Preface and Acknowledgements

This books stems from my desire to publish Cuban National Mathematical Olympiad problems with elegant solutions and illustrations. It encompasses all problems from the 2001 to 2016 Olympiads, except for 2002, with thorough, in-depth solutions. This work is much more than a compilation of problems; it is a meticulous exposition with complete solutions to every problem, allowing readers to grasp the problem-solving techniques discussed.

Many thanks to Ercole Suppa, Javier Garcia Capitan, and Daniel Lasaosa for their helpful contributions. I also thank Manuel Candales for his elegant solutions to some discrete mathematics problems. I express my gratitude to Eduardo Perez for his work "Cuatro Décadas de Olimpiadas de Matemáticas en Cuba".

This book is being published because of the great opportunity offered by Titu Andreescu and the rigorous and enhanced translation to English by Enrique Treviño.

I hope you all enjoy the problems.

Robert Bosch

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Problems

National Olympiad (2001)

Day 1

1. Each entry in a 3×3 table is a real number. The entry in the i^{th} row and the j^{th} column is equal to the absolute value of the difference of the sum of the elements in the i^{th} row and the sum of the elements in the j^{th} column. Prove that every entry in the table is equal to the sum or difference of two other entries in the table.

2. In convex quadrilateral $ABCD$, let M be the intersection of diagonals AC and BD. Let K be the intersection of the extension of side AB with the angle bisector of $\measuredangle ACD$. If

$$MA \cdot MC + MA \cdot CD = MB \cdot MD,$$

   prove that $\measuredangle BKC = \measuredangle CDB$.

3. Let n be a positive integer.

   a) Prove that $(2n+1)^3 - (2n-1)^3$ is the sum of three perfect squares.

   b) Prove that $(2n+1)^3 - 2$ is the sum of $3n-1$ perfect squares greater than 1.

Day 2

1. Let f be a linear function satisfying $f(0) = -5$ and $f(f(0)) = -15$. Find all values of $k \in \mathbb{R}$ for which the set of x satisfying the inequality $f(x) \cdot f(k-x) > 0$ is an interval of length 2.

2. Let $ABCD$ be a square. Let M and K be points on the line segments BC and CD, respectively, satisfying $MC = KD$. Let P be the intersection of the line segments MD and BK. Prove that $AP \perp MK$.

3. Prove that no natural number n exists such that the sum of the digits of $n(2n-1)$ is equal to 2000.

4. Let P_1, P_2, P_3, P_4 be four points on an arc of a circle measuring less than $180°$. The tangents to P_1, P_2, P_3, P_4 intersect to form a convex quadrilateral $ABCD$. Prove that two of the vertices of $ABCD$ belong to an ellipse that has the other two vertices as its focal points.

5. Let p and q be positive integers such that $1 \le q \le p$. Let

$$a = \left(p + \sqrt{p^2 + q}\right)^2.$$

   a) Prove that a is irrational.
   b) Prove that $\{a\} > 0.75$.

6. The roots of the equation $ax^2 - 4bx + 4c = 0$ with $a > 0$ belong to the interval $[2, 3]$. Prove that:

   a) $a \le b \le c < a + b$;
   b) $\dfrac{a}{a+c} + \dfrac{b}{b+a} > \dfrac{c}{b+c}$.

7. Prove that the equation $x^{19} + x^{17} = x^{16} + x^7 + a$ has at least two complex roots with nonzero imaginary part for all $a \in \mathbb{R}$.

8. Find all real solutions to the equation $x + \cos x = 1$.

9. In right triangle ABC, with right angle at C, let F be the intersection of the altitude CD and the angle bisector AE (where E is on BC). Let G be the intersection of ED and BF. Prove that the area of quadrilateral $CEGF$ is equal to the area of triangle BDG.

National Olympiad (2003)

Day 1

1. Consider the following list of numbers:

$$1990, 1991, 1992, \ldots, 2002, 2003, 2003, 2003, \ldots, 2003,$$

   where 2003 appears 12 times. Is it possible to write these numbers in some order in such a way to obtain a prime number?

