

Contents

Preface	vii
I Problems	1
1 Admission Tests 2022	3
2 Admission Tests 2023	13
3 Admission Tests 2024	27
II Solutions	43
4 Solutions to Admission Tests 2022	45
5 Solutions to Admission Tests 2023	77
6 Solutions to Admission Tests 2024	131
III Glossary	187

Chapter 4

Solutions to Admission Tests 2022

Test A

Part I

1. The number 2022 has precisely three equal digits. How many 4-digit numbers have this property?

First Solution. If 0 is the digit that appears three times, then we have 9 numbers: 1000, 2000, ..., 9000. If the nonzero digit d appears precisely three times, then we have $\overline{addd}$, $\overline{dbdd}$, $\overline{ddbd}$, $\overline{dddb}$, where a can take 8 values and b 9 values. So the answer is $9 + 8 \cdot 9 + 9 \cdot 9 + 9 \cdot 9 + 9 \cdot 9 = 324$.

Second Solution. A 4-digit number with precisely three equal digits must have one of the four forms $\overline{abbb}$, $\overline{abaa}$, $\overline{aaba}$, or $\overline{aaab}$. For each such form there are 9 possibilities for the digit a (since it can be any non-zero digit), and given a , there are exactly 9 choices for the digit b (since it can be any digit other than a). Thus there are $4 \cdot 9 \cdot 9 = 324$ such numbers.

2. For positive integers n prove that $\frac{2022}{n} + 4n$ is a perfect square if and only if $\frac{2022}{n} - 8n$ is a perfect square.

Solution. If n is not a divisor of 2022, then $\frac{2022}{n} + 4n$ and $\frac{2022}{n} - 8n$ are not integers, and hence not perfect squares. Thus we need only consider the case where n is a positive divisor of 2022. These divisors are 1, 2, 3, 6, 337, 674, 1011, and 2022. Because $\frac{2022}{n} + 4n$ and $\frac{2022}{n} - 8n$ are perfect squares for $n = 6$ only, the desired conclusion follows.

3. Starting with the 48-digit number 20112012...2022, delete 32 of its digits in order to obtain:

- (a) the least possible 16 digit number;
- (b) the greatest possible 16 digit number.

No permutation of digits is allowed.

Solution.

- (a) The answer is 1000000000001022. To see that this is attainable and minimal, note that to get the smallest possible result, we first of all want to get a single 1 followed by as many zeros as possible. The first 1 is in the third digit, so we must delete the two digits before it. The next 39 digits contain 11 zeros and 28 non-zero digits. Hence we can delete all 28 of these non-zero digits and leave a number with a single 1 followed by 11 zeros. This leaves us with the 18-digit number 100000000000212022 and able to do two more deletions. Since we can only delete two out of the next four digits, which are 2120, clearly the smallest result we can leave is to delete the twos, getting 10 for the next two digits and a final value of 1000000000001022.
 - (b) The answer is 8219202020212022. To see that this is attainable and maximal, note that the largest digit among the first 33 digits is an 8 and the only such 8 is the ones digit of 2018. Thus the largest possible first digit we can get is an 8, and to get it we must delete the first 31 digits. This leaves us with the 17-digit number 82019202020212022 and able to delete just one more digit. Since we can only delete one digit from the next three digits, which are 201, clearly the largest result we can leave is to delete the zero, getting 21 for the next two digits and a final value of 8219202020212022.
4. There are n girls in a chat group. Every day, each girl sends precisely one text message to each of the other girls. If three girls withdraw from the group, the total number of daily text messages decreases by 23%. Find n .

Solution. With n girls each texting $n - 1$ other girls, we get a total of $n(n - 1)$ daily text messages. If we reduce the number of girls by 3, this will become $(n - 3)(n - 4)$. Therefore the problem statement is equivalent to the equation

$$(n - 3)(n - 4) = \left(1 - \frac{23}{100}\right) n(n - 1).$$

Thus is equivalent to $100n^2 - 700n + 1200 = 77n^2 - 77n$, which reduces to

$$23n^2 - 623n + 1200 = 0,$$

whose solution in integers is $n = 25$.

