Preface

"Why wasn't this book available when my kids were those ages !!??"

Kathy Cordeiro, Frisco, TX

We welcome you to the world of mathematical problem solving! Once you enter it, the chances are you will love it!

Why did we write this book? This book is the result of our desire to create a set of introductory units for students showing interest in mathematical problem solving and competitions, as well as for their parents, teachers, and mentors. In doing this, we drew on our own experience working with young mathletes and on the collective wisdom of mathematics educators around the world. We aim to help parents and mentors challenge and teach their aspiring young math problem solvers.

Who is this book for? By their very nature, these units are not targeted at students of a particular age. Our experience shows that the topics contained in this book are best suited for advanced sixth-graders. Also, we know of many students who discovered competitive mathematics in later grades, and have benefited from the units in this book. Furthermore, the concepts and problems presented could be used as enrichment material by teachers in classrooms, parents teaching their kids at home, math team coaches, or in math clubs and circles.

What prior knowledge is needed? This work is an introductory book for mathematical problem solving and assumes very little prior knowledge. If a student shows interest in competitive math, then he or she most likely knows about integers, even and odd numbers, primes and composites, and solving simple equations. While not much is expected in terms of prior mathematical knowledge, a lot more is needed in order to follow the pace of the book. We expect the student to be highly motivated and to have support and guidance from an enthusiastic parent, teacher, or mentor. We would like to emphasize the importance of mentoring at every stage of mathematical education, especially at this early stage.

What does this book teach? This book will help you advance in several directions important in competitive mathematics: algebra, combinatorics, geometry, number theory, and games. You will learn a variety of problem solving strategies and will be challenged to explain your solutions, write proofs, and explore connections with other problems. From the spotlights you will learn about famous mathematicians and their discoveries.

In support of the learning process, each unit first discusses new concepts, illustrates them with examples, and then proceeds to exercises and problems.

Detailed solutions to all exercises and problems are provided in the second part of the book. In order to teach students a variety of problem solving skills and to instill the importance of multiple solutions to a problem, we give more than one method of approach to numerous problems. The solutions we feature provide good examples of reasoning and proof-writing. These are invaluable skills for anyone who wants to pursue a career in mathematics, computer science, engineering, or science.

How were the exercises and problems chosen? There are more than 330 fully solved exercises and problems in this book and numerous examples preceding them. They were drawn from a vast mathematical literature and were inspired by various competitions, problem books, and journals from around the world. They were carefully selected so that they promote ingenuity, creativity, an open mind, and desire to tackle interesting and meaningful questions. This book is unique because it is a collection of topics and problems used in high quality programs for young gifted children. The *Math Leads for Mathletes* is the first book series containing such diverse ideas, examples, and challenges at this level.

What comes after this book? This book will give you a big boost in your mathematical knowledge and problem solving skills, but there is much more to learn. For example, in one of the spotlight articles we talk about what Galois discovered about fifth-order equations but we do not really explain much about how Galois figured it out. To understand that and many other interesting results in mathematics, and to solve math problems beyond the AMC 8 competition, students will need to continue to learn more math and acquire more problem solving skills.

To support further interest from students, parents, and teachers, we continue to work on making more interesting mathematical topics and problems available – this book is the second in the *Math Leads for Mathletes* series. The following volumes will include more elaborate and complex concepts.

Acknowledgments: We would like to thank Richard Stong, an outstanding mathematical researcher, for his invaluable insights and feedback. Another big thank you goes to Alok Kumar, coach of the Indian IMO Team, for his detailed review. We would also like to thank Chris Jeuell, a math enthusiast and educator, for his detailed suggestions.

We are also grateful to our mathletes Adrian, Andreea, Milena, and Nikola for their support and inspiration. Last, but certainly not the least, we would like to thank our wives, Alina and Aleksandra, for their patience and support!