2. Let KL and KN be two tangents drawn from point K to the circle Γ, where L and N are points of tangency. Let M be an arbitrary point on the extension of the line segment KN through N. Let P be the second point of intersection of Γ with the circumcircle of $\triangle KLM$. Let Q be the foot of the perpendicular to ML through N. Prove that $\measuredangle MPQ = 2\measuredangle KML$.

3. A 4×4 chessboard has all its squares painted white. A possible move is to select a rectangle with three squares and change the colors of the three squares in either of the following ways:

   i) If a square is white, then paint it black.

   ii) If a square is black, then paint it white.

   Prove that it is not possible to obtain a chessboard with all its squares black by applying any number of moves.

Day 2

1. The roots of the equation $x^2 + (3a + b)x + a^2 + 2b^2 = 0$ are x_1 and x_2, with $x_1 \neq x_2$. Determine all values of a and b for which the roots of the equation $x^2 - 2a(3a + 2b)x + 5a^2b^2 + 4b^4 = 0$ are x_1^2 and x_2^2.

2. Prove that if $\dfrac{p}{q} = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \ldots - \dfrac{1}{1334} + \dfrac{1}{1335}$ with $p, q \in \mathbb{Z}^+$, then p is divisible by 2003.

3. Let ABC be an acute triangle and let T be an interior point such that $\measuredangle ATB = \measuredangle BTC = \measuredangle CTA$. Let M, N, and P be the feet of the perpendiculars from T to BC, CA, and AB, respectively. Prove that if the circumcircle of $\triangle MNP$ intersects (again) BC, CA, and AB in M_1, N_1, and P_1, respectively, then $\triangle M_1N_1P_1$ is equilateral.

4. Let $f : \mathbb{N} \to \mathbb{N}$ be such that $f(p) = 1$ for each prime p, and
$$f(ab) = bf(a) + af(b) \text{ for all } a, b \in \mathbb{N}.$$
Prove that if $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ is the prime factorization of n and p_i does not divide a_i for $i = 1, 2, \dots, k$, then $\frac{n}{\gcd(n, f(n))}$ is squarefree, i.e., it is not divisible by any square greater than 1.

5. Let $a_1, a_2, \dots, a_9$ be nonnegative real numbers such that $a_1 = a_9 = 0$ and at least one of the terms is not 0.
   a) Prove that for some i ($i = 2, \dots, 8$), the following is true:
   $$a_{i-1} + a_{i+1} < 2a_i.$$
   b) Would the inequality from a) remain true if we changed the $2a_i$ to $1.9a_i$ above?

6. Let P_1, P_2, P_3, P_4 be four points on a circle, let I_1 be the incenter of $\triangle P_2P_3P_4$, I_2 be the incenter of $\triangle P_1P_3P_4$, I_3 be the incenter of $\triangle P_1P_2P_4$, and I_4 be the incenter of $\triangle P_1P_2P_3$. Prove that $I_1I_2I_3I_4$ is a rectangle.

7. Let $S(n)$ be the sum of the digits of the positive integer n. Determine
$$S\left(S\left(S\left(S\left(2003^{2003}\right)\right)\right)\right).$$

8. Let $\mathbb{R}^+$ be the set of all nonnegative real numbers and let $\mathbb{C}$ be the set of all complex numbers. Find all functions $f : \mathbb{C} \to \mathbb{R}^+$ that satisfy all of the following conditions:
   i) $f(uv) = f(u)f(v)$ for all $u, v \in \mathbb{C}$;
   ii) $f(\alpha u) = |\alpha|\, f(u)$ for all $\alpha \in \mathbb{R}, u \in \mathbb{C}$;
   iii) $f(u) + f(v) \le |u| + |v|$ for all $u, v \in \mathbb{C}$.