5. Solve in positive real numbers the equation

$$\lfloor \sqrt[3]{x} \rfloor = \frac{x^3 - 1}{2022}.$$

Solution. Let $k = \lfloor \sqrt[3]{x} \rfloor$. Since x is a positive real number, we have $k \in \mathbb{N}$. From $\frac{x^3 - 1}{2022} = k$, we have $x^3 = 2022k + 1$, which gives

$$x = \sqrt[3]{2022k + 1}.$$

Since $k \leq \sqrt[3]{x} < k + 1$, we get $k^3 \leq x < (k + 1)^3$. It follows that

$$k^3 \leq \sqrt[3]{2022k + 1} < (k + 1)^3, \text{ i.e., } k^9 \leq 2022k + 1 < (k + 1)^9.$$

These inequalities fail for $k = 0$ (being $0 \leq 1 < 1$) and $k = 1$ (being $1 \leq 2023 < 512$). They hold for $k = 2$ (being $512 \leq 2023 < 19683$). For $k \geq 3$, we have $k^9 \geq 3^8 k > 2023k > 2022k + 1$, so the inequalities fails. Thus we get $k = 2$ and so $x = \sqrt[3]{4045}$.

6. Let $ABCD$ be a cyclic quadrilateral with $AB = 29\sqrt{2}$, $AC = \sqrt{2022}$, $AD = 34\sqrt{2}$, and $BC = CD$. Find the length of BD .

Solution. Since $ABCD$ is cyclic and $BC = CD$, the angles $\angle BAC$ and $\angle CAD$ subtend congruent arcs of the circumcircle, and hence $\angle BAC \cong \angle CAD$. From the law of cosines in triangles ABC and ACD , we therefore get

$$\frac{AB^2 + AC^2 - BC^2}{2AB \cdot AC} = \cos \angle BAC = \cos \angle CAD = \frac{AC^2 + AD^2 - CD^2}{2AC \cdot AD},$$

hence

$$\frac{3704 - BC^2}{58\sqrt{4044}} = \frac{4334 - BC^2}{68\sqrt{4044}}.$$

Solving this gives $BC^2 = 50$ and hence $BC = CD = 5\sqrt{2}$. From this Ptolemy's Theorem gives $BD = \frac{AD \cdot BC + AB \cdot CD}{AC} = \frac{630}{\sqrt{2022}}$.

7. Let $a_n = n + 2 - \sqrt{n^2 + 4}$, $n = 1, 2, 3, \dots$. Given that

$$2(\sqrt{a_1} + \sqrt{a_4} + \sqrt{a_9} + \sqrt{a_{16}} + \sqrt{a_{25}} + \sqrt{a_{36}} + \sqrt{a_{49}}) = \sqrt{q} - \sqrt{r} + \sqrt{s}$$

for some integers q, r, s , find $\sqrt{q+r+s}$.

Solution. We have

$$b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 = \sqrt{q} - \sqrt{r} + \sqrt{s},$$

where $b_n = 2\sqrt{n^2 + 2} - \sqrt{n^4 + 4}$. Because

$$n^4 + 4 = (n^2 - 2n + 2)(n^2 + 2n + 2),$$

we have

$$(b_n)^2 = 2 \left(\sqrt{n^2 + 2n + 2} - \sqrt{n^2 - 2n + 2} \right)^2.$$

Hence

$$b_n = \sqrt{2} \left(\sqrt{n^2 + 2n + 2} - \sqrt{n^2 - 2n + 2} \right),$$

which we can write as $b_n = c_{n+2} - c_n$, where $c_n = \sqrt{2((n-1)^2 + 1)}$. Thus

$$\sqrt{q} - \sqrt{r} + \sqrt{s} = c_9 + c_8 - c_2 - c_1 = \sqrt{130} + 10 - 2 - \sqrt{2} = \sqrt{130} - \sqrt{2} + \sqrt{64}.$$

The answer is $\sqrt{196} = 14$.

8. Solve in integers the equation

$$x^3 + 30xy + y^3 + 2(x + y) = 2022.$$

Solution. Let $x + y = s$ and $xy = p$. Note that x and y have the same parity, so $s = 2q$ for some positive integer q . We have

$$(2q)^3 - 3(2q)p + 30p + 4q = 2022,$$

implying

$$3p(q - 5) = 4q^3 + 2q - 1011.$$

It follows that

$$3p = 4q^2 + 20q + 102 - \frac{501}{q - 5}.$$

Hence $q - 5$ divides $501 = 3 \cdot 167$, so $q - 5$ is a divisor of 501 and hence is in $\{-501, -167, -3, -1, 1, 3, 167, 501\}$. The only case that produces integer values for x and y is $q - 5 = 1$, yielding $\{x, y\} = \{-3, 15\}$. (One can reduce the case-checking by noting that if $q - 5$ is a multiple of 3, then every term except the last one is a multiple of 3, which is impossible. Also since the AM-GM inequality gives $p \leq q^2$, it is easy to see that $q - 5 = \pm 167$ are impossible. Thus only $q - 5 = \pm 1$ need checked.) Thus the solutions are $(x, y) = (-3, 15)$ and $(x, y) = (15, -3)$.

