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## Part I

# Theory, Examples, and Problems



# Chapter 1

## On the Form $x^d P\left(\frac{1}{x}\right)$

On some occasions, changing the original setting of the problem will make the problem more approachable.

For instance, instead of working with the polynomial  $a_d x^d + \dots + a_0$ , we may work with the polynomial  $a_0 x^d + \dots + a_d$ . This, may lead us to discover some unseen consequences of the hypotheses. We can easily transform

$$P(x) = a_d x^d + \dots + a_0 \quad \text{to} \quad a_0 x^d + \dots + a_d.$$

We just take the polynomial  $x^d P\left(\frac{1}{x}\right)$ .

### 1.1 Basic properties

#### Reciprocal polynomial

Let  $P(x) = a_d x^d + \dots + a_0$ . The polynomial

$$x^d P\left(\frac{1}{x}\right) = a_0 x^d + \dots + a_d$$

is called the *reciprocal polynomial* or *inverse polynomial* of  $P(x)$ .

From now on, we will call  $x^d P\left(\frac{1}{x}\right)$  the reciprocal polynomial.

For example, the reciprocal polynomial of  $P(x) = 2x^3 - 3x^2 + 1$  is  $x^3 - 3x + 2$ . The reciprocal polynomial of  $Q(x) = 4x^3 - 3x$  is  $-3x^2 + 4$ .

In short, the reciprocal polynomial reverses the order of coefficients of the original polynomial.

Sometimes, the reciprocal polynomial has lower degree than the original polynomial. Crudely speaking, it depends on zero-multiplicity of the polynomial. That is, if  $P(x)$  is a polynomial of degree  $d$  such that  $0$  is a root with multiplicity  $r$ , so  $P(x) = x^r Q(x)$  with  $Q(0) \neq 0$ , we easily find that

$$x^d P\left(\frac{1}{x}\right) = x^d x^{-r} Q\left(\frac{1}{x}\right) = x^{d-r} Q\left(\frac{1}{x}\right).$$

The reciprocal polynomial  $x^d P(x) = x^{d-r} Q\left(\frac{1}{x}\right)$  is of degree  $d - r$ .

**Example 1.1.** Let  $P(x)$  be a polynomial of degree 5 with nonnegative integer coefficients such that for all  $x \neq 0$ ,  $P(x) = x^6 P\left(\frac{1}{x}\right)$  and  $P(2) = 10P(1)$ . Find the greatest possible value of  $\frac{P(3)}{P(2)}$ .

**Solution.** Let  $P(x) = ax^5 + \dots + c$ . If  $c \neq 0$ , then the polynomial  $x^6 P\left(\frac{1}{x}\right)$  has degree 6 and it cannot be equal to a polynomial  $P(x)$  of degree 5. Hence  $c = 0$ . Putting  $P(x) = xQ(x)$  for some polynomial  $Q(x)$  of degree 4, we get

$$P(x) = xQ(x) = x^6 P\left(\frac{1}{x}\right) = x^5 Q\left(\frac{1}{x}\right).$$

We deduce that  $Q(x) = x^4 Q\left(\frac{1}{x}\right)$ , which gives  $Q(x) = ax^4 + bx^3 + cx^2 + bx + a$  for some non-negative integers  $a, b, c$ . Moreover, from  $P(2) = 10P(1)$ , we find

$$2Q(2) = 10Q(1).$$



Thus  $17a + 10b + 4c = 5(2a + 2b + c)$ , which yields  $7a = c$ . Moreover,

$$\begin{aligned} \frac{P(3)}{P(2)} &= \frac{3Q(3)}{2Q(2)} = \frac{3(82a + 30b + 9c)}{10(2a + 2b + c)} \\ &= \frac{3(145a + 30b)}{10(9a + 2b)} \\ &= \frac{3(29a + 6b)}{2(9a + 2b)} \\ &= \frac{6a + 3(27a + 6b)}{2(9a + 2b)} \\ &= \frac{9}{2} + \frac{3a}{9a + 2b} \\ &= \frac{9}{2} + \frac{1}{3 + \frac{2b}{3a}}. \end{aligned}$$

Since  $\frac{2b}{3a} \geq 0$ , then  $\frac{9}{2} + \frac{1}{3 + \frac{2b}{3a}} \leq \frac{9}{2} + \frac{1}{3} = \frac{29}{6}$ . The equality case occurs for  $b = 0$  and the polynomial  $P(x) = x(ax^4 + 7ax^2 + a) = ax(x^4 + 7x^2 + 1)$ . ■

## 1.2 Sum of squares of coefficients of a polynomial

Let  $P(x) = a_d x^d + \dots + a_0$ . It is instructive to consider the product

$$P(x)P\left(\frac{1}{x}\right) = (a_d x^d + \dots + a_1 x + a_0)(a_d x^{-d} + \dots + a_1 x^{-1} + a_0).$$

The above product is a rational function. The constant term of the above product is of special interest. Constant terms arise from the products of the form  $a_r x^r \cdot a_r x^{-r} = a_r^2$ . Hence the constant term of the above product is the sum of squares of the coefficients of the polynomial.