9. Let D be the midpoint of the base AB of the acute and isosceles triangle ABC (where $AC = BC$). Let E be a point on AB, and let O be the circumcenter of triangle ACE. Prove that the line perpendicular to DO that passes through D, the line perpendicular to BC that passes through E, and the line parallel to AC that passes through B are concurrent (i.e., they all intersect in one point).

National Olympiad (2004)

Day 1

1. A square is divided into 25 unit squares by drawing lines parallel to the sides of the square. Some diagonals of the unit squares are drawn in such a way that no two diagonals share a point. What is the maximum number of diagonals that can be drawn with this property?

2. When an integer $n > 2$ is written as a sum of two or more consecutive positive integers, we say that we have an *elegant decomposition of* n. Two elegant decompositions are considered different if one of them contains a summand the other does not. How many different elegant decompositions does 3^{2004} have?

3. An exam has 6 problems. Each problem was solved by exactly 1000 students. There is no pair of students that (when taken together) has solved all 6 problems. Determine the smallest number of participants that could have taken the exam.

Day 2

1. Find all real solutions to the following system of equations:

$$x_1 + x_2 + \ldots + x_{2004} = 2004$$
$$x_1^4 + x_2^4 + \ldots + x_{2004}^4 = x_1^3 + x_2^3 + \ldots + x_{2004}^3.$$

2. Write two 1's, then a 2 between them, then a 3 between numbers whose sum is 3, then a 4 between numbers whose sum is 4, as shown in what follows: $(1, 1)$, $(1, 2, 1)$, $(1, 3, 2, 3, 1)$, $(1, 4, 3, 2, 3, 4, 1)$. Continue in this fashion with 5, 6, 7, and so on. Prove that the number of times the positive integer $n \geq 2$ appears in the sequence is the number of positive integers less than or equal to n that are relatively prime to n.

3. In non-isosceles triangle ABC, we draw the interior angle bisectors through B and C, which intersect the sides AC and AB at E and F, respectively. The line EF intersects the extension of side BC at point T. Point D lies on BC such that $\dfrac{DB}{DC} = \dfrac{TB}{TC}$. Prove that AT is the exterior angle bisector of $\measuredangle A$.

4. Find all pairs of natural numbers (x, y) satisfying $x^2 = 4y + 3 \cdot \text{lcm}(x, y)$.

5. Let K be a circle with an inscribed quadrilateral $ABCD$, such that BD is not a diameter. Prove that the intersection of the tangent lines to K through B and D lies on line AC if and only if $AB \cdot CD = AD \cdot BC$.

6. Consider the equation $\dfrac{ax^2 - 24x + b}{x^2 - 1} = x$. Find all real numbers a and b for which the equation has exactly two distinct real solutions whose sum is 12.

7. For real numbers α, β, γ with $\beta\gamma \neq 0$, we have that $\dfrac{1-\gamma^2}{\beta\gamma} \geq 0$. Prove that
$$5(\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma^3) \geq \alpha\beta.$$

8. Let $\mathbb{R}^+$ be the set of nonnegative real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy all of the following conditions:

   i) $f(xf(y))f(y) = f(x+y)$ for $x, y \geq 0$;
   ii) $f(2) = 0$;
   iii) $f(x) \neq 0$ for $0 \leq x < 2$.

9. We are given $\measuredangle XOY = \alpha$ and points A, B on line OY such that $OA = a$ and $OB = b$, with $a > b$. Consider a circle tangent to line OX and containing the points A and B.

   a) Calculate the radius of the circle in terms of a, b, and α.
   b) If a and b are constants and α varies, show that the minimum value of the radius of the circle is $\dfrac{a-b}{2}$.

National Olympiad (2005)

Day 1

1. Find the smallest real number a such that there exists a square with side length a that contains five unit circles such that no pair of circles shares any interior points.

2. We have n light bulbs in a circle and one of them is marked.
   Let A be the following operation:

   > Choose a divisor d of n. Beginning with the marked light bulb, start counting clockwise around the circle from 1 to dn and change the state ("on" or "off") of the light bulbs corresponding to the multiples of d.

