each function f found in part (a) find all $k \in \mathbb{R}$ for which the equation $f(x) + mx + k = 0$ has a solution for any $m \in \mathbb{R}$.

25. Show that any set of 16 consecutive integers can be partitioned into two sets U and V of 8 elements each such that

$$\sum_{u \in U} u^k = \sum_{v \in V} v^k$$

for $k = 1, 2, 3$.

26. Consider the polynomial $P = \alpha X^3 - \dfrac{1}{6} X$, $\alpha \in \mathbb{R}$.

(a) Find all α for which $n \in \mathbb{Z}$ implies $P(n) \in \mathbb{Z}$.

(b) Show that for irrational α, and for any $0 \leq u < v \leq 1$ there exists $n \in \mathbb{N}$ for which

$$u < \{P(n)\} < v.$$

27. Let $x_1, \ldots, x_n$ be nonnegative real numbers. Show that there exist $a_1, \ldots, a_n \in \{-1, 1\}$ for which

$$a_1 x_1^2 + \cdots + a_n x_n^2 \geq (a_1 x_1 + \cdots + a_n x_n)^2.$$

28. Find all triples of real numbers x, y, z from the interval $[4, 40]$ that satisfy

$$x + y + z = 62, \quad xyz = 2880.$$

29. Let A be a finite set of real numbers and let $f : A \to A$ be a function for which $|f(x) - f(y)| < |x - y|$ for all distinct $x, y \in A$. Show that f is not onto and that f has precisely one fixed point.

30. Let $f : \mathbb{R} \to \mathbb{R}$ be a function for which

$$f(f(x)) = \frac{x^9}{(x^2+1)(x^6+x^4+2x^2+1)}$$

for all $x \in \mathbb{R}$. Show that f has exactly one fixed point.

31. Let a, b, c be the lengths of the sides of a triangle. If the real numbers x, y, z satisfy $ax + by + cz = 0$, prove that $xy + xz + yz \leq 0$.

32. Prove that among any 19 distinct integers from the set $\{1, \ldots, 1022\}$ one can find three distinct numbers a, b, c for which $a < b + c < 4a$. If 1022 is replaced by n find the smallest possible m to replace 19, such that the result still holds true.

33. Let A be a non-empty set and let $h : A \times A \to A$ be a function. Show that the following statements are equivalent:

(i) There exists a bijection $f : A \to A$ such that $h(x, y) = f(x)$ for any $x, y \in A$ or $h(x, y) = f(y)$ for any $x, y \in A$.

(ii) For any two bijections $s : A \to A$, $t : A \to A$ the function $h_{s,t} : A \to A$ defined by $h_{s,t}(a) = h(s(a), t(a))$ is a bijection.

34. Let P be a real polynomial such that $P(\sin t) = P(\cos t)$ for every real number t. Prove that there exists a real polynomial T such that

$$P(X) = T(X^4 - X^2).$$

35. If $x_1, \ldots, x_n \in \mathbb{R}$, $a \in \left[0, \frac{\pi}{2}\right]$ and $\sum_{k=1}^{n} \sin x_k \geq n \sin a$, then prove that

$$\sum_{k=1}^{n} \sin(x_k - a) \geq 0.$$

36. Let p be an odd prime number, let $f \in \mathbb{Q}[X]$ be an irreducible polynomial of degree p over the field of rational numbers, and let $x_1, \ldots, x_p$ be the complex roots of f. Prove that for any non-constant polynomial g with rational coefficients, and of degree less than p, the numbers $g(x_1), \ldots, g(x_p)$ are pairwise distinct.

37. Let $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ be $n \geq 2$ functions. Show that there exist numbers $a_1, \ldots, a_n \in [0, 1]$ such that

$$\left| a_1 \ldots a_n - \sum_{i=1}^{n} f_i(a_i) \right| \geq \frac{1}{2^n}.$$

38. Find all pairs $(p, q) \in \mathbb{R} \times \mathbb{R}$ for which the inequality

$$\left| \sqrt{1 - x^2} - px - q \right| \leq \frac{\sqrt{2} - 1}{2}$$

holds for all $x \in [0, 1]$.

39. Find all positive integers $n \geq 2$ for which the equation

$$a_n x^2 - 2\sqrt{a_1^2 + \cdots + a_n^2}\, x + a_1 + \cdots + a_{n-1} = 0$$

has real roots now matter how the reals $a_1, \ldots, a_n$ (with $a_n \neq 0$) are chosen.