### Sum of squares of the coefficients of a polynomial

The sum of squares of the coefficients of a polynomial  $P(x)$  is the coefficient of the constant term in the product  $P(x)P\left(\frac{1}{x}\right)$ .

**Example 1.2.** Let  $P_{2n}(x) = (6x^2 + 5x + 1)^n$  and  $Q_{2n}(x) = (3x^2 + 7x + 2)^n$ . Prove that the sum of squares of the coefficients of  $P_{2n}(x)$  and  $Q_{2n}(x)$  are the same.

**Solution.** It is known that the sum of squares of the coefficients of  $P_{2n}(x)$  and  $Q_{2n}(x)$  are equal to the coefficient of  $x^0$  in the products  $P_{2n}(x)P_{2n}\left(\frac{1}{x}\right)$  and  $Q_{2n}(x)Q_{2n}\left(\frac{1}{x}\right)$ . Note that

$$P_{2n}(x) = (6x^2 + 5x + 1)^n = (3x + 1)^n(2x + 1)^n$$

and

$$Q_{2n}(x) = (3x^2 + 7x + 2)^n = (3x + 1)^n(x + 2)^n.$$

Therefore

$$\begin{aligned} P_{2n}(x)P_{2n}\left(\frac{1}{x}\right) &= (3x + 1)^n(2x + 1)^n \left(\frac{3}{x} + 1\right)^n \left(\frac{2}{x} + 1\right)^n \\ &= \frac{(3x + 1)^n(2x + 1)^n(x + 3)^n(x + 2)^n}{x^{2n}} \end{aligned}$$

and

$$\begin{aligned} Q_{2n}(x)Q_{2n}\left(\frac{1}{x}\right) &= (3x + 1)^n(x + 2)^n \left(\frac{3}{x} + 1\right)^n \left(\frac{1}{x} + 2\right)^n \\ &= \frac{(3x + 1)^n(x + 2)^n(3 + x)^n(2x + 1)^n}{x^{2n}}. \end{aligned}$$

Comparing these we see that

$$Q_{2n}(x)Q_{2n}\left(\frac{1}{x}\right) = P_{2n}(x)P_{2n}\left(\frac{1}{x}\right).$$

Thus the coefficient of  $x^0$  is the same in both products. ■

**Example 1.3.** Assume that for all  $x \neq 0$ :

$$\left(x + \frac{1}{x} + \sqrt{2}\right)^{12} = \sum_{k=0}^{24} c_k x^{k-12}.$$

Find the value of  $\sum_{k=0}^{24} (-1)^k c_k^2$ .

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**Solution.** If we use the substitution  $x \mapsto -\frac{1}{x}$ , we find that

$$\left(x + \frac{1}{x} - \sqrt{2}\right)^{12} = \sum_{k=0}^{24} c_k (-1)^{k-12} x^{12-k} = \sum_{k=0}^{24} c_k (-1)^k x^{12-k}.$$

Multiplying this equality by the original equality, we get

$$\begin{aligned} \left(x + \frac{1}{x} + \sqrt{2}\right)^{12} \left(x + \frac{1}{x} - \sqrt{2}\right)^{12} &= \left(\left(x + \frac{1}{x}\right)^2 - 2\right)^{12} \\ &= \left(x^2 + \frac{1}{x^2}\right)^{12} \\ &= \left(\sum_{k=0}^{24} c_k x^{k-12}\right) \left(\sum_{k=0}^{24} (-1)^k c_k x^{k-12}\right). \end{aligned}$$

Examining the coefficient of  $x^0$  in the last expression, we see that

$$[x^0] \left(\sum_{k=0}^{24} c_k x^{k-12}\right) \left(\sum_{k=0}^{24} (-1)^k c_k x^{k-12}\right) = \sum_{k=0}^{24} (-1)^k c_k^2$$

is exactly the sum we want. Thus the answer is the coefficient of  $x^0$  in

$$\left(x^2 + \frac{1}{x^2}\right)^{12} = \frac{(x^4 + 1)^{12}}{x^{24}}$$

which is the same as the coefficient of  $x^{24}$  in the numerator. Since

$$(x^4 + 1)^{12} = \sum_{k=0}^{12} \binom{12}{k} x^{4k},$$

we obtain that the coefficient of  $x^{24}$  is  $\binom{12}{6} = 924$ , so  $\sum_{k=0}^{24} (-1)^k c_k^2 = 924$ . ■