Contents

Part I

Theory, Examples, and Problems

Chapter 1

On the Form $x^d P\left(\frac{1}{x}\right)$

On some occasions, changing the original setting of the problem will make the problem more approachable.
For instance, instead of working with the polynomial $a_d x^d + \ldots + a_0$, we may work with the polynomial $a_0 x^d + \ldots + a_d$. This, may lead us to discover some unseen consequences of the hypotheses. We can easily transform

$$P(x) = a_d x^d + \ldots + a_0 \quad \text{to} \quad a_0 x^d + \ldots + a_d.$$

We just take the polynomial $x^d P\left(\frac{1}{x}\right)$.

1.1 Basic properties

Reciprocal polynomial

Let $P(x) = a_d x^d + \ldots + a_0$. The polynomial

$$x^d P\left(\frac{1}{x}\right) = a_0 x^d + \ldots + a_d$$

is called the *reciprocal polynomial* or *inverse polynomial* of $P(x)$.

From now on, we will call $x^d P\left(\frac{1}{x}\right)$ the reciprocal polynomial.

For example, the reciprocal polynomial of $P(x) = 2x^3 - 3x^2 + 1$ is $x^3 - 3x + 2$. The reciprocal polynomial of $Q(x) = 4x^3 - 3x$ is $-3x^2 + 4$.

In short, the reciprocal polynomial reverses the order of coefficients of the original polynomial.

Sometimes, the reciprocal polynomial has lower degree than the original polynomial. Crudely speaking, it depends on zero-multiplicity of the polynomial. That is, if $P(x)$ is a polynomial of degree d such that 0 is a root with multiplicity r, so $P(x) = x^r Q(x)$ with $Q(0) \neq 0$, we easily find that

$$x^d P\left(\frac{1}{x}\right) = x^d x^{-r} Q\left(\frac{1}{x}\right) = x^{d-r} Q\left(\frac{1}{x}\right).$$

The reciprocal polynomial $x^d P(x) = x^{d-r} Q\left(\frac{1}{x}\right)$ is of degree $d - r$.

Example 1.1. Let $P(x)$ be a polynomial of degree 5 with nonnegative integer coefficients such that for all $x \neq 0$, $P(x) = x^6 P\left(\dfrac{1}{x}\right)$ and $P(2) = 10P(1)$. Find the greatest possible value of $\dfrac{P(3)}{P(2)}$.

Solution. Let $P(x) = ax^5 + \ldots + c$. If $c \neq 0$, then the polynomial $x^6 P\left(\dfrac{1}{x}\right)$ has degree 6 and it cannot be equal to a polynomial $P(x)$ of degree 5. Hence $c = 0$. Putting $P(x) = xQ(x)$ for some polynomial $Q(x)$ of degree 4, we get

$$P(x) = xQ(x) = x^6 P\left(\frac{1}{x}\right) = x^5 Q\left(\frac{1}{x}\right).$$

We deduce that $Q(x) = x^4 Q\left(\dfrac{1}{x}\right)$, which gives $Q(x) = ax^4 + bx^3 + cx^2 + bx + a$ for some non-negative integers a, b, c. Moreover, from $P(2) = 10P(1)$, we find

$$2Q(2) = 10Q(1).$$

Thus $17a + 10b + 4c = 5(2a + 2b + c)$, which yields $7a = c$. Moreover,

$$\begin{aligned}\frac{P(3)}{P(2)} = \frac{3Q(3)}{2Q(2)} &= \frac{3(82a + 30b + 9c)}{10(2a + 2b + c)} \\ &= \frac{3(145a + 30b)}{10(9a + 2b)} \\ &= \frac{3(29a + 6b)}{2(9a + 2b)} \\ &= \frac{6a + 3(27a + 6b)}{2(9a + 2b)} \\ &= \frac{9}{2} + \frac{3a}{9a + 2b} \\ &= \frac{9}{2} + \frac{1}{3 + \frac{2b}{3a}}.\end{aligned}$$

Since $\frac{2b}{3a} \geq 0$, then $\frac{9}{2} + \frac{1}{3 + \frac{2b}{3a}} \leq \frac{9}{2} + \frac{1}{3} = \frac{29}{6}$. The equality case occurs for $b = 0$ and the polynomial $P(x) = x(ax^4 + 7ax^2 + a) = ax(x^4 + 7x^2 + 1)$. ■

1.2 Sum of squares of coefficients of a polynomial

Let $P(x) = a_d x^d + \ldots + a_0$. It is instructive to consider the product

$$P(x)P\left(\frac{1}{x}\right) = (a_d x^d + \ldots + a_1 x + a_0)(a_d x^{-d} + \ldots + a_1 x^{-1} + a_0).$$

The above product is a rational function. The constant term of the above product is of special interest. Constant terms arise from the products of the form $a_r x^r \cdot a_r x^{-r} = a_r^2$. Hence the constant term of the above product is the sum of squares of the coefficients of the polynomial.

