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Preface

Mathematics competitions often feature problems involving angles, some specifically asking students the measures of angles in triangles or quadrilaterals. This book responds to the need of those who would like to improve their knowledge and skills in geometric problems in which angles play an important role.

This is Book 1 which is divided into three chapters that are designed to engage readers in the study of angles. The first chapter begins by presenting essential concepts concerning angles in the plane and provides a multitude of applications. Throughout this chapter the readers will encounter relevant examples that illustrate key concepts and principles, which will help them grasp the material and solve the proposed problems around the book.

The second chapter shifts focus to other classical results in Euclidean geometry. It delves into the tools that use angles as their primary elements, unveiling the power of angles in problems related to triangle similarity, quadrilaterals, and circles. This chapter was written to show math enthusiasts how angles are critical in solving many mathematics contest problems. Numerous significant examples and problems have been carefully selected from recent mathematical Olympiads, providing the readers with the opportunity to put their knowledge to test.

This awesome journey culminates with the third chapter where a collection of thought-provoking problems are being offered. These problems draw upon the insights and skills acquired in previous chapters, inviting the reader to explore angles in novel ways. All solutions to the proposed problems in the book are included at the end.

The intended audience includes students who are preparing for mathematical Olympiads and those who want to enhance their geometry skills. Teachers and coaches who train students for competitions or organize mathematical circles can certainly benefit from this book as well.

Titu Andreescu — Alessandro Ventullo

Chapter 1

Basic Facts

The goal of this chapter is to apply elementary plane geometry knowledge in order to study problems related with angles. We assume that the reader is familiar with the basic notions of point, line, parallelism, perpendicularity, complementary and supplementary angles and angle bisectors.
In order to prove that two angles are equal we can argue in the following ways:

1. we prove that they are opposite angles at a vertex;
2. we prove that they have the same supplementary angle;
3. we prove that they are corresponding angles in two congruent triangles;
4. we prove that they are the base angles of an isosceles triangle;
5. we prove that they are both angles of an equilateral triangle;
6. we use the transitive property of the equality;
7. we use our knowledge on the angle bisector;
8. we use our knowledge on two parallel lines cut by a transversal.

1.1 Awesome angles in the plane

Our discussion on angles will start by dealing with angles in the plane.
We state and prove a few fundamental facts that every student should know.

Theorem 1.1 *The sum of three interior angles of a triangle is* $180°$.

Proof. Let ABC be a triangle and draw the line passing through A parallel to the line BC. Let D and E be two points on this line so that A is between D

and E. Since $DE \parallel BC$ and AB is a secant, we have $\angle DAB = \angle ABC$ (alternate interior angles). Since $DE \parallel BC$ and AC is a secant, we get $\angle EAC = \angle ACB$ (alternate interior angles). Now, since D, A, E are collinear, we get $\angle DAE = 180^\circ$. Hence,

$$180^\circ = \angle DAE = \angle DAB + \angle BAC + \angle EAC = \angle ABC + \angle BAC + \angle ACB,$$

and we are done. ❑

Theorem 1.2 *The sum of all interior angles of a convex n-gon is* $(n-2)\cdot 180^\circ$.

Proof. We prove the statement by induction on n. For $n = 3$ this is the previous theorem. Assume that the statement is true for a convex n-gon and let's prove it for a convex $(n+1)$-gon. Let $A_1A_2\ldots A_{n+1}$ be a convex $(n+1)$-gon. Draw the diagonal A_1A_n and consider the convex n-gon $A_1A_2\ldots A_n$. By the inductive hypothesis the sum of all interior angles of $A_1A_2\ldots A_n$ is $(n-2)\cdot 180^\circ$. Moreover, the sum of the interior angles of triangle $A_nA_{n+1}A_1$ is 180°. So, the sum of all interior angles of $A_1A_2\ldots A_{n+1}$ is

$$(n-2)\cdot 180^\circ + 180^\circ = (n-1)\cdot 180^\circ,$$

and the conclusion follows by the Principle of Mathematical Induction. ❑

Corollary 1.3 *The size of each internal angle of a regular n-gon is*

$$\frac{(n-2)\cdot 180^\circ}{n}.$$

Theorem 1.4 *The sum of all exterior angles of an n-gon is* 360°.