Titu Andreescu — Branislav Kisačanin

January, 2018 — Plano, TX 75025

Note on our ZIP code

We learned from one of our students that the ZIP code in which we live is a Fibonacci number. The US ZIP codes range from 00501 to 99950. If we allow leading zeros, there are 7 matches with Fibonacci numbers out of the possible 11 (F_{15} through F_{25}):

$F_{15} = 00610$ Anasco, Puerto Rico

$F_{16} = 00987$ Carolina, Puerto Rico

$F_{17} = 01597$ not a valid ZIP code

$F_{18} = 02584$ Nantucket, MA

$F_{19} = 04181$ not a valid ZIP code

$F_{20} = 06765$ not a valid ZIP code

$F_{21} = 10946$ Wallkill, NY

$F_{22} = 17711$ not a valid ZIP code

$F_{23} = 28657$ Newland, NC

$F_{24} = 46368$ Portage, IN

$F_{25} = 75025$ Plano, TX

Contents

Part 1

Concepts, Exercises, and Problems

The impossible ancient Greek problems. The ancient Greek mathematicians introduced a tradition of solving geometric construction problems using a very limited set of tools: an unmarked straight edge (unmarked ruler) and compasses (tool for drawing circles). Such constructions are called *Euclidean constructions* after Euclid, who lived around 300 BCE and wrote the most famous geometry textbook of all time, the *Elements*.

Ancient Greek mathematicians perfected the art of Euclidean constructions and were able to construct regular polygons with 3, 4, 5, 6, 8, and 10 sides, but were unable to find the construction procedure for the regular 7-sided polygon – the heptagon, nor for the regular 9-sided polygon – the enneagon (nonagon). Similarly, they were able to bisect (divide in two equal parts) arbitrary angles, but did not find a way to trisect them (divide them into three equal parts). Moreover, they were able to construct a square with twice the area of the given square and to construct the square with area equal to the area of an arbitrary polygon. However, they were not able to construct a cube with double the volume of the given cube or to construct a square with area equal to the area of a given circle.

Ancient mathematicians and many generations of later mathematicians kept going back to trying to solve these four problems using Euclidean tools. They remain unsolved even after more than two thousand years and now we know why – they are impossible to solve using Euclidean tools only. This realization was made only after algebra advanced enough through the works of Viete and Descartes in the 16th century and with later contributions by Gauss, Hermite, Lambert, and Lindemann in 18th and 19th centuries. Here is a simplified account of how algebra helps us with impossibility proofs in geometry:

- Duplication of a cube[1] is impossible because it involves the construction of $\sqrt[3]{2}$, which is a solution of a cubic equation, $x^3 = 2$. However, lines and circles can only solve first and second order equations, themselves being first and second order curves.
- Angle trisection[2] also turns out to be associated with a third order equation and therefore cannot be solved using lines and circles.
- Squaring a circle requires construction of the number π, which was shown to be a transcendental number – it is a solution of a transcendental (non-algebraic) equation, which turns out to be beyond what lines and circles can be used for.
- Construction of regular polygons was not progressing since ancient times until nineteen year old Gauss found a way to construct a regular heptadecagon, a 17-sided polygon. Later he proved that a regular polygon is constructible if and only if its number of sides can be written as a product of distinct Fermat primes[3] and a power of 2. Since 7 is a prime but not a Fermat prime, it cannot be constructed using Euclidean tools.

[1]Duplication of a cube is also known as the Delian problem, because in 430 BCE the Athenians asked the oracle at Delos (an island where Apollo and Artemis were born) how to stop the approaching plague. The answer was to double in size the cubical altar of Apollo.

[2]Archimedes looked beyond the limitations of Euclidean constructions: he found a way to trisect an arbitrary angle after allowing himself to use two marks on the ruler. We can call his solution the earliest out-of-the-(tool)box solution!

[3]Fermat primes are primes of the form $2^{2^n}+1$. Only five are known: 3, 5, 17, 257, 65537.

1.1 Counting I

In this unit we talk about the basic principles for counting various arrangements of objects. First, we show two very simple examples. The examples will be so simple, that the reader might wonder why we discuss them at all. We will show how these examples are generalized into counting principles that are used to solve a whole range of problems. We also define a factorial, a very useful shorthand notation for a product of integers between 1 and n.

