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Chapter 1

Real Numbers

We begin our study of calculus with a discussion of the real-number system. Several axiomatic constructions are used to define real numbers. Although these matters are an important part of the foundations of mathematics, we will not delve into these problems here. Therefore, we shall take the real numbers themselves as undefined objects satisfying certain axioms from which further properties will be derived. We assume that the reader is familiar with most of the properties of real numbers discussed in the next few pages, so the presentation will be concise.

For convenience we use some elementary set notation. Let A denote a set (defined intuitively as a collection of objects). The notation $x \in A$ means that the element x is in the set A , and we write $x \notin A$ to indicate that the element x is not in A . A set A is said to be a *subset* of B , and we write $A \subseteq B$, if every element in A is also in B . A set is called *nonempty*, and we write $A \neq \emptyset$, if it contains at least one element. If $A \subseteq B$ and $B \subseteq A$, we write $A = B$. Otherwise, we write $A \neq B$. We also use some logic notation.

- Any statement S has a *negation* $\neg S$ ('not S ') defined by

$\neg S$ is true if and only if S is false and $\neg S$ is false if and only if S is true

- Let $P(x)$ denote a property P of the object x . We write $\exists$ for the quantifier '*there exists*'. The expression

$$\exists x \in X : P(x)$$

means that 'there exists (at least) one object x in the class X which has the property P '. If X is clear from context, we may omit writing it. The symbol $\exists$ is called the *existential quantifier*. Furthermore, the symbol $\exists!$ means that 'there exists and it is unique'.

- Let $P(x)$ denote a property P of the object x . We write $\forall$ for the quantifier ‘for all’. The expression

$$\forall x \in X : P(x)$$

means that ‘for each object x in the class X , x has the property P ’. Again, we will omit X if it is clear from the context. The symbol $\forall$ is called the *universal quantifier*.

- Let A and B be statements. A *implies* B will be denoted by $A \implies B$. If A implies B , we take this to mean that if we wish to prove B , it suffices to prove A (A is a sufficient condition for B).
- The equivalence $A \iff B$ (‘ A and B are equivalent’ or ‘ A if and only if B ’) of the statements A and B is defined by

$$A \iff B \text{ means } A \implies B \text{ and } B \implies A.$$

A is a necessary and sufficient condition for B , or vice versa.

We assume that there exists a nonempty set $\mathbb{R}$ of elements, called *real numbers*, which satisfy the eight axioms listed below. The axioms fall in a natural way into three groups which we call the *field axioms*, the *order axioms*, and the *completeness axiom*.

1.1 The field axioms

Along with the set $\mathbb{R}$ of real numbers we assume the existence of two operations, called addition and multiplication, such that for every pair of real numbers x and y the sum $x + y$ and the product xy are real numbers uniquely determined by x and y satisfying the following axioms.

Axiom 1.1 (Commutative Laws) For all real numbers x, y , we have

$$x + y = y + x, \quad xy = yx.$$

Axiom 1.2 (Associative Laws) For all real numbers x, y, z , we have

$$x + (y + z) = (x + y) + z, \quad x(yz) = (xy)z.$$

Axiom 1.3 (Distributive Law) For all real numbers x, y, z , we have

$$x(y + z) = xy + xz.$$

Axiom 1.4 (Existence of Identity Elements) There exist two real numbers, which we denote 0 and 1, such that for any real number x , we have

$$x + 0 = 0 + x = x, \quad x \cdot 1 = 1 \cdot x = x.$$

Axiom 1.5 (Existence of an Inverse Element) Given any real number x , there exists a real number y , called the *additive inverse*, such that $x + y = y + x = 0$. We denote this element y by $-x$. Given any real number $x \neq 0$, there exists a real number y , called the *multiplicative inverse*, such that $xy = yx = 1$. We denote this element y by $1/x$ or x^{-1} .

A nonempty set A endowed with two binary operations which satisfy these axioms (where we replace “real number(s)” with “element(s) of A ”) is called a *field*. This means that $\mathbb{R}$ with addition and multiplication is a field.