40. Let f be a real polynomial such that $f(x) > 0$ for all $x \in \mathbb{R}$. Show that there exists a polynomial g such that fg is a polynomial with all coefficients positive.

41. Let a and b be two integers. Find all the polynomials f from $\mathbb{Z}[X]$ for which

$$xf(x-b) = (x-a)f(x)$$

for all $x \in \mathbb{N}$.

42. For integer $n \geq 2$ let $P_n(X) = 2^{n-1}(X^n + 1) - (X+1)^n$. Show that

(a) All the roots of $P_n(X)$ have the same absolute value.

(b) For any $n \geq 1$ we have

$$\frac{P_{2n}(X)}{X^n} = Q_n\left(X + \frac{1}{X}\right)$$

where Q_n is a polynomial of degree n whose roots are all real and belong to the interval $[-2, 2]$.

43. Find all pairs of real functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) - f(y) = (x-y)(g(x) + g(y))$$

for all $x, y \in \mathbb{R}$.

44. Show that there exist no sequences $(a_n)_{n\geq 1}$ of positive integers such that

$$a_{n-1} \leq (a_{n+1} - a_n)^2 \leq a_n$$

for all $n \geq 2$.

45. Let $f : \mathbb{N}^* \to \mathbb{N}^*$ be a surjection and let $g : \mathbb{N}^* \to \mathbb{N}^*$ be an injection, such that $f(n) \geq g(n)$ holds for all $n \in \mathbb{N}^*$. Prove that $f = g$.

46. Let a, b, x, y, z be positive real numbers. Show that

$$\frac{x}{ay+bz} + \frac{y}{az+bx} + \frac{z}{ax+by} \geq \frac{3}{a+b}.$$

47. Let $n \geq 2$ be a positive integer. Find all possible values for the positive integer $p \geq n$ such that the system

$$\sum_{j=1}^{n} x_j^k = n, \ k = 1, \ldots, n-1, p$$

has the unique (and obvious) solution (in $\mathbb{C}^n$) $x_1 = \cdots = x_n = 1$.

48. Show that if n is an even positive integer and $a_0, a_1, \ldots, a_n$ are positive real numbers that form an arithmetic progression with common difference $d \in [-2a_n, a_0]$, then

$$a_0 + a_1 x + \cdots + a_n x^n > 0$$

for all $x \in \mathbb{R}$.

49. The sequence $(x_n)_{n \geq 0}$ satisfies the relations

$$\sqrt{x_{n+2} + 2} \leq x_n \leq 2$$

for all $n \in \mathbb{N}$. What are the values that x_{1986} can assume?

50. Determine all bijective and monotonic functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) + f^{-1}(x) = 2x$$

for all $x \in \mathbb{R}$. (Of course, f^{-1} denotes the inverse of the function f).

Chapter 2

Number Theory and Combinatorics

a. Number Theory

1. Find a polynomial P with integer coefficients, such that $P(n) + 4^n$ is divisible by 27 for any positive integer n.

2. Prove that the equation

$$x_1^2 + \cdots + x_n^2 - \frac{(x_1 + \cdots + x_n)^2}{n} = 1$$

admits solutions in integers if and only if $n = 4$.

3. Let $n \geq k$ be positive integers, and let p be a prime. Prove that if $\binom{n}{k}$ is divisible by p^s (for some natural number s), then $n \geq p^s$.

4. Prove that every infinite arithmetic progression of positive integers contains an infinite geometric progression.

5. Solve in non-negative integers the equation

$$3^x + 4^y = 5^z.$$

6. Let $p > 2$ be a prime number. Show that

$$(-1)^{\lfloor\sqrt{p}\rfloor}\prod_{j=2}^{p-1}\sin\frac{p\pi}{j} > 0.$$

7. If n is an odd positive integer, show that

$$\frac{2\cdot(3n)!}{n!(2n)!}$$

is divisible by $3(3n-1)(3n-2)$.

8. Let b_n be the last digit of the sum $\sum_{k=1}^{n} k^k$. Show that $b_{n+100} = b_n$ for any $n \in \mathbb{N}$.

9. Let n be a positive integer. An $n \times n$ matrix M_n is filled with zeros and ones in the following way: for each $1 \le i \le n$ and $1 \le j \le n$, the entry at the intersection of the ith row with the jth column is 1 if i divides j and 0 otherwise. For a given positive integer p, find the greatest possible integer n for which there are in M_n precisely p columns such that the sum of entries in each of them is odd.

