Preface

Hundreds of mathematical competitions are organized these days all around the world. Almost every country holds its national Olympiad, and in some of them are town or regional competitions[1]. Also, there are regional competitions, such as the Balkan Mathematical Olympiad (and its younger sister the Junior Balkan Mathematical Olympiad), Asian Pacific Mathematical Olympiad, Baltic Way, Nordic Mathematical Contest, South East Asian Mathematics Competition, etc.

The International Mathematical Olympiad, queen of all math competitions, started in 1959, in Romania, with 7 attending countries. In 2009, in Germany, students from 104 countries were fighting for a medal.

The number of those interested in math competitions is constantly increasing and the popularity of websites like www.artofproblemsolving.com (a math forum with over 130000 members) is a proof for this.

It all started in 1894, in Hungary, when the Eötvös Competition, a math contest for secondary school students, was held for the first time. The competitors were given four hours to solve three problems individually (almost the same happens today at the International Mathematical Olympiad).

In the neighboring Romania, the first issue of the monthly Gazeta Matematica was published in 1895. The journal organized a competition for school students, which improved in format over the years and eventually became the Romanian National Mathematical Olympiad.

A math competition was first held in Sankt Petersburg, Russia in 1934 and in Poland in 1947. The Mathematical Association of America organized a competition for Metropolitan New York in 1950 and extended this to the entire country in 1957.

In the last decade of July 1959 students from Bulgaria, Czechoslovakia, German Democratic Republic, Hungary, Poland, Romania, and Soviet Union gathered together in Romania to compete in the first International Mathematical Olympiad. Ever since, the IMO has developed a rich legacy and has

[1] In Romania, for instance, there are around 50 regional competitions every year, each involving hundreds of students.

established itself as the most important mathematical competition for high school students.

In 1983, the IMO was held in Paris, France. It was then when the leaders from Bulgaria, Greece, and Romania decided to organize, starting the next year, another math competition, the Balkan Countries Mathematical Olympiad.

Although the rules of IMO are very encouraging for the students, given that approximately half of them can win prizes, the competition is difficult and many students come to be disappointed for different reasons. Therefore, a preliminary competition was considered very helpful. The aims of the Balkan Mathematical Olympiad (BMO) include:

a. challenge, encouragement, and development of mathematically gifted school students in all participating countries;

b. fostering friendly relationships among students and teachers of the member countries;

c. creation of opportunities for the exchange of information on school syllabi and practice within the member countries;

d. gaining experience and preparation for the IMO.

The first Balkan Mathematical Olympiad was organized in Athens, Greece, in 1984. The participating countries were Bulgaria, Greece, and Romania. The rules of the competition were approximately the same as the IMO's. The competition extended since, and 11 countries are nowadays official members of the contest, the list being not closed. In the last years, other teams such as Hungary, United Kingdom, Kazakhstan, France, Italy, Saudi Arabia, etc. took part as invited countries.

It is important to mention that BMO problems are usually original, but less difficult than in the IMO's. Therefore, many young and/or less experienced students are encouraged to solve them. Even more, doing this successfully these students are motivated to involve themselves more in mathematics.

The authors of the book, attending several times the BMO as leader and/or deputy leader, present to the readers a complete description of the evolution of BMO's since their creation up to today. All problems are presented with complete solutions. Many problems have several alternative solutions and we also present some extensions. An additional preparatory addendum, containing concepts and classical useful results has been added to the end of the book.

The Authors

Contents

Part I

Problems and Solutions

The 1st BMO

The first Balkan Mathematical Olympiad for high-school students was held between May 6th and May 10th , 1984, in Athens, Greece. The competition was organized by the Greek Mathematical Society. The participating countries were Bulgaria, Greece, and Romania, the founder countries of this competition.

Problems

1.1. Let $n \geq 2$ be a positive integer and $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \ldots + a_n = 1$. Show that the following inequality holds:

$$\frac{a_1}{1 + a_2 + a_3 + \ldots + a_n} + \frac{a_2}{1 + a_1 + a_3 + \ldots + a_n} + \ldots + \frac{a_n}{1 + a_1 + \ldots + a_{n-1}} \geq \frac{n}{2n-1}.$$

(Greece)

1.2. Let $ABCD$ be a cyclic quadrilateral and let H_A, H_B, H_C, H_D be the orthocenters of the triangles BCD, CDA, DAB, and ABC, respectively. Show that the quadrilaterals $ABCD$ and $H_AH_BH_CH_D$ are congruent.

