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Preface

Each year, the AwesomeMath Summer Program (AMSP) brings together mathematics students and teachers from all over the globe. The program has been created to provide a place where math enthusiasts can interact with each other and share the same love for the subject. Meaningful problem solving has been at the front row of the AwesomeMath programs. In today's competitive environment, developing problem-solving and critical thinking skills has become more important than ever.

Over the years, AMSP has become a prestigious program, not only for its rigorous curriculum and experienced instructors but also because of its selective admission process. Students who aspire to participate in the program need to take one of the three admissions tests offered every year. Based on the performance on the test, the personal essays, and the recommendation letters, students' overall application is evaluated and an admission decision is made for each applicant.

Since 2006, when the summer program started, there have been 48 admission tests featuring a total of 510 problems. The vast majority of the problems were created by Dr. Titu Andreescu, Co-founder and Director of AwesomeMath. The problems are original, carefully designed, and cover all four traditional areas of competition math: Algebra, Geometry, Number Theory, and Combinatorics.

This book brings to light a "behind the scenes" story: what happens when a creative mind and extensive involvement in mathematics competitions come together. Dr. Andreescu has more than 30 years of experience teaching, coaching, and mentoring the brightest mathematical minds of our times. He is the author or co-author of over 50 mathematics books, essential for every math enthusiast's library. Here are some of the highlights of his career:

- Head coach and leader of the USA International Mathematical Olympiad (IMO) for 8 years
- Director of the Mathematical Association of America (MAA) American Mathematics Competitions for 5 years

- Contributed hundreds of problems to various math competitions including USA(J)MO and IMO, having his problems selected for the USA(J)MO for every single year since 1994 (27 years! Wow!)
- Co-founder of the free online Purple Comet! Math Meet, the first international team-based math competition, that annually gathers over 3,300 teams from over 50 countries (purplecomet.org)
- Founder and editor-in-chief of the free online Mathematical Reflections journal (https://www.awesomemath.org/mathematical-reflections/)

The problems and solutions are divided in two volumes. Volume I focuses on the years since the start of the summer program in 2006 through 2014. Volume II includes the years 2015 to 2021, inclusively. Each volume starts with the statements of the test problems. Complete and enhanced solutions to all problems are then presented, numerous problems having multiple solutions.

We advise our summer program applicants not to be discouraged when approaching these problems. All admission tests accommodate different ages and experience of students, applicants' submissions being evaluated by the admission committee accordingly. Some of the problems involve complex mathematical ideas, but all can be solved using only elementary techniques, admittedly combined in clever ways. Students should include all significant steps in their reasoning and detailed computation as we are interested in their ability to showcase their work.

Through complete and thorough solutions to high level competition-like problems, this book teaches students the invaluable skill of proof writing. The multitude and variety of problems, not only in terms of difficulty but also with respect to the areas of mathematics being covered, comprise a rich source of beautiful mathematics and bright ideas. Advanced middle and high school students preparing for mathematics competitions or anyone simply looking to expand their mathematical horizons will find this book the perfect companion.

Afterthought
Students should not attend this great mathematics program only to prepare for mathematics competitions. The main reason for participating in such a program is to immerse yourself in mathematics that brings joy to your mind and soul. You will be solving challenging and intriguing problems and be surrounded by people from around the world who share the same enthusiasm for problem-solving. You will be developing skills that will make you a stronger, more confident, resilient, and successful person in your personal life and future career.

Alina Andreescu
AwesomeMath Director of Operations
awesomemath.org

This reduces to

$$900 = \frac{(x-1009)^2}{784},$$

yielding

$$30 = \frac{x-1009}{28} \quad \text{or} \quad -30 = \frac{x-1009}{28}.$$

Thus $x = 1849$ and $x = 169$, both satisfying the given equation.

3. Find all pairs (p, q) of twin primes such that

$$(2p+q)^3 = p^3 + 2q^3 + 2018.$$

First Solution. If $q = p + 2$, then

$$(3p+2)^3 = p^3 + 2(p+2)^3 + 2018,$$

which reduces to

$$24p^3 + 42p^2 + 12p = 2026.$$

This equation does not have a solution in integers as the left-hand side is divisible by 3, while 2026 is not.

If $p = q + 2$, then

$$(3q+4)^3 = (q+2)^3 + 2q^3 + 2018,$$

which reduces to

$$24q^3 + 102q^2 + 132q - 1962 = 0$$

or

$$3(q-3)(8q^2 + 58q + 218) = 0.$$

It follows that $q = 3$ and so the only pair is $(p, q) = (5, 3)$.

Second Solution. Observe that $(2p+q)^3 - p^3 - q^3 = q^3 + 2018$ is divisible by $(2p + q - p - q) = p$. So $p \mid (p \pm 2)^3 + 2018$, i.e., $p \mid 2010$ or $p \mid 2026$. In the first case, $p = 2, 3, 5, 67$, which gives $q = 0, 1, 3, 65$, respectively. In the second case, $p = 2, 1013$, which gives $q = 4, 1015$, respectively. Thus $(p, q) = (5, 3)$.