Proof. Let S_E and S_I be the sum of all exterior and internal angles, respectively. Observe that the sum of an internal angle and its exterior angle is equal to 180°. Since we have n angles, we have $S_E + S_I = n\cdot 180^\circ$. As

$$S_I = (n-2)\cdot 180^\circ,$$

we have

$$S_E + (n-2)\cdot 180^\circ = n\cdot 180^\circ \implies S_E = 360^\circ,$$

and we are done. ❑

Example 1.1. Draw two adjacent angles $\angle AOB$ and $\angle BOC$ such that $\angle AOB = 72^\circ$ and $\angle BOC$ is acute.

(a) Prove that the angle determined by the angle bisectors of $\angle AOC$ and $\angle BOC$ does not depend on $\angle BOC$.

(b) Let OD be the ray opposite to the ray OC and suppose that

$$2\angle BOC = 3\angle AOD.$$

Find $\angle BOC$, $\angle AOD$ and $\angle BOD$.

Solution. (a) Let OD_1 be the angle bisector of $\angle AOC$ and let OE be the angle bisector of $\angle BOC$. We have

$$\angle AOD_1 = \angle D_1OC = \frac{\angle AOC}{2} \text{ and } \angle BOE = \angle EOC = \frac{\angle BOC}{2}.$$

Hence,

$$\angle D_1OE = \angle D_1OC - \angle EOC = \frac{\angle AOC - \angle BOC}{2} = \frac{72^\circ}{2} = 36^\circ,$$

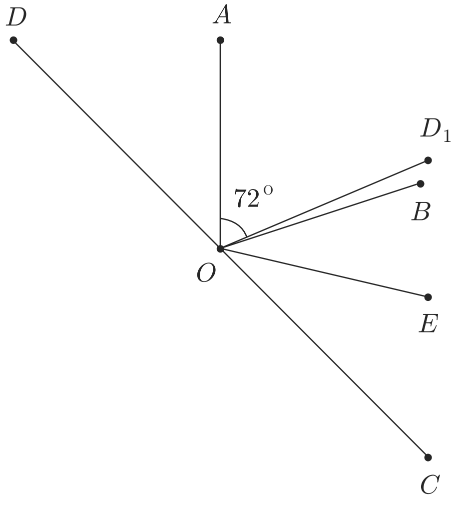
which doesn't depend on $\angle BOC$.

(b) Since OD is the ray opposite to the ray OC, we have $\angle DOC = 180^\circ$. Thus

$$\angle AOD + \angle AOB + \angle BOC = 180^\circ \implies \angle AOD + \angle BOC = 108^\circ.$$

Since $2\angle BOC = 3\angle AOD$, by multiplying the above equality by 2, we get $5\angle AOD = 216^\circ$, i.e., hence $\angle AOD = 43^\circ 12'$,

$$\angle BOC = 180^\circ - (72^\circ + 43^\circ 12') = 64^\circ 48' \text{ and } \angle BOD = 115^\circ 12'.$$



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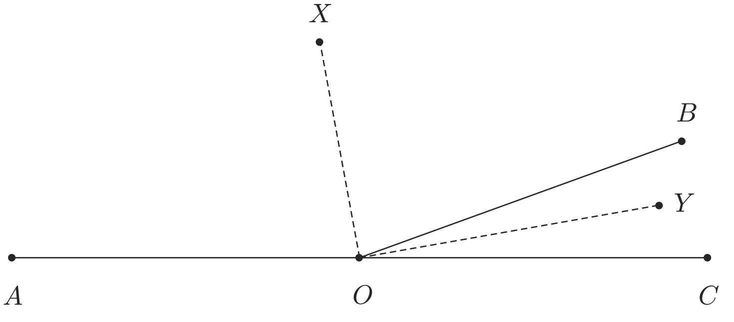
Example 1.2. Two angles are adjacent and their angle bisectors are perpendicular. Find the angles knowing that one angle is eight times greater than the other.