10. Let m be an integer not divisible by 3. Show that there are infinitely many positive integers N such that both N and $N+1$ have the sums of digits divisible by m.

11. Consider all numbers $k \in \mathbb{Z}$ such that

$$1981 \mid 7^n + k\cdot 290^n$$

for all positive integers n. Find the least possible value of $|k|$ for the numbers k with this property.

12. For each positive integer x let $f(x)$ be the largest positive integer k for which there exist k integers $1 \le a_1 < \cdots < a_k \le x$ such that $a_i + a_j$ does not divide $a_i a_j$ whenever $1 \le i < j \le k$. Show that

$$f(2^n) \ge 2^{n-1} + n.$$

13. For an integer $m = \overline{m_1m_2m_3m_4m_5m_6m_7}$ $(0 \le m_i \le 9,\ m_1 \ne 0)$, define

$$f(m) = \frac{m}{\overline{m_1m_2m_3} + \overline{m_4m_5m_6} + m_7}.$$

Find the least possible value of $f(m)$ for m running over all 7-digits natural numbers. (All numbers are written in base 10. We denote by $\overline{ab\ldots l}$ the number with digits $a, b, \ldots, l$.)

14. Let p be a prime number, and let n, a be two positive integers.

(a) If $a = a_0 + a_1p + a_2p^2 + \cdots$, $n = n_0 + n_1p + n_2p^2 + \cdots$ are the expansions of a and n respectively in base p, show that $\binom{n}{a}$ is divisible by p if and only if $n_i < a_i$ for at least one $i \in \mathbb{N}$.

b) If $t \in \mathbb{N}$ and $p^t \le a < p^{t+1}$ then show that

$$\binom{n + p^{t+1}}{a} \equiv \binom{n}{a} \pmod{p} \text{ for any } n \in \mathbb{N}.$$

15. Let us denote by $P(n)$ the following statement: There exist numbers of n digits in decimal expansion with the sum of digits equal to the product of digits. Show that $P(n)$ is true for infinitely many $n \in \mathbb{N}^*$, and false for infinitely many $n \in \mathbb{N}^*$.

16. Let p be an odd prime, let $m_1, \ldots, m_p$ be p consecutive integers, and let σ be a permutation of $\{1, \ldots, p\}$. Show that there exist $k, l \in \{1, \ldots, p\}$, $k \ne l$, such that

$$m_k m_{\sigma(k)} \equiv m_l m_{\sigma(l)} \pmod{p}.$$

17. Find all triples (x, y, z) of non-negative integers that satisfy

$$3^x \cdot 2^y + 1 = z^2.$$

18. For a positive integer $n \ge 2$ let $h(n)$ denote the largest prime divisor of n. (For instance, $h(7) = h(21) = 7$, or $h(32) = 2$.) Show that there are infinitely many n for which we have

$$h(n) < h(n+1) < h(n+2).$$

19. Find all triples (x, y, z) of non-negative integers satisfying the relation

$$(x+y)^2 + 3x + y + 1 = z^2.$$

20. (a) Find all primes p, q and non-negative integers k that satisfy the equation

$$p^2 - q^2 = 2^k.$$

(b) If $n \geq 3$ is an integer, and k is a non-negative integer, show that no primes p, q satisfy the equation

$$p^n - q^n = 2^k.$$

21. Let m, n be positive integers. Consider the set A of all m-tuples $(k_1, \ldots, k_m)$ of non-negative integers such that

$$k_1 + \cdots + k_m = n \text{ and } \frac{n!}{k_1! \cdots k_m!} \text{ is odd.}$$

Show that the cardinality of A is a power of m.

22. Find all non-negative integer solutions to the equation

$$x^2y + y^2z + z^2x - 3xyz = 1.$$

23. Let k be a natural number. The sequence $(x_n)_{n\geq 0}$ is defined by

$$x_0 = 0,\ x_1 = 1,\ x_{n+2} = kx_{n+1} + x_n \text{ for each } n \geq 0.$$

Show that among the numbers $x_1, \ldots, x_{1986}$ there are two whose product is divisible by $19 \cdot 86$.

24. Let the positive integers s, p, q be such that q divides s, p divides $s - 1$, $q > p$, and $s \leq pq$. Show that

$$s \leq pq - \frac{q(p-1)}{q-p}.$$

For given p, provide an example of s, q for which the equality is achieved.

25. Find all pairs (a, b) of integers for which the equation

$$(ax - by)^2 + (bx - ay)^2 = xy$$

has a solution in non-zero integers.