(Romania)

1.3. Show that for any positive integer m, there exists a positive integer n, so that in the decimal representations of the numbers 5^m and 5^n, the representation of 5^n ends in the representation of 5^m.

(Bulgaria)

1.4. Let a, b, c be positive real numbers. Find all real solutions (x, y, z) of the system:

$$\begin{aligned} ax + by &= (x-y)^2 \\ by + cz &= (y-z)^2 \\ cz + ax &= (z-x)^2 . \end{aligned}$$

(Romania)

Solutions

1.1. Since $\sum_{k=1}^{n} a_k = 1$, the given inequality can be written as follows:

$$\sum_{k=1}^{n} \frac{a_k}{2-a_k} \geq \frac{n}{2n-1}. \quad (1)$$

Note that

$$\frac{a_k}{2-a_k} = \frac{2}{2-a_k} - 1,$$

therefore, inequality (1) is equivalent to

$$2\sum_{k=1}^{n} \frac{1}{2-a_k} - n \geq \frac{n}{2n-1},$$

or

$$\sum_{k=1}^{n} \frac{1}{2-a_k} \geq \frac{n^2}{2n-1}. \quad (2)$$

The latter follows easily from the HM-AM inequality: observe that $2 - a_k > 0$, so that we have

$$\frac{n}{\sum_{k=1}^{n} \frac{1}{2-a_k}} \leq \frac{1}{n}\sum_{k=1}^{n} (2-a_k) = \frac{1}{n}\left(2n - \sum_{k=1}^{n} a_k\right) = \frac{2n-1}{n}.$$

Second solution. We prove (1) by using Jensen's inequality.
Consider the convex function $f : (0,1) \to \mathbb{R}$, $f(x) = \dfrac{x}{2-x}$. Then we have

$$\frac{1}{n}\sum_{k=1}^{n} \frac{a_k}{2-a_k} = \frac{1}{n}\sum_{k=1}^{n} f(a_k) \geq f\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right) = f\left(\frac{1}{n}\right) = \frac{1}{2n-1}.$$

Third solution. We prove (1) using Jensen's inequality and Cauchy-Schwartz inequality. Consider the convex function $f : (0,1) \to \mathbb{R}$, $f(x) = \dfrac{1}{2-x}$, and apply Jensen's inequality to the numbers $a_1, \ldots, a_n$, and weights $a_1, \ldots, a_n$, satisfying the condition $\sum_{k=1}^{n} a_k = 1$. We obtain

$$\sum_{k=1}^{n} a_k f(a_k) \geq f\left(\sum_{k=1}^{n} a_k^2\right).$$

Explicitly, we have

$$\sum_{k=1}^{n}\frac{a_k}{2-a_k} \geq \frac{1}{2-\sum\limits_{k=1}^{n}a_k^2},$$

therefore, it is sufficient to prove that

$$\frac{1}{2-\sum\limits_{k=1}^{n}a_k^2} \geq \frac{n}{2n-1},$$

which is equivalent to

$$\sum_{k=1}^{n}a_k^2 \geq \frac{1}{n}.$$

This inequality can be obtained by using Cauchy-Schwartz inequality, as follows:

$$1=\left(\sum_{k=1}^{n}a_k\right)^2 \leq (1+\ldots+1)\left(a_1^2+\ldots+a_n^2\right)=n\sum_{k=1}^{n}a_k^2.$$

1.2. Let O be the circumcenter of the quadrilateral $ABCD$, M be the midpoint of the segment AB and G_A, G_B, G_C, G_D be the centroids of triangles BCD, CDA, DAB, and ABC, respectively.

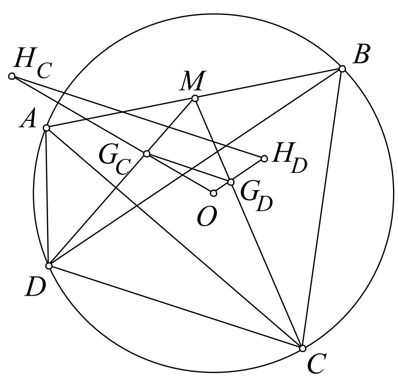

Figure 1.1

It is known that G_C lies on the segment DM and that $MG_C=\frac{1}{3}MD$. Similarly, G_D lies on the segment CM and $MG_D=\frac{1}{3}MC$. Therefore, in the triangle CMD, the segment G_CG_D is parallel to CD and $G_CG_D=\frac{1}{3}CD$ (see Figure 1.1).