9. Parallelogram $\mathcal{P}$ has a 30° angle and side-lengths $mn + 2$ and $2(mn - 2)$ while rectangle $\mathcal{R}$ has side-lengths $m + n$ and $2(m - n)$ for some positive integers m and n . Given that $\text{area}(\mathcal{P}) + \text{area}(\mathcal{R}) = 2022$, find m and n .

Solution. We have

$$2 \cdot \frac{1}{2} \cdot (mn + 2)2(mn - 2) \sin 30^\circ + (m + n)2(m - n) = 2022,$$

implying

$$m^2n^2 - 4 + 2(m^2 - n^2) = 2022.$$

This can be rewritten as $(m^2 - 2)(n^2 + 2) = 2 \cdot 3 \cdot 337$. Thus $m^2 - 2$ and $n^2 + 2$ must be complementary divisors of 2022. The divisors of 2022 are $\{1, 2, 3, 6, 337, 674, 1011, 2022\}$. The only two which are two less than a square are $2 = 2^2 - 2$ and $674 = 26^2 - 2$. The only ones that are two more than a square are $2 = 0^2 + 2$, $3 = 1^2 + 2$, and $6 = 2^2 + 2$. Thus the only complementary pair is 674 and 3, and we see that $m = 26$ and $n = 1$.

10. Find all rational numbers x such that $x[x]\{x\} = 2022$, where $[a]$ and $\{a\}$ are the greatest integer less than or equal to a and the fractional part of a , respectively.

First Solution. Let $[x] = n$ and $\{x\} = f$. Then, $(n + f)nf = 2022$, which can be rewritten as

$$nf^2 + (n^2)f - 2022 = 0.$$

This is a quadratic equation in f and, in order to have rational roots, its discriminant $\Delta = n^4 + 8088n = n(n^3 + 2^3 \cdot 3 \cdot 337)$ must be a perfect square. Note that $n = 0$ (which does make Δ a perfect square) does not lead to a solution for f , so we can assume $n \neq 0$. Write $n = rs^2$ where r is a square-free integer (possibly negative) and s is a positive integer. Then since $\Delta = rs^2(r^3s^6 + 2^3 \cdot 3 \cdot 337)$ is a perfect square, it must be a multiple of $(rs)^2$, and hence r must divide $2 \cdot 3 \cdot 337$. Writing $\Delta = (rst)^2$, we get

$$t^2 = (rs^3)^2 + \frac{2^3 \cdot 3 \cdot 337}{r},$$

which we can rewrite as

$$(t - rs^3)(t + rs^3) = \frac{2^3 \cdot 3 \cdot 337}{r}.$$

Since $t - rs^3$ and $t + rs^3$ have the same parity and their product is even (since 4 cannot divide r), both must be even. Thus

$$d_1 = \frac{|t - rs^3|}{2}, \quad d_2 = \frac{|t + rs^3|}{2},$$

and $|r|$ must be a triple of complementary factors of $2022 = 2 \cdot 3 \cdot 337$. Note that for some choice of signs, we must have $\pm d_1 \pm d_2 = |r| \cdot s^3$.

If $|r| = 1$, then we need $\pm d_1 \pm d_2 = s^3$ to be a perfect cube. We have four possibilities for $\{d_1, d_2\}$, namely, $\{1, 2022\}$, $\{2, 1011\}$, $\{3, 674\}$, and $\{6, 337\}$. Taking signs into account, we find that only the possibility is $\{d_1, d_2\} = \{6, 337\}$ and $d_1 + d_2 = 7^3$. Thus $n = \pm 7^2$, and checking we see that we get only the solution $n = -49$, $\Delta = (7 \cdot 331)^2$.

If $|r| = 2$, then we need $\pm d_1 \pm d_2 = 2s^3$. However neither of the pairs $\{d_1, d_2\} = \{1, 1011\}$, $\{3, 337\}$ makes this hold for any choice of signs.

If $|r| = 3$, then we need $\pm d_1 \pm d_2 = 3s^3$. However neither of the pairs $\{d_1, d_2\} = \{1, 674\}$, $\{2, 337\}$ makes this hold for any choice of signs.

If $|r| = 6$, then we need $\pm d_1 \pm d_2 = 6s^3$. However the only pair is $\{d_1, d_2\} = \{1, 337\}$ which doesn't make this hold for any choice of signs.