   Let B be the following operation:

   > Start with all light bulbs off. Apply operation A to all divisors of n, in order.

   Find all positive integers n, such that after applying operation B, all light bulbs are in the "on" position.

3. We have two piles of cards, one with n cards and the other with m cards. Players A and B play alternating turns by making any the following moves:

   i) Remove one card from one pile.
   ii) Remove one card from each pile.
   iii) Move one card from one pile to the other pile.

   Player A starts the game. The winner is whoever picks the last card. Determine if there is a winning strategy in terms of m and n so that a player would win every time that player follows such a strategy.

Day 2

1. Find all quadrilaterals that can be divided by a diagonal into two triangles that have the same area and the same perimeter.

2. Find all quadratic functions $f(x) = ax^2 + bx + c$ for which there exists an interval (h, k) such that for all $x \in (h, k)$, we have $f(x)f(x+1) < 0$ and $f(x)f(x-1) < 0$.

3. Find all quadruples of real numbers such that the product of any three of them plus the fourth one is a constant.

4. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}$ such that

$$f(x)f(y) = f(xy) + \frac{1}{x} + \frac{1}{y} \quad \text{for all positive reals } x, y.$$

5. Point P is chosen on the circumcircle of triangle ABC such that the perpendicular to AC through P intersects the circle at point Q, the perpendicular to AB through Q intersects the circle at R, and the perpendicular to BC through R intersects the circle at P. Let O be the circumcenter of triangle ABC. Prove that $\measuredangle POC = 90°$.

6. All the positive differences $a_i - a_j$ of five distinct positive integers a_1, a_2, a_3, a_4, a_5 are distinct. Let A be the set containing all the maximum elements from each group of five positive integers with this property. Find the smallest possible element of A.

7. Find all ordered triples (x, y, z) of positive integers satisfying $x < y < z$, $\gcd(x, y) = 6$, $\gcd(y, z) = 10$, $\gcd(z, x) = 8$, and $\operatorname{lcm}(x, y, z) = 2400$.

8. Find the smallest real number A such that there exist two distinct triangles with integer side lengths and with area equal to A.

9. Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be positive real numbers such that:

$$x_1 + x_2 + \ldots + x_n \geq y_i \geq x_i^2 \quad \text{for all } i = 1, 2, \ldots, n.$$

Prove that

$$\frac{x_1}{x_1y_1 + x_2} + \frac{x_2}{x_2y_2 + x_3} + \ldots + \frac{x_n}{x_ny_n + x_1} > \frac{1}{2n}.$$

National Olympiad (2006)

Day 1

1. Each one of n students in a class sent a letter to m classmates. Prove that if $2m+1>n$, then at least two students sent each other letters.

2. Suppose that n people numbered from 1 to n are placed in a row. An *admissible move* consists of each person either staying in place or changing his or her position with at most one person. For example,

Initial Position	1	2	3	4	5	6	...	$n-2$	$n-1$	n
Final Position	2	1	3	6	5	4	...	n	$n-1$	$n-2$

is an admissible move. Is it possible to reach the position

n	1	2	3	4	5	...	$n-3$	$n-2$	$n-1$

starting from

1	2	3	4	5	6	...	$n-2$	$n-1$	n

with two admissible moves?

3. In an $m\times n$ table, k squares are painted such that the following is true:

> If the centers of four squares are the vertices of a quadrilateral with sides parallel to the sides of the table, then at most two of these squares are painted.

In terms of m and n, find the greatest possible value of k.

Day 2

1. Find all monic polynomials $P(x)$ of degree 3 with integer coefficients that satisfy all of the following conditions:

   i) $x-1$ divides $P(x)$.

   ii) When $P(x)$ is divided by $x-5$, it leaves the same remainder as when it is divided by $x+5$.

   iii) $P(x)$ has a root between 2 and 3.

2. Let U be the center of the inscribed circle of triangle ABC. Let O_1, O_2, and O_3 be the respective circumcenters of triangles BCU, CAU, and ABU. Prove that the circumcircles of triangles ABC and $O_1O_2O_3$ have the same center.

