Preface

This book showcases the synthetic problem-solving methods which frequently appear in modern day Olympiad geometry, in the way we believe they should be taught to someone with little familiarity in the subject. In some sense, the text also represents an unofficial sequel to the recent problem collection published by XYZ Press, *110 Geometry Problems for the International Mathematical Olympiad*, written by the first and third authors; but, the two books can be studied completely independently of each other.

Lemmas in Olympiad Geometry is a project that started in the summer of 2011, when the third author first taught the Geometric Proofs course at the AwesomeMath Summer Camp. Some brief lecture notes were written back then (with the intention of getting expanded), but nothing substantial happened until last summer, when the second author came to the Cornell camp as a teaching assistant for the same course. Ever since, we have all been working together to make the current version of the manuscript possible, and are excited to announce that it is ready.

The work is designed as a medley of the important Lemmas in classical geometry in a relatively linear fashion: gradually starting from Power of a Point and common results to more sophisticated topics, where knowing a lot of techniques can prove to be tremendously useful. We treated each chapter as a short story of its own and included numerous solved exercises with detailed explanations and related insights that will hopefully make your journey very enjoyable. Each chapter is also accompanied by a short list of problems that we have carefully selected. These are problems that we have solved ourselves on our own at some point, and so we are convinced that you are going to appreciate them as well. The last chapter on three dimensional geometry is the only chapter which is not followed by such a list of problems, since we considered it as a bonus section, yet one that has beautiful problems which are also relevant in other subdomains of geometry.

We wish you a pleasant reading and hope that you will enjoy *Lemmas in Olympiad Geometry* as much as we enjoyed writing it.

The authors

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Chapter 1

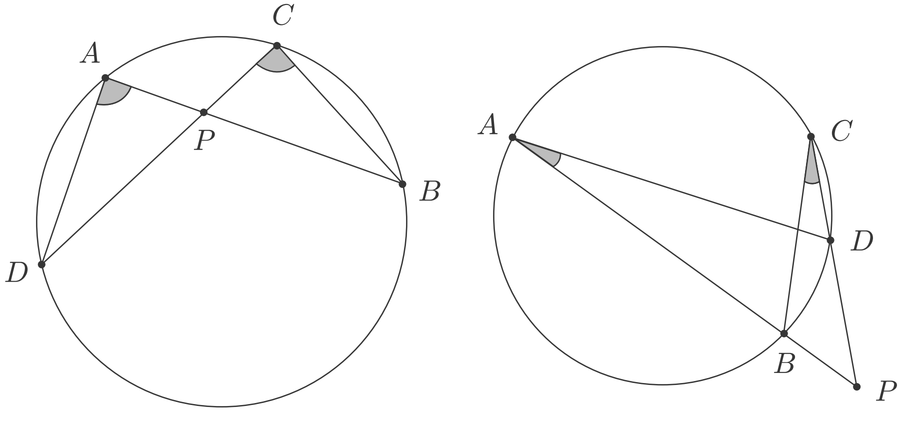
Power of a Point

One of the most important tools in Olympiad Geometry is the so-called Power of Point Theorem, our first Lemma.

Theorem 1.1. Let Γ be a circle, and P a point. Let a line through P meet Γ at points A and B, and let another line through P meet Γ at points C and D. Then

$$PA \cdot PB = PC \cdot PD.$$

We announce the reader that we will be labeling our Lemmas as Theorems not to follow any convention, but rather to emphasize their importance, since after all they represent the main stars of our show.



Proof. Of course, there are two configurations to consider here, depending on whether P lies inside the circle or outside the circle. In the case when P lies inside the circle, we have $\angle PAD = \angle PCB$ and $\angle APD = \angle CPB$, so that triangles PAD and PCB are similar; hence

$$\frac{PA}{PD} = \frac{PC}{PB}.$$

Rearranging then yields $PA \cdot PB = PC \cdot PD$.

When P lies outside the circle, we again have $\angle PAD = \angle PCB$ and $\angle APD = \angle CPB$, so again triangles PAD and PCB are similar. We get the same result in this case. □

As a very important special case, when P lies outside the circle and PC is tangent to the circle, we have that

$$PA \cdot PB = PC^2.$$

Conversely, the above represents a very useful criterion for proving concyclities.