4. Let a, b, c be nonnegative real numbers such that $a^2 + b^2 + c^2 \le 1$.

Prove that

$$(9a + 16b + 41c)(5a + 12b + 43c) < 2018.$$

Solution. From the Cauchy-Schwarz Inequality,

$$(9a + 16b + 41c)^2 \le (9^2 + 16^2 + 41^2)(a^2 + b^2 + c^2) \le 2018$$

and

$$(5a + 12b + 43c)^2 \le (5^2 + 12^2 + 43^2)(a^2 + b^2 + c^2) \le 2018.$$

It follows that

$$(9a + 16b + 41c)^2(5a + 12b + 43c)^2 \le 2018^2.$$

We cannot have simultaneously $\frac{a}{5} = \frac{b}{12} = \frac{c}{43}$ and $\frac{a}{9} = \frac{b}{16} = \frac{c}{41}$, hence the conclusion.

5. Find all pairs (a, b) of positive integers such that $a^3 - 6b^2 = 2018$ and $b^3 - 6a^2 = 155$ hold simultaneously.

Solution. From the first equation, we have $a \ge 13$ and from the second, $b^3 \ge 6 \cdot 13^2 + 155$, so $b \ge 11$. By subtraction and factoring,

$$(a - b)(a^2 + ab + b^2 + 6(a + b)) = 1863.$$

The prime factorization of 1863 is $3^4 \cdot 23$. Since

$$a^2 + ab + b^2 + 6(a + b)) \ge 169 + 143 + 121 + 144 = 577,$$

we have $a - b \le 3$. It follows that $a - b = 1$ or $a - b = 3$. In the first case, $b^2 + 5b - 618 = 0$, equation with no integer solution, while in the second, $b^2 + 7b - 198 = 0$, whose positive solution is $b = 11$. Hence the only pair is $(a, b) = (14, 11)$.

6. Solve in positive real numbers the system of equations

$$\begin{cases} (x + 3y)\sqrt{x} = 2018 - \frac{1}{3}\sqrt{x} \\ (3x + y)\sqrt{y} = 2078 + \frac{1}{3}\sqrt{x}\,. \end{cases}$$

Solution. Let $\sqrt{x} = u$ and $\sqrt{y} = v$. Then

$$\begin{cases} u^3 + 3uv^2 = 2018 - \frac{1}{3}u \\ 3u^2v + v^3 = 2078 + \frac{1}{3}u. \end{cases}$$

Adding up we get $(u + v)^3 = 4096$ and subtracting,

$$(u - v)^3 = -60 - \frac{2}{3}u,$$

implying

$$(u+v)+(u-v)=16-\sqrt[3]{60+\frac{2}{3}u}.$$

It follows that

$$2u+\sqrt[3]{60+\frac{2}{3}u}=16.$$

Clearly, the function in the left-hand side is increasing, so the above equation has unique solution. It is easy to see that this is $u=6$.

Hence $(x,y)=(36,100)$, is the solution that satisfies both equations of the system.

7. Let $f:\mathbb{Z}^+\times\mathbb{Z}^+\to\mathbb{Z}^+\cup\{0\}$ be a function such that

$$f(1,1)=1,\ f(m+1,n)=f(m,n)+m,\ \text{and}\ f(m,n+1)=f(m,n)-n$$

for all positive integers m and n.

Find all pairs (a,b) such that $f(a,b)=2018$.

Solution. We have

$$\begin{aligned} f(a,b) &= f(a-1,b)+a-1=f(a-2,b)+(a-2)+(a-1)=\dots \\ &= f(1,b)+\frac{a(a-1)}{2}=f(1,b-1)-(b-1)+\frac{a(a-1)}{2}=\dots \\ &= f(1,1)-\frac{b(b-1)}{2}+a(a-1)=2018. \end{aligned}$$

It follows that

$$\frac{a(a-1)}{2}-\frac{b(b-1)}{2}=2017,$$

hence

$$(a-b)(a+b-1)=2\cdot 2017.$$

Note that 2017 is a prime and that $a-b<a+b-1$ for all a and b in $\mathbb{Z}^+$. We have the following two cases:

(i) $a-b=1$ and $a+b-1=4034$, implying $a=2018$ and $b=2017$;

(ii) $a-b=2$ and $a+b-1=2017$, implying $a=1010$ and $b=1008$.

Therefore $(a,b)=(2018,2017)$ or $(1010,1008)$.

8. Solve in positive integers the equation $x^3+y^3+3xyz=z^3+2018$.

Solution. Rewrite the equation as

$$(x+y-z)[(x-y)^2+(y+z)^2+(z+x)^2]=2\cdot 2018.$$