On the other hand, it is known that in the triangle ABD, the orthocenter H_C, the centroid G_C, and the circumcenter O lie on the same line (the Euler line-see Appendix) in such a way that $OG_C=\frac{1}{3}OH_C$.

In the same way, $OG_D = \frac{1}{3}OH_D$. It follows that in the triangle OH_CH_D, the side H_CH_D is parallel to the line G_CG_D and $H_CH_D = 3G_CG_D$.

Combining these two results, we obtain that the segments CD and H_CH_D are parallel and have equal lengths. Thus, the quadrilaterals $ABCD$ and $H_AH_BH_CH_D$ have the corresponding sides parallel and of equal lengths. This proves the statement.

Second solution. Other geometric proofs can be obtained by using in various ways the Euler line. For instance, it is known that in any triangle ABC the median CM intersects the segment OH at the centroid G such that $OG = \frac{1}{3}OH$. Since CH and OM are both perpendicular to AB, it follows that the triangles CGH and MGO are similar with ratio 2:1. Hence, CH and OM are parallel and $CH = 2OM$.

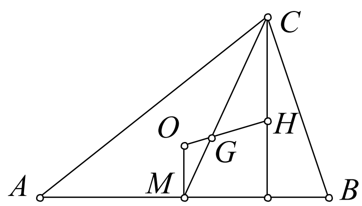

Figure 1.2

Applying the above argument to the triangles ABC and ABD, which are inscribed in the same circle, we obtain that the segments CH_D and DH_C are parallel and have the same length. Therefore, the quadrilateral CH_DH_CD is a parallelogram (see Figure 1.3). It follows that CD and H_CH_D are parallel and have equal lengths and the proof ends as the previous one.

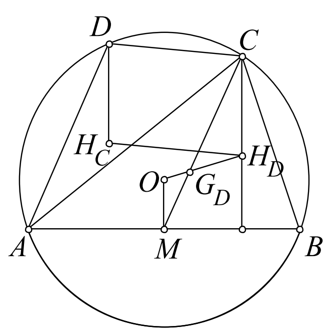

Figure 1.3

Third solution. Computational solutions are also possible, using either complex numbers, vectors or coordinates. For shortness, we will use complex numbers. Assume that the circumcenter O is the origin of the complex plane and denote by a, b, c, d the complex numbers corresponding to the points A, B, C, and D, respectively. Since O is the circumcenter of any of the triangles BCD, CDA, DAB, and ABC, the complex numbers corresponding to their orthocenters are $h_A = b + c + d$, $h_B = a + c + d$, $h_C = a + b + d$, and $h_D = a + b + c$, respectively.

Note that $h_B - h_A = a - b$, thus the vectors $\overrightarrow{H_A H_B}$ and $\overrightarrow{AB}$ are parallel, have the same length and distinct orientations. Using the same argument for the other sides of the quadrilaterals, we obtain the desired conclusion.

Observation. From the above solution one may easily obtain the following characterization of the quadrilateral $H_A H_B H_C H_D$. Let S be the point corresponding to the complex number $s = \frac{1}{2}(a + b + c + d)$. Then

$$a + h_a = b + h_b = c + h_c = d + h_d = 2s.$$

These equalities show that H_A, H_B, H_C, and H_D are the reflections of the points A, B, C, and D across the point S[2]. This, again, proves the statement of the problem (see Figure 1.4).

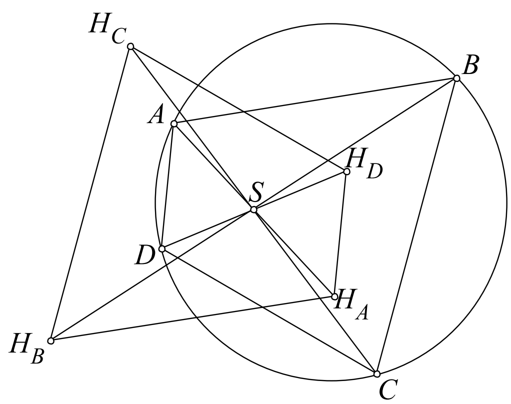

Figure 1.4

1.3. Assume that

$$5^m = \overline{a_{k-1} a_{k-2} \dots a_1 a_0},$$

that is, 5^m has k digits. We have to find some positive integer n such that

$$5^n = \overline{a_l a_{l-1 \dots} a_k a_{k-1} \dots a_1 a_0}.$$

This is equivalent to: $5^n \equiv 5^m \pmod{10^k}$. The condition

$$10^k | 5^m \left(5^{n-m} - 1\right)$$

[2]The point S is called the Mathot point of the quadrilateral and it is also the point of intersection between the perpendiculars dropped from the midpoint of each side to the opposite side.

requires $k \le m$. In the same time, we have

$$10^{k-1} < 5^m < 10^k,$$

implying

$$\frac{k-1}{m} < \lg 5 < \frac{k}{m} \le 1.$$

These conditions determine k (namely, $k = \lfloor m \lg 5 \rfloor + 1$) and we have to find n such that

$$2^k | 5^{n-m} - 1.$$

There are several ways to obtain such numbers n.