Theorem 1.2. Let A, B, C, D be four distinct points. Let the lines AB and CD intersect at P. Assume that either P lies on both line segments AB and CD, or P lies on neither line segment. Then A, B, C, D are concyclic if and only if $PA \cdot PB = PC \cdot PD$.

Proof. Going backwards, the relation $PA \cdot PB = PC \cdot PD$ is equivalent to

$$\frac{PA}{PD} = \frac{PC}{PB},$$

which combined with $\angle APD = \angle CPB$ (which holds in both configurations described above) yields that triangles APD and CPB are similar. Thus, we get that $\angle PAD = \angle PCB$, which in both cases implies that A, B, C, D are concyclic. □

This tells us that no matter what chord XY we take through P (with X, Y on the circle), the value $PX \cdot PY$ is constant. This constant is called the **power of P** with respect to the circle considered. In particular, if $\Gamma(O, R)$ is the circle with center O and radius R, then if we consider the chord XY that passes through the center O (i.e. we choose the diameter of the circle passing through P), we get that

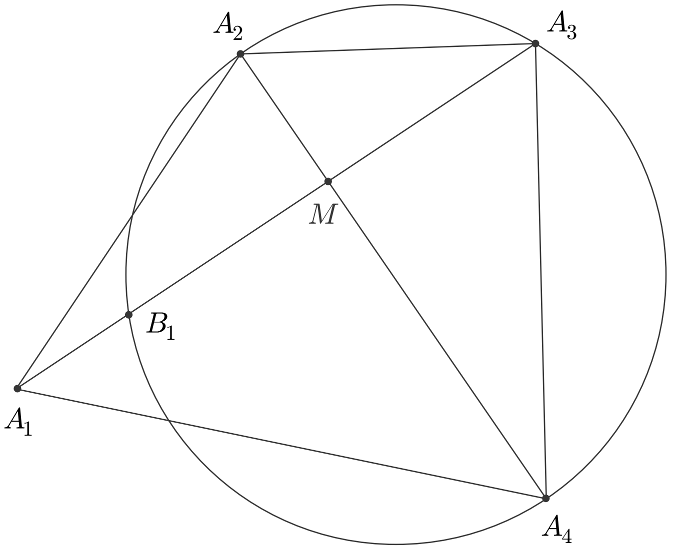
$$PX \cdot PY = \|OP^2 - R^2\|$$

//We say that the points lying on the circle Γ have zero power with respect to Γ!

We emphasize this interplay between products and differences of squares with the following exercise.

Delta 1.1. (IMO 2011 Shortlist) Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcenter and the circumradius of triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$



Proof. Let M be the point of intersection of the diagonals A_1A_3 and A_2A_4. On each diagonal choose a direction and let x, y, z, and w be the signed distances from M to the points A_1, A_2, A_3, A_4, respectively. Let ω_1 be the circumcircle of triangle $A_2A_3A_4$ and let B_1 be the second intersection of ω_1 and A_1A_3 (thus, $B_1 = A_3$ if and only if A_1A_3 is tangent to ω_1). Since the expression $O_1A_1^2 - r_1^2$ is the power of the point A_1 with respect to ω_1, we get

$$O_1A_1^2 - r_1^2 = A_1B_1 \cdot A_1A_3.$$

On the other hand, from the equality $MB_1 \cdot MA_3 = MA_2 \cdot MA_4$, we obtain

$$MB_1 = \frac{yw}{z}.$$

Hence, it follows that

$$O_1A_1^2 - r_1^2 = \left(\frac{yw}{z} - x\right)(z - x) = \frac{z-x}{z}(yw - xz).$$

Doing the same thing for the other three expressions, we then get that

$$\sum_{i=1}^{4} \frac{1}{O_iA_i^2 - r_i^2} = \frac{1}{yw - xz}\left(\frac{z}{z-x} - \frac{w}{w-y} + \frac{x}{x-z} - \frac{y}{y-w}\right) = 0,$$