We denote the sum of real numbers $x_1, x_2, \dots, x_n$ by

$$x_1 + x_2 + \dots + x_n = \sum_{k=1}^n x_k.$$

The summation notation was introduced in 1772 by the French mathematician Joseph Louis Lagrange. The index k is a *dummy variable*, so we can use any variable as the index without altering the value of the sum, that is

$$\sum_{i=m}^n x_i = \sum_{j=m}^n x_j = \sum_{k=m}^n x_k = x_m + x_{m+1} + \dots + x_n.$$

We have the following.

Theorem 1.6 Let n be a positive integer and let c be a real number. Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be real numbers. Then

$$\sum_{k=1}^n c = nc \tag{1.1}$$

$$\sum_{k=1}^n ca_k = c \sum_{k=1}^n a_k \tag{1.2}$$

$$\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k. \tag{1.3}$$

We denote the product of real numbers $x_1, x_2, \dots, x_n$ by

$$x_1 x_2 \dots x_n = \prod_{k=1}^n x_k.$$

Again, k is just a dummy variable. As for the summation notation, we have the following proposition.

Theorem 1.7 *Let n be a positive integer and let c be a real number. Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be real numbers. Then*

$$\prod_{k=1}^n c = c^n \quad (1.4)$$

$$\prod_{k=1}^n ca_k = c^n \prod_{k=1}^n a_k \quad (1.5)$$

$$\prod_{k=1}^n a_k b_k = \prod_{k=1}^n a_k \cdot \prod_{k=1}^n b_k. \quad (1.6)$$

This compact notation will be very useful in many proofs regarding sums or products of several real numbers.

1.2 The order axioms

We assume there exists a subset $P \subseteq \mathbb{R}$ whose elements are called *positive numbers* and satisfy the following properties:

Axiom 1.8 If $x, y \in P$, then $x + y \in P$ and $xy \in P$.

Axiom 1.9 (Trichotomy Law) For every real number x exactly one of the following relations holds:

$$x \in P, \quad -x \in P, \quad x = 0.$$

The real numbers with $-x \in P$ are called *negative numbers*. Thus the Trichotomy Law can be stated as saying that every real number is either positive, negative, or 0, and these three cases are mutually exclusive. Note that in particular, the Trichotomy Law says 0 is neither positive nor negative. We write $x > 0$ if x is positive and $x < 0$ if x is negative. We write $x > y$ or $y < x$ if $x - y > 0$. A real number x is called *nonnegative* if it is not negative, which by Trichotomy means that x is either positive or zero. Nonpositive is defined similarly. We write $x \geq 0$ if x is nonnegative and $x \geq y$ or $y \leq x$ if $x - y \geq 0$. We denote by $\mathbb{R}_+$ the set of all positive real numbers, and by $\mathbb{R}_-$ the set of all negative real numbers.

The following theorem is often used in proofs in calculus.

Theorem 1.10 Let a and b be real numbers such that for every $\varepsilon > 0$

$$a < b + \varepsilon. \quad (1.7)$$

Then $a \leq b$.

Proof. We argue by contradiction. Suppose that $b < a$. Then $(a - b)/2 > 0$. Since the inequality (1.7) holds for all $\varepsilon > 0$, it holds for $\varepsilon = \frac{a - b}{2}$. This implies

$$a < b + \frac{a - b}{2},$$

which simplifies to $a < b$, a contradiction. $\square$

Before showing the completeness axiom, we make some remarks.

1.3 Intervals

The real numbers are often represented geometrically as points on a line (called the *real line*). A point is selected to represent 0 and another to represent 1. This choice determines the scale. Under an appropriate set of axioms for Euclidean geometry, each point on the real line corresponds to one and only one real number and, conversely, each real number is represented by one and only one point on the line. This means that there is a *bijection* between the set of real numbers and the set of points on the real line. We can refer to the *point* x rather than the point representing the real number x . The order relation has a simple geometric interpretation. If $x < y$, the point x lies to the left of the point y . Positive numbers lie to the right of 0, and negative numbers to the left of 0. If $a < b$, a point x satisfies the inequalities $a < x < b$ if and only if x is between a and b .

The set of all points between a and b is called an *interval*.

Definition 1.11 Let a and b real numbers such that $a < b$. The *open interval* is defined to be the set

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

The *closed interval* is defined to be the set

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$

The *half-open intervals* are defined as

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}, \quad (a, b] = \{x \in \mathbb{R} : a < x \leq b\}.$$

For an interval I of any of these four types we define the width of I to be $b - a$ and denote it $|I|$.