3. Let a, b, c be distinct real numbers. Prove that

$$\left(\frac{2a-b}{a-b}\right)^2 + \left(\frac{2b-c}{b-c}\right)^2 + \left(\frac{2c-a}{c-a}\right)^2 \geq 5.$$

4. Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ (where $\mathbb{Z}^+$ includes 0) such that:

   i) $f(n+1) > f(n)$ for all $n \in \mathbb{Z}^+$, and

   ii) $f(m + f(n)) = f(m) + n + 1$ for all $m, n \in \mathbb{Z}^+$.

   Find $f(2006)$.

5. The sequence of positive integers $a_1, a_2, \ldots, a_{400}$ satisfies the relation $a_{n+1} = \tau(a_n) + \tau(n)$ for all $1 \leq n \leq 399$, where $\tau(k)$ is the number of positive divisors of k. Prove that the sequence contains at most 210 prime numbers.

6. Two concentric circles with radii equal to 1 and 2 are centered at point O. The vertex A of an equilateral triangle ABC lies on the larger circle. The midpoint of BC lies on the smaller circle. If B, O, and C are not collinear, find, with proof, all possible values of $\measuredangle BOC$.

7. The sequence $a_1, a_2, a_3, \ldots$ satisfies

$$a_1 = 3,\ a_2 = -1, \text{ and } a_n a_{n-2} + a_{n-1} = 2 \text{ for all } n \geq 3.$$

   Calculate $a_1 + a_2 + \ldots + a_{99}$.

8. Prove that for each integer $k \geq 2$, there exists a power of 2 such that the last k digits contain at least half as many nines. For example, for $k = 2$ and $k = 3$, we have $2^{12} = \ldots 96$ and $2^{53} = \ldots 992$, respectively.

9. In cyclic quadrilateral $ABCD$, diagonals AC and BD intersect at point P. Let O be the center of the circumcircle of $ABCD$, and let E be a point on the extension of OC through C. Through E, a line parallel to CD is drawn which intersects the extension of OD through D in F. Let Q be a point in the interior of $ABCD$ such that $\measuredangle AFQ = \measuredangle BEQ$ and $\measuredangle FAQ = \measuredangle EBQ$. Prove that $PQ \perp CD$.

National Olympiad (2007)

Day 1

1. We place tokens in some of the squares of an 8×8 table such that:

   i) There is at least one token in each 2×1 and 1×2 rectangle.

   ii) There are at least two neighboring squares with a token in each 7×1 and 1×7 rectangle.

   Find the smallest number of tokens that can be placed to satisfy the above conditions.

2. A prism is called *binary* if its vertices can be assigned numbers from the set $\{-1, 1\}$ in such a way that the product of the numbers assigned to the vertices of each face is -1.

   a) Prove that the number of vertices of a binary prism is divisible by 8.

   b) Prove that a prism with 2000 vertices is binary.

3. A tennis competition takes place for four days and the number of participants is $2n$, with $n \geq 5$. Each participant plays exactly once a day (it is possible for a pair of participants to encounter each other several times). Prove that the competition could end with exactly one winner and exactly three people in second place such that no players lost all four games.

Day 2

1. Find all ordered pairs of real numbers (x, y) such that

$$x^3 - y^3 = 7(x - y) \quad \text{and} \quad x^3 + y^3 = 5(x + y).$$

2. Find three distinct positive integers whose sum is as small as possible given that the sum of any pair of them is a perfect square.

3. Let $ABCD$ be a quadrilateral that can be inscribed in a circle and has perpendicular diagonals. Let P and Q be the respective feet of the perpendiculars through D and C to line AB. Let X be the intersection of AC and DP, and let Y be the intersection of BD and CQ. Prove that quadrilateral $XYCD$ is a rhombus.

4. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$x^2(f(x) + f(y)) = (x + y)f(yf(x)) \text{ for all positive reals } x, y.$$