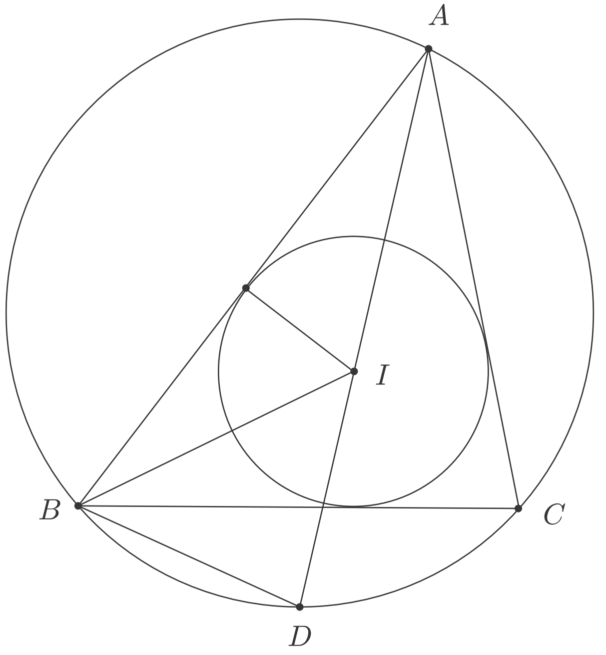
as claimed. This completes the proof. □

By the way, this will not be the only time we will make use of signed distances in this material. Usually, we can assume without loss of generality a certain position of the points in our diagram - however, in problems involving lots of circles, the computations involving the Power of Point Theorem are not the same for all configurations; hence, we often need to take extra care when dealing with signs.

Some warm-up problems now! We begin with another simple interplay between the two formulas for the power of a point.

Delta 1.2. (Euler's Theorem) In a triangle ABC with circumcenter O, incenter I, circumradius R, and inradius r, prove that

$$OI^2 = R(R - 2r).$$



Proof. Let AI meet the circumcircle again at D. In this case, the Power of Point Theorem applied for I yields

$$IA \cdot ID = R^2 - OI^2.$$

Thus, we would like to show that $IA \cdot ID = 2Rr$. First, note that $IA = \frac{r}{\sin \frac{A}{2}}$ (draw the perpendicular from I to AB and apply the Law of Sines in the right triangle that you obtain). Next, note that

$$\angle BID = \angle BAD + \angle ABI = \angle DAC + \angle IBC = \angle DBC + \angle IBC = \angle IBD;$$

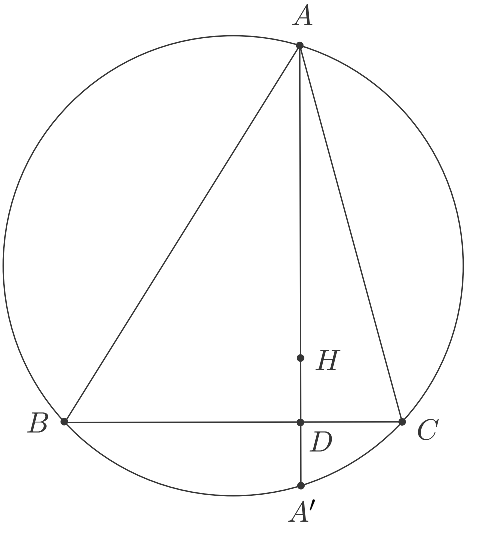
hence $ID = BD = 2R\sin\frac{A}{2}$, where the last equality comes from the (extended) Law of Sines in triangle ABD. Hence, we get that

$$IA \cdot ID = \frac{r}{\sin\frac{A}{2}} \cdot 2R\sin\frac{A}{2} = 2Rr,$$

as desired. This completes the proof. □

Note that for any given point P in plane, the above method can be extended to generate an identity for OP^2.

Delta 1.3. Let ABC be an acute-angled triangle and let D be the foot of the A-altitude. Let H be a point on the segment AD. Prove that H is the orthocenter of triangle ABC if and only if $DB \cdot DC = AD \cdot HD$.