Example. In one of my drawers I have three t-shirts: red, blue, and green, while in the other drawer I have four more: yellow, white, purple, and black. If I can choose a t-shirt from either drawer, how many choices do I have?

We probably don't need to explain much here, perhaps only emphasize that the two sets of t-shirts are disjoint with respect to colors, so the number of choices is $3 + 4 = 7$.

Example. How many different 2-letter words can be formed using a 6-letter alphabet $\{A, B, C, D, E, F\}$, if all letters in the word must be different? To further clarify, let us say that these words do not have to be actual words in any language, so for example, FB is acceptable as a word in this problem.

Let us solve this by building the words letter by letter. If our first letter is A, we get the following five words:

$$AB, AC, AD, AE, AF$$

Similarly, if our first letter is B, we get these five words:

$$BA, BC, BD, BE, BF$$

Continuing like this, we get five more words for each of the remaining cases for the first letter:

$$CA, CB, CD, CE, CF \qquad DA, DB, DC, DE, DF$$

$$EA, EB, EC, ED, EF \qquad FA, FB, FC, FD, FE$$

We see that for each of six choices for the first letter, we have five choices for the second letter, for a total of $6 \cdot 5 = 30$ possible words.

Addition principle. If set A has m elements, set B has n elements, and they have no elements in common, then their union has $m + n$ elements.

Multiplication principle. If set A has m elements, set B has n elements, then the number of different ordered pairs (a, b) such that $a \in A$ and $b \in B$ is equal to $m \cdot n$.

Note. In some of the following problems we will use a subtle extension of the multiplication principle:

Suppose we have a two-step process. At the first step we have m choices. At the second step, the list of choices we have may depend on what we chose at the first step, but regardless of that decision, the number of choices at the second step is n. Then the number of possibilities is mn.

Factorial. In combinatorics we often encounter a product of consecutive integers from 1 to n, so often, in fact, that we like to use the following shorthand notation for it:

$$n! = 1 \cdot 2 \cdot \ldots \cdot n$$

and even have a special name for such products, *factorial*. In particular, $n!$ is read "n factorial."

It is also useful to define the factorial of zero: $0! = 1$. Note that as n grows, the value of $n!$ grows very fast[4]. Here are the values up to $n = 10$.

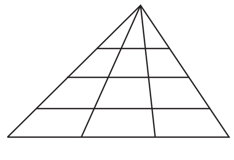
n	0	1	2	3	4	5	6	7	8	9	10
$n!$	1	1	2	6	24	120	720	5040	40320	362880	3628800

Example. A group of six students is about to form a mock government for their history class and they need to select the president, the vice president, the secretaries of agriculture, transportation, state, and defense. In how many ways can they assign these roles between them?

The role of the president can be taken by one of 6 people, the role of the vice president by one of remaining 5, etc. There are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6! = 720$ possible ways for them to assign the roles.

Problems

1. How many triangles are in the figure below?

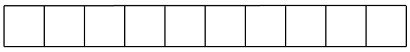


[4]This growth is so fast that most modern calculators cannot display the value of 70!. This growth is captured by an important formula, the famous Stirling approximation from 1730 (actually discovered by de Moivre):

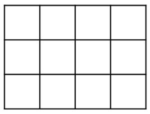
$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ (π and e are important mathematical constants, $\pi \approx 3.14$, and $e \approx 2.72$).

2. How many rectangles with horizontal and vertical sides are in the figures below?

   (a)

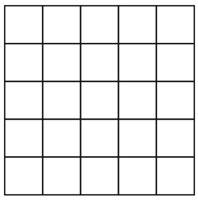
   

   (b)

3. (a) How many squares with horizontal and vertical sides are there in the figure below?

   (b) How many non-square rectangles with horizontal and vertical sides are there in the figure below?

4. How many diagonals does a regular octagon have?

5. How many three-digit numbers are there such that the middle digit is the average of the other two digits?

6. How many nine-digit numbers consisting of distinct digits 1, 2, …, 9 are there such that the sum of any two adjacent digits is odd?