If $337 \mid |r|$, then d_1 and d_2 are at most 6 and hence $d_1 + d_2 < 337$. Thus the only way for $\pm d_1 \pm d_2$ to even be a multiple of r is if it equals zero. However, we already excluded the case $n = 0$ above.

Thus the only possibility is $n = -49$, which gives

$$f = \frac{-49^2 \pm 7 \cdot 331}{2 \cdot (-49)} = \frac{343 \mp 331}{2 \cdot 7}.$$

Since we need $0 \leq f < 1$, we must have $n = -49$ and $f = 6/7$. Thus the only solution is $x = -49 + 6/7 = -337/7$. It indeed satisfies the given equation.

Second Solution. Since $x = \lfloor x \rfloor + \{x\} \in \mathbb{Q}$, $\{x\} \in \mathbb{Q}$. Let $k = \lfloor x \rfloor$ and $\{x\} = \frac{m}{n}$, where $k, m, n \in \mathbb{Z}$. It's easy to see that $k, m, n \neq 0$. Since $0 \leq \{x\} < 1$, we can assume that m and n are positive integers, $m < n$ and $\gcd(m, n) = 1$. We have two cases.

(i) If $x > 0$, then $k > 0$. So,

$$\left(k + \frac{m}{n}\right) \cdot k \cdot \frac{m}{n} = 2022 \implies \frac{k^2 mn + km^2}{n^2} = 2022.$$

It follows that

$$n^2 \mid (k^2 mn + km^2), \quad (4.1)$$

which gives $n \mid (k^2 mn + km^2)$, i.e., $n \mid km^2$. Since $\gcd(m, n) = 1$, we have $n \mid k$, which gives $k = nt$ for some $t \in \mathbb{N}^*$. From (4.1) we get $n^2 \mid (n^3 t^2 m + n t m^2)$, so $n \mid t m^2$, which gives $n \mid t$. Hence, $t = ns$ for some $s \in \mathbb{N}^*$ and thus $k = n^2 s$. The initial relation becomes

$$\left(n^2 s + \frac{m}{n}\right) \cdot n^2 s \cdot \frac{m}{n} = 2022 \implies (n^3 s + m)sm = 2022.$$

This implies s and m are both divisors of 2022.

Also from $(n^3m)s^2 + sm^2 = 2022$, we see that s^2 and m^2 are both at most 2022. Hence we get $s, m \in \{1, 2, 3, 6\}$. Testing these numbers, we find no solutions.

- (ii) If $x < 0$, then $k < 0$, so $k = n^2s$ with $s < 0$ and $(n^3s+m)sm = 2022$. We find

$$m \in \{1, 2, 3, 6, 337, 674, 1011, 2022\}$$

and

$$s \in \{-1, -2, -3, -6, -337, -674, -1011, -2022\}.$$

Checking these, we get $(m, n, s) = (6, 7, -1)$, so

$$x = -49 + \frac{6}{7} = -\frac{337}{7}.$$

So, $x = -\frac{337}{7}$.

Part II

1. Solve the equation

$$4^x + 2^{1+x} + 2^{1-x} + 4^{-x} = 2022.$$

Solution. The equation is equivalent to

$$(2^x + 1 + 2^{-x})^2 = 2025$$

and so $2^x + 1 + 2^{-x} = 45$. Setting $2^x = y$ yields the quadratic equation

$$y^2 - 44y + 1 = 0,$$

whose roots, $y_{1,2} = 22 \pm \sqrt{483}$, are positive. Hence the solutions are $\log_2(22 + \sqrt{483})$ and $\log_2(22 - \sqrt{483})$.

2. Find all triples (p, q, r) of primes such that $p \leq q \leq r$ and

$$3^{q+2} - pqr = 2022.$$

Solution. The equation can be rewritten as

$$9 \frac{3^q - 3}{q} - \frac{1995}{q} = pr.$$

Because q divides $3^q - 3$ and $1995 = 3 \cdot 5 \cdot 7 \cdot 19$, we get $q \in \{3, 5, 7, 19\}$. If $q = 3$, then $pr < 0$, while $q = 5$ makes $pr = 33$, so $p = 3$ and $r = 11$. If $q = 7$, then $pr = 3 \cdot 29^2$, with no solution, while for $q = 19$, $pr = 3 \cdot 7 \cdot 8941 \cdot 55711$, again with no solution.

In conclusion, $(p, q, r) = (3, 5, 11)$.