The first idea is to use Euler's theorem: since $\gcd\left(5, 2^k\right) = 1$, it follows that

$$5^{\varphi\left(2^k\right)} \equiv 1 \left(\bmod 2^k\right),$$

and hence, setting $n = m + \varphi\left(2^k\right)$ yields the required result.

The second idea is to obtain the exponent n by induction. Indeed, the following statement can be easily proved: for any $s \ge 1$,

$$5^{2^s} \equiv 1 \left(\bmod 2^{s+1}\right).$$

Obviously, this is true for $s = 1$, and the induction step follows from the factorization

$$5^{2^{s+1}} - 1 = \left(5^{2^s} - 1\right)\left(5^{2^s} + 1\right).$$

Setting $n = m + 2^{k-1}$ ends the proof.

1.4. Adding up the equalities yields

$$ax + by + cz = \frac{1}{2}\left[(x-y)^2 + (y-z)^2 + (z-x)^2\right].$$

Using this equality and the given equations we obtain

$$\begin{cases} ax = (x-y)(x-z) \\ by = (y-z)(y-x) \\ cz = (z-x)(z-y) \end{cases} \tag{3}$$

Multiplying these equalities by $z-y$, $x-z$, and $y-x$, respectively, we get

$$\begin{cases} ax(z-y) = (x-y)(y-z)(z-x) \\ by(x-z) = (x-y)(y-z)(z-x) \\ cz(y-x) = (x-y)(y-z)(z-x) \end{cases} \tag{4}$$

Denote by $P = (x-y)(y-z)(z-x)$; then, from (4) it follows that

$$x(z-y) = \frac{P}{a},\ y(x-z) = \frac{P}{b},\ z(y-x) = \frac{P}{c}.$$

Adding up these equalities yields

$$P\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=0,$$

and since a, b, and c are positive numbers, it follows that $P=0$. Hence at least two of the numbers x, y, z are equal. If, for instance, $x=y$, we derive from (3) that $x=y=0$ and $cz=z^2$, yielding the solutions $(0,0,0)$ and $(0,0,c)$. Analogously, we obtain the solutions $(a,0,0)$ and $(0,b,0)$.

Second solution. Assume that $x \le y \le z$. Then, from (3) we derive that $ax \ge 0$, $by \le 0$, and $cz \ge 0$. Since a, b, c are positive numbers, we deduce that $0 \le x \le y \le 0$. Hence $x=y=0$ and either $z=0$ or $z=c$. Considering the other possible orderings of the numbers x, y, z, we obtain the other solutions.

Third solution. The system (3) can be solved by brute force: denote by $u=x-z$, $v=y-z$. Then (3) is equivalent to

$$\begin{cases} az = u\,(u-v-a) \\ bz = v\,(v-u-b) \\ cz = uv \end{cases} \tag{5}$$

We analyze four cases.

Case 1. $u=v=0$. Then from (5) it follows that $x=y=z=0$, yielding the solution $(0,0,0)$.

Case 2. $u=0$ and $v \ne 0$. We immediately derive that $x=z=0$ and the second equation in (5) becomes $v-u-b=0$, hence $y=b$. We obtain thus the solution $(0,b,0)$.

Case 3. $u \ne 0$ and $v=0$. As in the previous case, we obtain the solution $(a,0,0)$.

Case 4. $u \ne 0$ and $v \ne 0$. From (5) we obtain

$$\frac{a}{c}=\frac{u-v-a}{v}$$

and

$$\frac{b}{c}=\frac{v-u-b}{u}.$$

These equalities can be translated into a system of linear equations:

$$\begin{cases} cu-(a+c)\,v = ac \\ (b+c)\,u - cv = -bc \end{cases}$$

Solving for u and v, we get $u=v=-c$, hence $z=c$ and $x=y=0$. Thus, we obtain the last solution $(0,0,c)$.