7. A theater stage has eight lights and each can be separately turned on or off. In how many ways can we illuminate the stage?

8. Seven students run an 800 m race. In how many different ways can they end up in the final ranking? Ties are not allowed.

9. Let $n = \overline{abc}$ be a three-digit number.

   (a) How many numbers n are there with $a > b > c$?

   (b) How many numbers n are there with $a < b < c$?

10. My bank account comes with a six-digit PIN (personal identification number) as a password. How many different PINs can I form if I decide to use only two different digits and each has to appear at least once?

1.2 Pascal's Triangle and Binomial Coefficients

In this unit we will learn about Pascal's triangle and Newton's binomial theorem, and we will see how binomial coefficients connect them together and to combinatorics.

- Binomial coefficients, i.e., the numbers of the form

  $$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!}$$

  which solve problems such as "how many k-element subsets are there in an n-element set?" or "how many different committees of k people can be formed from a group of n people?"

- Pascal's triangle

  $$\begin{array}{ccccccccccccc} &&&&&&1&&&&&& \\ &&&&&1&&1&&&&& \\ &&&&1&&2&&1&&&& \\ &&&1&&3&&3&&1&&& \\ &&1&&4&&6&&4&&1&& \\ &1&&5&&10&&10&&5&&1& \\ 1&&6&&15&&20&&15&&6&&1 \\ &&&&&&\dots&&&&&& \end{array}$$

  a table of numbers formed row-by-row as follows: each row begins and ends with a 1, while each of the other entries is a sum of the two entries just above it. For example, $15 = 5 + 10$.

- Newton's binomial theorem

  $$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{n}a^0 b^n$$

  which generalizes the following identities:

  $$\begin{aligned} (a+b)^0 &= 1 \\ (a+b)^1 &= a+b \\ (a+b)^2 &= a^2+2ab+b^2 \\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\ (a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\ (a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 \\ &\dots \end{aligned}$$

Problems

1. If the number of k-element subsets of an n-element set is denoted by $\binom{n}{k}$ (this is read as "n choose k"), prove that

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

2. Calculate the values of the following binomial coefficients: $\binom{5}{0}$, $\binom{5}{1}$, $\binom{5}{2}$, $\binom{5}{3}$, $\binom{5}{4}$, $\binom{5}{5}$.

3. Prove that Pascal's triangle consists of binomial coefficients.

4. Prove that Pascal's triangle is symmetric with respect to a vertical line through its top vertex, i.e., $\binom{n}{n-k} = \binom{n}{k}$.

5. Prove the following identities:

$$\binom{n}{0} = \binom{n}{n} = 1 \quad \text{and} \quad \binom{n}{1} = \binom{n}{n-1} = n.$$

6. Observe the correspondence between the coefficients (numbers) in the identities for $(a+b)^0$, $(a+b)^1$, $(a+b)^2$, $(a+b)^3$, $(a+b)^4$, $(a+b)^5$ and the entries in Pascal's triangle. Try to guess the coefficients for the expansion of $(a+b)^6$ by looking at Pascal's triangle. Finally, do the actual expansion of $(a+b)^6$ to check if the hint we got from Pascal's triangle was correct.

7. Prove Newton's binomial theorem:

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n}a^0 b^n.$$

8. How many different subsets does the set $\{a, b, c, d, e, f\}$ have?

9. Use Newton's binomial formula

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n}a^0 b^n$$

to prove the following two identities:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n,$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n} = 0.$$

10. Prove that p divides $\binom{p}{k}$ for $1 \le k < p$ and p prime.

1.3 Probability I

In this unit we will learn how to compute probabilities of various events. The following definition, even though not mathematically rigorous, is very intuitive and will work well for us as we learn the basics of probability. After all, it had been used by mathematicians like Pascal, Fermat, the Bernoulli brothers, and others for centuries. Eventually, we will want to learn how to properly define probability, the way Kolmogorov did, but let us not rush with that.