Solution. If the angle bisectors of two adjacent angles are perpendicular, then these two angles are adjacent and supplementary. Indeed, by denoting $\angle AOB$ and $\angle BOC$ these two angles, where B is inside $\angle AOC$ and OX, OY are the angle bisectors of $\angle AOB$, $\angle BOC$, respectively, then $\angle AOX = \angle XOB$, $\angle BOY = \angle YOC$. Since $OX \perp OY$, we have $\angle XOY = 90°$, So

$$\angle AOC = \angle AOB + \angle BOC = 2\angle XOB + 2\angle BOY = 2\angle XOY = 180°.$$

If $\angle AOB = 8\angle BOC$, then adding to both sides of this equality $\angle BOC$, we get $9 \cdot \angle BOC = 180°$, i.e., $\angle BOC = 20°$ and $\angle AOB = 160°$.



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Example 1.3. Let B and C be two points on the same side of the line AD and O a point on the line between A and D with $\angle AOB$, $\angle BOC$ and $\angle COD$ directly proportional to the numbers 7, 9 and 20.

(a) Find $\angle AOB$, $\angle BOC$ and $\angle COD$.

(b) If OE is the ray opposite to the ray OC and OM and ON are angle bisectors of $\angle AOE$ and $\angle DOE$, respectively, determine $\angle MON$.

Solution. (a) Let

$$\frac{\angle AOB}{7} = \frac{\angle BOC}{9} = \frac{\angle COD}{20} = k.$$

We have $\angle AOB = 7k$, $\angle BOC = 9k$ and $\angle COD = 20k$, so

$$\angle AOB < \angle BOC.$$

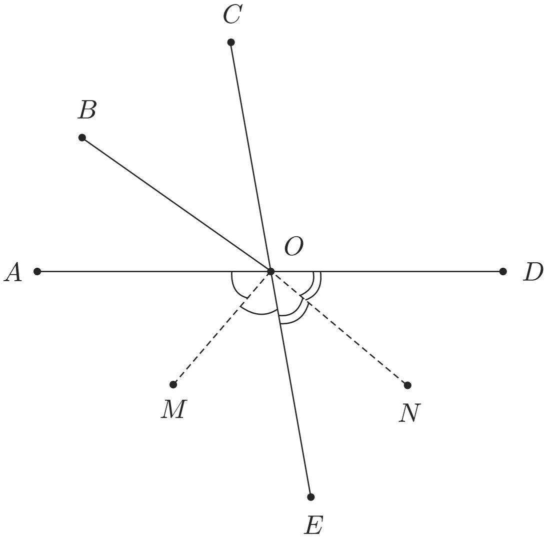
Thus B is inside $\angle AOC$. As $\angle AOD = 180°$, we get

$$7k + 9k + 20k = 180° \implies k = 5°.$$

Thus

$$\angle AOB = 35^\circ,\ \angle BOC = 45^\circ,\ \angle COD = 100^\circ.$$

(b) The angles $\angle AOE$ and $\angle DOE$ are supplementary adjacent angles, so their angle bisectors are perpendicular and $\angle MON = 90^\circ$.



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Example 1.4. Let $\angle AOB$ be an acute angle. In the half-plane determined by the line AO that does not contain the point B, consider the points C and D such that $OC \perp OA$, $OD \perp OB$ and OP is the angle bisector of $\angle AOD$.

(a) If $\angle COP = \angle AOB + 24^\circ$, find $\angle AOB$ and $\angle DOP$.

(b) If OD is the angle bisector of $\angle COP$, prove that OA is the angle bisector of $\angle BOP$.

Solution. **(a)** Since $OC \perp OA$, we have $\angle AOC = 90^\circ$ and since $OD \perp OB$, we have $\angle BOD = 90^\circ$. Since $\angle AOB$ and $\angle COD$ have the same complement $\angle AOD$, we get $\angle AOB = \angle COD$. As

$$\angle COP = \angle AOB + 24^\circ,$$

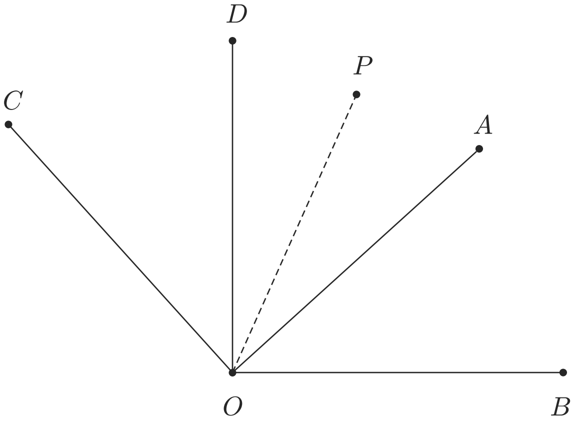
we get

$$\angle COD + \angle DOP = \angle AOB + 24^\circ \implies \angle DOP = 24^\circ.$$