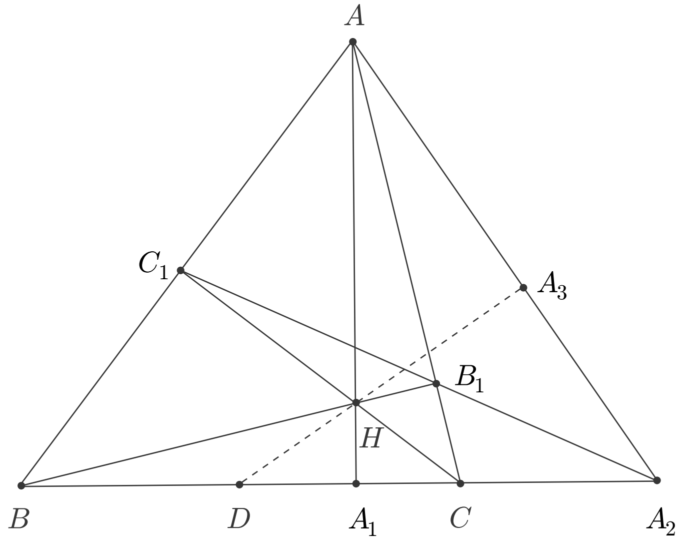
Proof. Let A' be the second intersection of the line AD with the circumcircle of triangle ABC. We know that A' is the reflection of the orthocenter across BC (if not, try angle chasing). Thus, if H is the orthocenter of ABC, then the computing power of D with respect to the circumcircle gives us

$$DB \cdot DC = AD \cdot DA' = AD \cdot HD,$$

as desired. Conversely, we have that $DB \cdot DC = AD \cdot HD$ and also $DB \cdot DC = AD \cdot HA'$ (the power of D with respect to the circumcircle); thus $HD = HA'$, and so H needs to be the orthocenter of ABC, as claimed. □

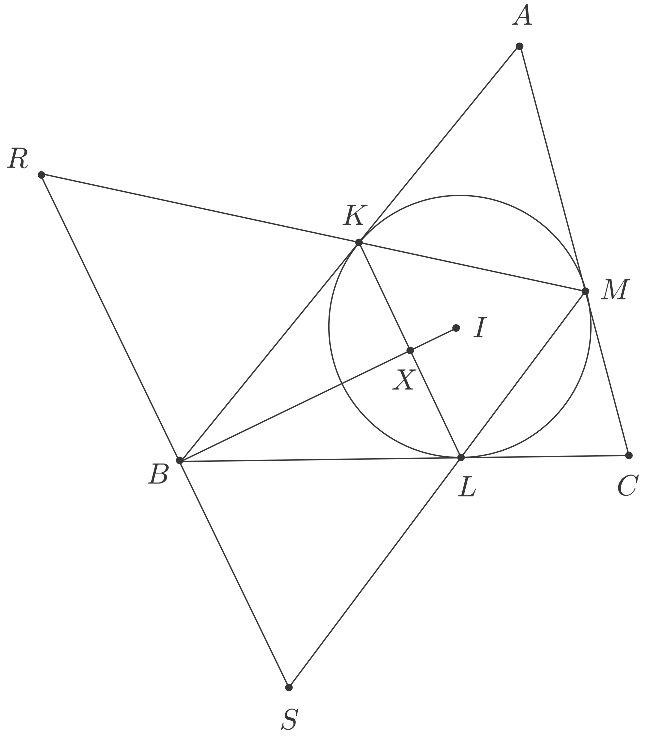
Although very simple, this proves to be a very useful criterion for showing that a point lying on an altitude of a triangle is the orthocenter. Let's see a couple of problems where this may come in handy.

Delta 1.4. (USA TSTST 2012) In scalene triangle ABC, let the feet of the perpendiculars from A to BC, B to CA, C to AB be A_1, B_1, C_1, respectively. Denote by A_2 the intersection of lines BC and B_1C_1. Define B_2 and C_2 analogously. Let D, E, F be the respective midpoints of sides BC, CA, AB. Show that the perpendiculars from D to AA_2, E to BB_2 and F to CC_2 are concurrent.



Proof. Let H be the orthocenter of triangle ABC. We claim that H is the desired point of concurrency. Let A_3 be the foot of perpendicular from D to line AA_2. Since $AA_1 \perp BC$ and $DA_3 \perp AA_2$, quadrilateral A_3A_1DA is cyclic. By Power of a Point, we have $A_2C_1 \cdot A_2B_1 = A_2A_3 \cdot A_2A$. Again, by Power of a Point (this time with respect to the nine point circle of triangle ABC) $A_2A_1 \cdot A_2D = A_2C_1 \cdot A_2B_1$, so combining these equations, $A_2C_1 \cdot A_2B_1 = A_2A_3 \cdot A_2A$, implying quadrilateral $A_3C_1B_1A$ is cyclic by **Theorem 1.2**. But H lies on the circumcircle of this quadrilateral, since $HC_1 \perp AB$ and $HB_1 \perp AC$. It follows that $\angle HA_3A = 180° - \angle HB_1A = 90°$, so points D, H, A_3 are collinear. Defining B_3 and C_3 analogously, similar arguments show that points E, H, B_3 and F, H, C_3 are also collinear, so the lines in the problem are concurrent at H as claimed. □

