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Preface

Many of the problems in this book were originally featured in the online CleverMath program, offered by AwesomeMath in 2015. This program was designed to provide students with high-quality math training through carefully selected problems that would challenge their problem-solving abilities and deepen their understanding of key mathematics concepts. Over the years, these problems have been refined and compiled into this book to create a structured and comprehensive learning resource.

This book is part of a five-volume series, with the other volumes focusing on geometry and trigonometry, algebra, number theory, and calculus. Each volume is dedicated to a fundamental branch of mathematics, providing learners with a comprehensive foundation in problem-solving techniques essential for mathematics competitions and beyond.

All problems presented have a one to three-digit answer, making them ideal preparation for the AIME (American Invitational Mathematics Examination). Nevertheless, the problem set is diverse, covering a wide range of difficulty levels – some questions align with the AMC (American Mathematics Competitions) 10/12 while others are more advanced, making them suitable for Olympiad-level training.

Each problem is accompanied by a detailed and rigorous solution, ensuring that students not only arrive at the correct answer but also understand the underlying reasoning. Many solutions go beyond standard explanations, offering insightful remarks, alternative approaches, and connections to broader mathematics concepts. These additional observations encourage deeper thinking, allowing students to develop flexibility in their problem-solving strategies.

This book is tailored to students preparing for math competitions, their coaches, or anyone with a passion for problem-solving. With a carefully curated collection of challenging and thought-provoking problems, students will refine their analytical skills, hone their problem-solving techniques, and gain confidence in tackling even more complex mathematics problems.

Titu Andreescu

Problems

Problem 1. Find the number of ways of distributing 20 identical donuts to four people, so that each person gets at least one donut.

Problem 2. The equation

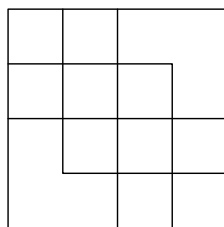
$$x_1 + x_2 + \cdots + x_{251} = 708$$

has a certain number of solutions in positive integers $x_1, x_2, \dots, x_{251}$. The equation

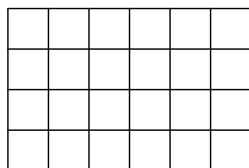
$$y_1 + y_2 + \cdots + y_n = 708$$

has the same number of solutions in positive integers $y_1, y_2, \dots, y_n$, where $n \neq 251$. Find n .

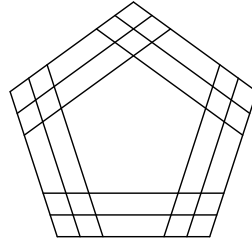
Problem 3. Find the number of squares (of any size) in the diagram.



Problem 4. How many squares are there in the following diagram?



Problem 5. How many pentagons are in the following figure?



Problem 6. You draw two cards at random from a standard deck of 52 cards. Then the probability that the two cards are from the same suit can be expressed in the form m/n , where m and n are relatively prime positive integers. Find $m + n$.

Problem 7. Find the number of five-digit numbers, where the sum of the digits is 2.

Problem 8. If $(x_1 + x_2 + \cdots + x_9)^2$ is expanded and simplified, how many different terms appear in the final expression?

Problem 9. Let the entries of 3×3 grid be numbers 0 or 1, such that the total sum of all 9 numbers is an even number. How many such squares are there?

Problem 10. Holly chooses at random a two-digit number, where both digits are odd. The probability that the number is prime can be expressed in the form m/n , where m and n are relatively prime positive integers. Find $m + n$.

Problem 11. A committee of nine people wishes to elect a chair-person, secretary, and treasurer. No person can hold more than one position. How many ways can this be done?

Problem 12. A car has one driver's seat and four different passenger seats. A group of five students wants to use the car to go to school, but only two of those five students are legally allowed to drive the car. What is the number of ways that the five students can be seated in the car for the drive?

Problem 13. In a class of 50 students, 32 are males, 41 are right-handed, and 26 are right-handed males. How many students are left-handed females?

Problem 14. Lady and the Tramp are enjoying a bowl of spaghetti which contains 20 noodles. Each randomly picks a different end of a noodle. Then the probability that they have chosen the same noodle can be expressed in the form m/n , where m and n are relatively prime positive integers. Find $m + n$.

Problem 15. How many different words can be formed from the letters M, A, T, and H, where each letter is used at most once? Examples include HT, M, and AHMT.

Problem 16. Three vertices of a regular dodecagon (12-sided polygon) are chosen at random. Then the probability that the three vertices form an equilateral triangle can be expressed in the form m/n , where m and n are relatively prime positive integers. Find $m + n$.

Problem 17. How many 3-digit positive integers have at least two equal digits?

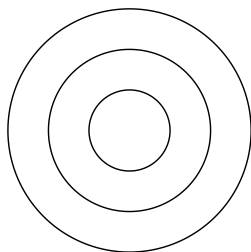
Problem 18. There are nine squares in a row, and we want to color each square either red, green, or blue, so that no two adjacent squares have the same color. How many different ways can we color the nine squares?

Problem 19. Cosmin has a bag containing 4 red balls, 2 yellow balls, and 7 blue balls. He draws the balls out of the bag at random, one at a time, without replacement. The probability that the fifth ball is blue can be expressed in the form m/n , where m and n are relatively prime positive integers. Find $m + n$.

Problem 20. Find the number of 14-digit numbers such that:

- (i) each digit is 1, 2, or 3,
- (ii) every digit of 2 is next to a 1 and a 3,
- (iii) any two consecutive digits are different.

Problem 21. A dart board, shown below, is divided into three regions by circles of radius 1, 2, and 3. Then the probability that a dart lands in the region that is inside the circle of radius 2 but outside the circle of radius 1 can be expressed in the form m/n , where m and n are relatively prime positive integers. Find $m + n$.



Problem 22. I have three collars for my dog, one blue, one green, and one red. I have 5 leashes for my dog, one each in green, red, purple, orange, and black. In how many ways can I put a collar on my dog, then put him on a leash so that the leash and the collar are different colors?

Problem 23. In a lottery, a bin is filled with balls numbered from 1 to 49, and five balls are drawn, one at a time, at random (without replacement).