As OP is the angle bisector of $\angle AOD$, we get $\angle AOD = 2\angle DOP = 48^\circ$, so

$$\angle AOB = \angle BOD - \angle AOD = 90^\circ - 48^\circ = 42^\circ.$$

(b) Since OD is the angle bisector of $\angle COP$, we get $\angle COD = \angle DOP$. As $\angle AOB = \angle COD$ and $\angle AOP = \angle DOP$, we get $\angle AOB = \angle AOP$. Since A is inside $\angle BOP$, this says OA is the angle bisector of $\angle BOP$.



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Example 1.5. The angles $\angle AOB$ and $\angle COD$ are right, C is in the interior of $\angle AOB$, OP and OQ are angle bisectors of $\angle AOC$ and $\angle BOD$, respectively.

(a) Prove that $\angle AOC = \angle BOD$.

(b) Find $\angle POQ$.

Solution. (a) We know that $\angle AOB = \angle COD = 90°$, so

$$\angle AOC = \angle AOB - \angle BOC = \angle COD - \angle BOC = \angle BOD.$$

(b) Since OP is the angle bisector of $\angle AOC$, we get

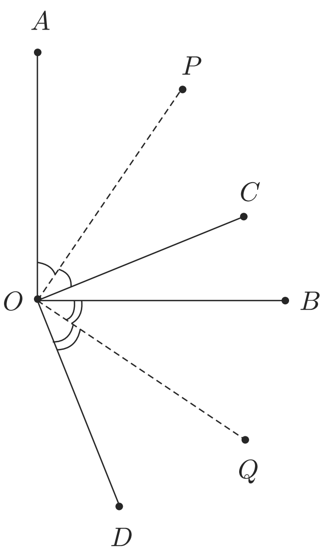
$$\angle AOP = \frac{\angle AOC}{2}$$

and since OQ is the angle bisector of $\angle BOD$, we get

$$\angle BOQ = \frac{\angle BOD}{2}.$$

From part (a), we immediately see that $\angle AOP = \angle BOQ$. Hence,

$$\begin{aligned} \angle POQ &= \angle POB + \angle BOQ \\ &= \angle POB + \angle AOP \\ &= \angle AOB = 90°. \end{aligned}$$



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Example 1.6. Let OC be the angle bisector of $\angle AOB$, let OD be the angle bisector of $\angle AOC$, let OE be the angle bisector of $\angle BOD$, and let OF be the angle bisector of $\angle AOE$. If $\angle DOF = 10^\circ$, compute $\angle AOB$.

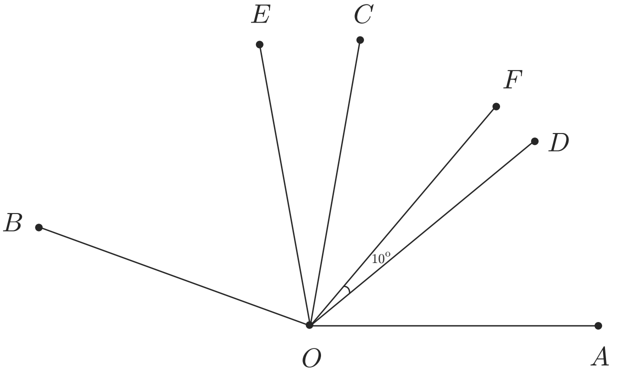
Solution. Let $x = \angle AOB$. Then, $\angle AOC = \angle BOC = \dfrac{x}{2}$, so

$$\angle AOD = \angle COD = \frac{x}{4},$$

which gives $\angle BOD = \dfrac{3x}{4}$. Therefore, $\angle BOE = \angle DOE = \dfrac{3x}{8}$ and $\angle AOE = \dfrac{5x}{8}$, which implies $\angle AOF = \dfrac{5x}{16}$. Hence, $\angle DOF = \dfrac{5x}{16} - \dfrac{x}{4} = \dfrac{x}{16}$, i.e.

$$\frac{x}{16} = 10^\circ \implies x = 160^\circ.$$

We conclude that $\angle AOB = 160^\circ$.



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