Delta 1.5. (IMO Shortlist 1998) Let I be the incenter of triangle ABC. Let K, L and M be the points of tangency of the incircle of triangle ABC with sides AB, BC, and CA, respectively. The line ℓ passes through B and is parallel to KL. The lines MK and ML intersect ℓ at the points R and S respectively. Prove that $\angle RIS$ is acute.



Proof. First note that

$$\angle KRB = \angle MKL = \angle MLC = \angle SLB$$

and

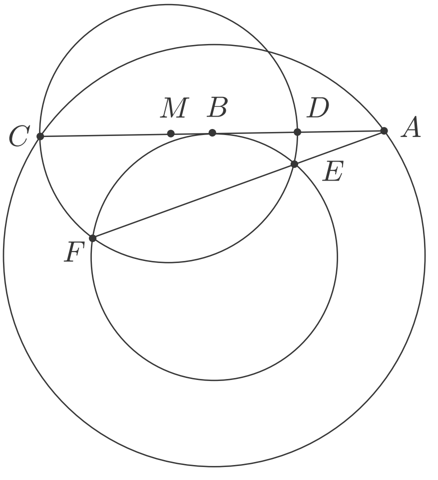
$$\angle RKB = \angle AKM = \angle KLM = \angle LSB$$

Thus, triangle BKS is similar to triangle BRL. This means that $BS \cdot BR = BL^2$. Now let X be the midpoint of segment KL. We have that X lies on the altitude from I to RS and also that $BX = BL \cos \frac{B}{2}$ and $BI = \frac{BL}{\cos \frac{B}{2}}$ which means that $BX \cdot BI = BR \cdot BS$. Hence, by **Delta 1.3**, X is the orthocenter of triangle RIS. But since X is the projection of I onto line KL it's clear that X lies inside of triangle RIS which implies that this triangle is acute as desired. □

//Another way to prove that X is the orthocenter of triangle RIS is to prove that triangle RXS is self-polar with respect to the incircle of triangle ABC.

We continue with a computational problem from the USA Mathematical Olympiad from 1998.

Delta 1.6. (USAMO 1998) Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent AB to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let C be the second point of intersection of AB with $\mathcal{C}_1$, and let D be the midpoint of AB. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio $\frac{AM}{MC}$.

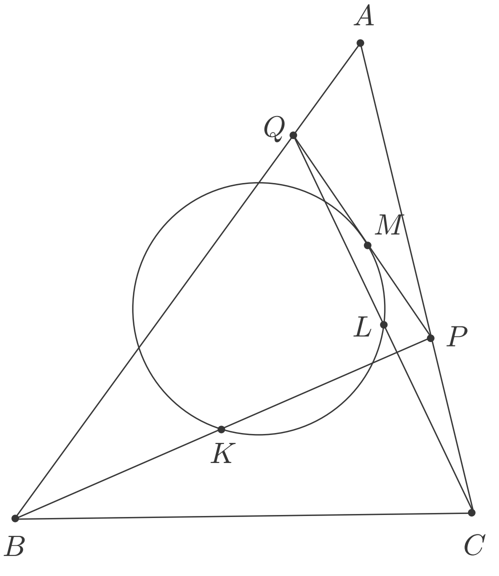


Proof. Let O be the common center of the concentric circles $\mathcal{C}_1, \mathcal{C}_2$. The tangency point B is the midpoint of the chord AC, because AC is perpendicular to the radius OB of the circle $\mathcal{C}_2$, and O is also the center of the circle $\mathcal{C}_1$. The power of the point A with respect to circle $\mathcal{C}_2$ is $AE \cdot AF = AB^2$. But since B is the midpoint of AC and D the midpoint of AB, we have that $AD \cdot AC = \frac{AB}{2} \cdot 2AB = AB^2$ as well. Hence, by **Theorem 1.2**, quadrilateral $CDEF$ is cyclic. The intersection M of the perpendicular bisectors of its diagonals CE, DF is its circumcenter. If this circumcenter is to be on its side CD, it must be the midpoint of this side, hence $DM = MC = \frac{DC}{2}$. Since $DC = \frac{3}{2}AB$, we now have $DM = MC = \frac{3}{4}AB$ and $AM = AD + DM = \frac{AB}{2} + \frac{3}{4}AB = \frac{5}{4}AB$ and so $\frac{AM}{MC} = \frac{5}{3}$. □