Sum of squares of the coefficients of a polynomial

The sum of squares of the coefficients of a polynomial $P(x)$ is the coefficient of the constant term in the product $P(x)P\left(\frac{1}{x}\right)$.

Example 1.2. Let $P_{2n}(x) = (6x^2 + 5x + 1)^n$ and $Q_{2n}(x) = (3x^2 + 7x + 2)^n$. Prove that the sum of squares of the coefficients of $P_{2n}(x)$ and $Q_{2n}(x)$ are the same.

Solution. It is known that the sum of squares of the coefficients of $P_{2n}(x)$ and $Q_{2n}(x)$ are equal to the coefficient of x^0 in the products $P_{2n}(x)P_{2n}\left(\frac{1}{x}\right)$ and $Q_{2n}(x)Q_{2n}\left(\frac{1}{x}\right)$. Note that

$$P_{2n}(x) = (6x^2 + 5x + 1)^n = (3x + 1)^n(2x + 1)^n$$

and

$$Q_{2n}(x) = (3x^2 + 7x + 2)^n = (3x + 1)^n(x + 2)^n.$$

Therefore

$$\begin{aligned} P_{2n}(x)P_{2n}\left(\frac{1}{x}\right) &= (3x+1)^n(2x+1)^n\left(\frac{3}{x}+1\right)^n\left(\frac{2}{x}+1\right)^n \\ &= \frac{(3x+1)^n(2x+1)^n(x+3)^n(x+2)^n}{x^{2n}} \end{aligned}$$

and

$$\begin{aligned} Q_{2n}(x)Q_{2n}\left(\frac{1}{x}\right) &= (3x+1)^n(x+2)^n\left(\frac{3}{x}+1\right)^n\left(\frac{1}{x}+2\right)^n \\ &= \frac{(3x+1)^n(x+2)^n(3+x)^n(2x+1)^n}{x^{2n}}. \end{aligned}$$

Comparing these we see that

$$Q_{2n}(x)Q_{2n}\left(\frac{1}{x}\right) = P_{2n}(x)P_{2n}\left(\frac{1}{x}\right).$$

Thus the coefficient of x^0 is the same in both products. ■

Example 1.3. Assume that for all $x \neq 0$:

$$\left(x + \frac{1}{x} + \sqrt{2}\right)^{12} = \sum_{k=0}^{24} c_k x^{k-12}.$$

Find the value of $\displaystyle\sum_{k=0}^{24}(-1)^k c_k^2$.

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Solution. If we use the substitution $x \mapsto -\dfrac{1}{x}$, we find that

$$\left(x+\frac{1}{x}-\sqrt{2}\right)^{12}=\sum_{k=0}^{24} c_k(-1)^{k-12}x^{12-k}=\sum_{k=0}^{24} c_k(-1)^k x^{12-k}.$$

Multiplying this equality by the original equality, we get

$$\begin{aligned}\left(x+\frac{1}{x}+\sqrt{2}\right)^{12}\left(x+\frac{1}{x}-\sqrt{2}\right)^{12} &= \left(\left(x+\frac{1}{x}\right)^2-2\right)^{12}\\ &= \left(x^2+\frac{1}{x^2}\right)^{12}\\ &= \left(\sum_{k=0}^{24} c_k x^{k-12}\right)\left(\sum_{k=0}^{24}(-1)^k c_k x^{k-12}\right).\end{aligned}$$

Examining the coefficient of x^0 in the last expression, we see that

$$[x^0]\left(\sum_{k=0}^{24} c_k x^{k-12}\right)\left(\sum_{k=0}^{24}(-1)^k c_k x^{k-12}\right)=\sum_{k=0}^{24}(-1)^k c_k^2$$

is exactly the sum we want. Thus the answer is the coefficient of x^0 in

$$\left(x^2+\frac{1}{x^2}\right)^{12}=\frac{(x^4+1)^{12}}{x^{24}}$$

which is the same as the coefficient of x^{24} in the numerator. Since

$$(x^4+1)^{12}=\sum_{k=0}^{12}\binom{12}{k}x^{4k},$$

we obtain that the coefficient of x^{24} is $\dbinom{12}{6}=924$, so $\displaystyle\sum_{k=0}^{24}(-1)^k c_k^2=924$. ■