For example, the set $[1, 2)$ is the set of real numbers x which satisfy $1 \leq x < 2$ and it is an interval of width $2 - 1 = 1$.

1.4 The extended real number system

We can extend the real line by adjoining two ‘ideal points’ denoted by the symbols $+\infty$ and $-\infty$ (‘plus infinity’ and ‘minus infinity’), where we think of $+\infty$ as being farther to the right than any point on the real line and $-\infty$ as being farther to the left. The symbols $+\infty$ and $-\infty$ are a convenient notation, but are not to be considered as being real numbers. One immediate consequence of this choice is that we can define infinite intervals.

Definition 1.12 *Infinite intervals* are defined as follows:

$$(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}, \quad (-\infty, b) = \{x \in \mathbb{R} : x < b\}$$

$$[a, +\infty) = \{x \in \mathbb{R} : a \leq x\}, \quad (a, +\infty) = \{x \in \mathbb{R} : a < x\}.$$

Moreover, we set $(-\infty, +\infty) = \mathbb{R}$. Let $a \in \mathbb{R}$. Every open interval $(a, +\infty)$ is called a *neighborhood* of $+\infty$. Every open interval $(-\infty, a)$ is called a *neighborhood* of $-\infty$.

In many cases we can unambiguously extend our arithmetic to $+\infty$ and $-\infty$, which leads to the following definition.

Definition 1.13 By the *extended real number system* $\overline{\mathbb{R}}$ (which is sometimes denoted by $[-\infty, +\infty]$) we mean the set of real numbers $\mathbb{R}$ together with two symbols $+\infty$ and $-\infty$ which satisfy the following properties:

(i) If $x \in \mathbb{R}$, then we have

$$x + (+\infty) = +\infty, \quad x + (-\infty) = -\infty$$

$$x - (+\infty) = -\infty, \quad x - (-\infty) = +\infty$$

$$\frac{x}{+\infty} = \frac{x}{-\infty} = 0.$$

(ii) If $x > 0$, then we have

$$x(+\infty) = +\infty, \quad x(-\infty) = -\infty.$$

(iii) If $x < 0$, then we have

$$x(+\infty) = -\infty, \quad x(-\infty) = +\infty.$$

(iv) $(+\infty) + (+\infty) = (+\infty)(+\infty) = (-\infty)(-\infty) = +\infty$ and
 $(-\infty) + (-\infty) = (+\infty)(-\infty) = -\infty$.

(v) If $x \in \mathbb{R}$, then we have $-\infty < x < +\infty$.

Notice that it is not possible to add, subtract, multiply, and divide arbitrary elements of the extended reals. For example, none of the operations

$$(+\infty) + (-\infty), (+\infty) - (+\infty), 0 \cdot (+\infty), \text{ or } \frac{+\infty}{+\infty}$$

are meaningful.

1.5 Rational and irrational numbers

Definition 1.14 A *rational number* is any number that can be expressed as a quotient $\frac{m}{n}$ of two integers m and n , with $n \neq 0$. The set of the rational numbers is denoted by

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}.$$

Since we can always take $n = 1$, we observe that any integer is also a rational number, and thus $\mathbb{Z} \subseteq \mathbb{Q}$.

Definition 1.15 An *irrational number* is any number that can't be expressed as a quotient $\frac{m}{n}$ of two integers m and n , with $n \neq 0$.

For example, $\frac{1}{2}, -1, 2$ are rational numbers, $\sqrt{2}, \pi, e$ are irrational numbers.

The following theorems can be proved by using elementary techniques like proof by contradiction, the Fundamental Theorem of Arithmetic and induction, so we omit the proofs.

Theorem 1.16 If n is a positive integer which is not a perfect square, then $\sqrt{n}$ is irrational.

Theorem 1.17 If $b_1, \dots, b_n$ are non-zero integers, and $a_1, \dots, a_n$ are distinct square-free positive integers, then

$$b_1\sqrt{a_1} + b_2\sqrt{a_2} + \dots + b_n\sqrt{a_n}$$

is irrational.