Definition. If we perform an experiment with n equally likely outcomes, and a desired event E happens in m out of those n outcomes, the probability of that event, $P(E)$, is given by

$$P(E) = \frac{m}{n}.$$

Example. If our experiment involves rolling a fair, standard six-sided die, there are $n = 6$ possible outcomes, all equally likely to happen. If the desired event is "the cast number is divisible by 3," we see that it happens in $m = 2$ out of those $n = 6$ outcomes (when the die yields 3 or 6), so we can calculate the probability of that event, $P(E) = \frac{m}{n} = \frac{2}{6} = \frac{1}{3}$.

The probability of any event is a number between 0 and 1 (inclusive). If an event is *impossible*, its probability is 0. If an event is *certain*, its probability is 1. In an experiment, some event will either occur or not. An event happening and not happening are two *complementary* events. If the event is denoted by A, the complementary event "not A" is denoted by $\bar{A}$ and we can write

$$P(A) + P(\bar{A}) = 1.$$

This fact comes in handy in many problems.

Problems

1. We are rolling a fair, standard six-sided die. What is the probability of rolling (a) a 3? (b) a prime number? (c) an even number?

2. We draw a card from a standard deck of 52 cards. What is the probability of drawing (a) a queen? (b) a black card? (c) an ace or a two?

3. A kitty named Meow jumps on a computer keyboard that only has 26 keys corresponding to the letters of the English alphabet. Meow's paws land on four different keys. What is the probability that the keys Meow jumped on can spell its name?

4. A pair of 8-sided dice have sides numbered 1 through 8. Each side has the same probability of landing face up. What is the probability that the product of the two numbers on the sides that land face up exceeds 36?

5. From a regular hexagon, three vertices are selected at random. What is the probability that these three vertices form an isosceles triangle?

6. If we roll two fair, standard six-sided dice, what is the probability that the sum of the resulting numbers is greater than 10?

7. There are equal numbers of pennies, nickels, dimes, and quarters in a bag. Four coins are pulled out, one at a time, and each coin is replaced before the next is drawn. What is the probability that the total value of the four coins is less than 20 cents?

8. There are 23 students in my classroom and a total of 1101 students at my school.

   (a) What is the probability that two or more people in the classroom have the same birthday?

   (b) What is the probability that at least four students in this school have the same birthday?

   To simplify things, disregard the existence of leap years and therefore the possibility of someone having February 29 as a birthday.

9. Three fair, standard six-sided dice are rolled. What is the probability that the sum of the numbers shown is at least 6?

10. Four fair, standard six-sided dice are rolled. What is the probability that the product of the numbers shown is even?

1.4 Mathematical Induction

Mathematical induction is a very important proof technique that allows us to solve a wide range of problems. A good way to visualize how mathematical induction works is to compare it to a domino effect, in which we arrange dominos in a row and knock down the first. As the first domino falls, it knocks down the second, which in turn knocks down the third, and so on until the end of the row.

How do we ensure that the domino effect works? The domino effect will work if we can ensure the following two things:

(a) we can knock down the first domino;

(b) for any $n \geq 1$ the following is true: if the n-th domino falls, then the next domino will also fall.

Let us now talk about how mathematical induction works. If we want to prove that some formula, property, or statement is true for any integer $n \geq n_0$, we proceed as follows:

Step 1. (Base case) We prove that it is true for $n = n_0$;

Step 2. (Inductive hypothesis) We assume it is true for some $n = k \geq n_0$;

Step 3. (Inductive step) We try to prove it for $n = k + 1$ by relying on the assumption from Step 2.