We continue with a beautiful IMO problem, where Power of Point can be used in a surprising way.

Delta 1.7. (IMO 2009) Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB respectively. Let

K, L and M be the midpoints of the segments BP, CQ and PQ. respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.



Proof. Since line PQ is tangent to Γ, we have that $\angle QMK = \angle MLK$. Since MK is the P-midline of triangle PQB we have that $MK \parallel AB$ so $\angle QMK = \angle AQM$. Hence, $\angle AQP = \angle MLK$. Similarly we get that $\angle MKL = \angle APQ$, so triangles MKL and APQ are similar. Therefore

$$\frac{AQ}{ML} = \frac{AP}{MK} \Longrightarrow \frac{AP}{BQ} = \frac{AQ}{PC} \Longrightarrow AP \cdot PC = AQ \cdot BQ.$$

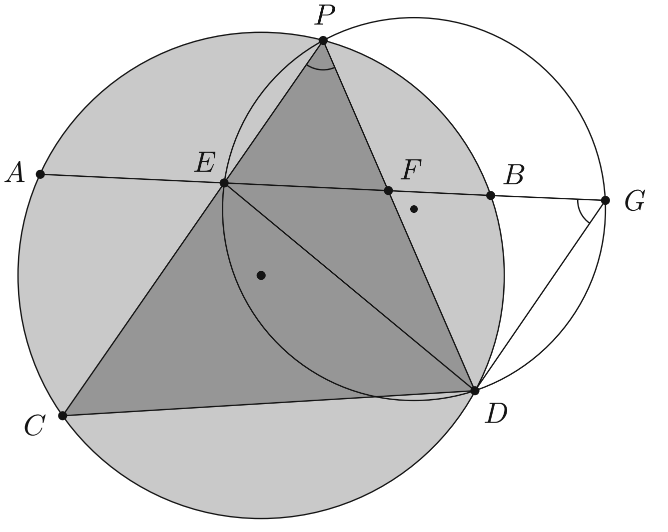
Thus, P and Q have the same power with respect to the circumcircle of triangle ABC, so $OP = OQ$ as desired. □

We end this section with a cute result due to Hiroshi Haruki (according to [18]).

Delta 1.8. (Haruki's Lemma) Given two non-intersecting chords AB and CD in a circle and a variable point P on the arc AB remote from points C and D, let E and F be the intersections of chords PC, AB, and of PD, AB, respectively. Prove that the value of

$$\frac{AE \cdot BF}{EF}$$

does not depend on the position of P.



Proof. The proof relies on the fact that the angle $\angle CPD$ is constant. We begin by constructing the circumcircle of triangle PED. Define point G to be the intersection of this circle with the line AB. Note that $\angle EGD = \angle EPD$ as they are subtended by the same chord ED of the circumcircle of triangle PED; these angles remain constant as P varies on the arc AB. Hence, for all positions of P, $\angle EGD$ remains fixed and, therefore, point G remains fixed on the line AB. It follows that BG is constant. On the other hand, by Power of Point, we have that $AF \cdot FB = PF \cdot FD$ and $EF \cdot FG = PF \cdot FD$. Hence,

$$(AE + EF) \cdot FB = EF \cdot (FB + BG),$$

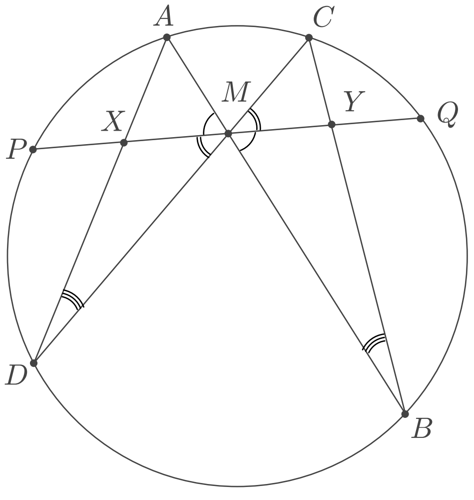
and $AE \cdot FB = EF \cdot BG$. Therefore, we conclude that

$$\frac{AE \cdot BF}{EF} = BG,$$

a constant. □

Haruki's Lemma can be used to give a very short proof of the so-called Butterfly Theorem, a very popular result in projective geometry.