Let us illustrate this technique with two examples:

Example. Use mathematical induction to prove that for any $n \geq 1$ the sum of the first n positive integers,

$$S(n) = 1 + 2 + 3 + \ldots + n,$$

can be written as

$$S(n) = \frac{n(n+1)}{2}.$$

We proceed step by step:

Step 1. For $n = 1$ the formula is true because

$$1 = \frac{1 \cdot 2}{2}.$$

Step 2. Let us assume that the formula is true for $n = k$:

$$S(k) = 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}.$$

Step 3. Can we prove the analogous formula for the next n? Let us see what happens for $n = k + 1$:

$$\begin{aligned} S(n) &= S(k+1) = 1 + 2 + 3 + \ldots + k + k + 1 \\ &= (1 + 2 + 3 + \ldots + k) + k + 1 = \frac{k(k+1)}{2} + k + 1. \end{aligned}$$

In the last step we used our assumption from Step 2. Finally, for $n = k + 1$,

$$\begin{aligned} S(n) &= S(k+1) = \frac{k(k+1)}{2} + k + 1 \\ &= \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2} = \frac{n(n+1)}{2}. \end{aligned}$$

In the last step we substituted n instead of $k+1$ and got the same formula that we wanted to prove. This concludes our proof by mathematical induction.

Example. Use mathematical induction[5] to prove that $n^3 + 2n$ is divisible by 3 for any integer n.

Let us first prove this for $n \geq 0$:
Step 1. For $n = 0$ the statement is true because

$$0^3 + 2 \cdot 0 = 0 \quad \text{is divisible by 3.}$$

Step 2. Let us assume that the statement is true for $n = k$, i.e, $k^3 + 2k$ is a multiple of 3:

$$k^3 + 2k = 3m.$$

Step 3. Let us see what happens for $n = k + 1$:

$$n^3 + 2n = (k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2 = 3m + 3k^2 + 3k + 3.$$

In the last step we used the assumption from Step 2. Finally, for $n = k + 1$

$$n^3 + 2n = 3(m + k^2 + k + 1) = 3t,$$

where t is some integer, therefore $n^3 + 2n$ is a multiple of 3 for $n = k + 1$.

This completes our proof of the statement for $n \geq 0$. If we want to use induction for $n \leq 0$ as well, we can slightly modify the induction procedure and see if the dominos will fall in the other direction as well: show the statement is true for $n = 0$ (we already discussed that), and then assuming it is true for $n = k$ see what happens for $n = k - 1$.

In this way we proved that the domino labeled with $n = 0$ can fall and that it can knock down all dominos labeled with positive integers as well as all dominos labeled with negative integers.

[5]Note that induction is not the easiest way to prove this (can you find a better way?), but we do it here to illustrate how induction works.

Problems

1. Use mathematical induction to prove the following identities:

   (a) $1 \cdot 2 + 2 \cdot 3 + \ldots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

   (b) $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + n(n+1)(n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$

2. Use mathematical induction to prove the following identities:

   (a) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

   (b) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$

3. Use mathematical induction to prove that $n^7 - n$ is divisible by 7 for any integer n.

4. **Fermat's Little Theorem.** Prove that if p is a prime then $n^p - n$ is divisible by p for any integer n.

5. There are n lines in a plane; no two of them are parallel and no three of them intersect at the same point. They divide the plane into L_n areas. Determine L_n.

6. Try to discover the formula for the sum of the first n positive odd integers and then prove the formula by mathematical induction.

7. Prove that in base-9, a number written only with digit 1 is a triangular number, $T_m = \frac{m(m+1)}{2}$ for some m.

8. Use mathematical induction to prove Newton's binomial formula

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n} a^0 b^n.$$

9. Use mathematical induction to prove that any integer $n > 1$ can be written as a product of the number 1 and one or more prime numbers.

10. Sometimes the statement we are trying to prove, even though correct, when used as the inductive hypothesis, does not give enough support for the jump we need to make in the inductive step. In such cases we cannot use induction for the proof of the statement, but may be able to use induction to prove a more strict, stronger, and apparently harder statement. Here are two such problems. Prove:

    (a) $\dfrac{1}{2} \cdot \dfrac{3}{4} \cdot \dfrac{5}{6} \cdot \ldots \cdot \dfrac{2n-1}{2n} \leq \dfrac{1}{\sqrt{3n}}$

    (b) $\dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \ldots + \dfrac{1}{n^2} \leq 1$