Delta 1.9. (Butterfly Theorem). Let M be the midpoint of chord PQ of a given circle, through which two other chords AB and CD are drawn; AD cuts PQ at X and BC cuts PQ at Y. Then, M is also the midpoint of XY.



Proof. We think of A and C as being two positions of the variable point traversing the circle. Then, Haruki's lemma tells us that

$$\frac{XP \cdot MQ}{XM} = \frac{MP \cdot YQ}{YM},$$

which, because of $MP = MQ$, is simplified to

$$\frac{XP}{XM} = \frac{YQ}{YM}.$$

Adding 1 to both sides gives

$$\frac{XP + XM}{XM} = \frac{YQ + YM}{YM}.$$

Applying $MP = MQ$ again, we obtain the required $XM = YM$. This completes the proof. □

Assigned Problems

Epsilon 1.1. Let ABC be an acute triangle. Let the line through B perpendicular to AC meet the circle with diameter AC at points P and Q, and let the line through C perpendicular to AB meet the circle with diameter AB at points R and S. Prove that P, Q, R, S are concyclic.

Epsilon 1.2. Let ABC be an acute-angled triangle with circumcenter O and orthocenter H. Prove that

$$OH^2 = R^2(1 - 8\cos A \cos B \cos C).$$

Epsilon 1.3. Let ABC be a triangle and let D, E, F be the feet of the altitudes, with D on BC, E on CA, and F on AB. Let the parallel through D to EF meet AB at X and AC at Y. Let T be the intersection of EF with BC and let M be the midpoint of side BC. Prove that the points T, M, X, Y are concyclic.

Epsilon 1.4. (Kazakhstan MO 2008) Suppose that B_1 is the midpoint of the arc AC, containing B, of the circumcircle of triangle ABC, and let I_b be the B-excircle's center. Assume that the external angle bisector of $\angle ABC$ intersects AC at B_2. Prove that B_2I is perpendicular to B_1I_B, where I is the incenter of ABC.

Epsilon 1.5. (IMO 2000) Two circles Γ_1 and Γ_2 intersect at M and N. Let ℓ be the common tangent to Γ_1 and Γ_2 so that M is closer to ℓ than N is. Let ℓ touch Γ_1 at A and Γ_2 at B. Let the line through M parallel to ℓ meet the circle Γ_1 again at C and the circle Γ_2 again at D. Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

Epsilon 1.6. Let C be a point on a semicircle Γ of diameter AB and let D be the midpoint of the arc AC. Let E be the projection of D onto the line BC and F the intersection of the line AE with the semicircle. Prove that BF bisects the line segment DE.

Epsilon 1.7. Let A, B, C be three points on a circle Γ with $AB = BC$. Let the tangents at A and B meet at D. Let DC meet Γ again at E. Prove that the line AE bisects the segment BD.

Epsilon 1.8. (EGMO 2012) Let ABC be a triangle with circumcenter O. The points D, E, F lie in the interiors of the sides BC, CA, AB respectively, such that DE is perpendicular to CO and DF is perpendicular to BO. (By interior we mean, for example, that the point D lies on the line BC and D is between B and C on that line.) Let K be the circumcenter of triangle AFE. Prove that the lines DK and BC are perpendicular.

Epsilon 1.9. (IMO Shortlist 2013) Let ABC be a triangle with $\angle B > \angle C$. Let P and Q be two different points on line AC such that $\angle PBA = \angle QBA = \angle ACB$ and A is located between P and C. Suppose that there exists an interior point D of segment BQ for which $PD = PB$. Let the ray AD intersect the circumcircle of triangle ABC at $R \neq A$. Prove that $QB = QR$.