Preface

"Why wasn't this book available when my kids were those ages !!??"

Kathy Cordeiro, Frisco, TX

We welcome you to the world of mathematical problem solving! Once you enter it, the chances are you will love it!

Some math problems, such as the ancient Greek problem of constructing a regular heptagon, are so hard that it took more than 2000 years to fully understand them. Whether you find yourself inspired or discouraged by this fact, this book is for you! Why? Because throughout the book we teach an important problem solving strategy: If you cannot solve a problem, you can break it up into smaller problems that you *can* solve. As George Polya said, "If there is a problem you cannot solve, then there is an easier problem you *can* solve: find it."

Why did we write this book? This book is the result of our desire to create a set of introductory units for students showing interest in mathematical problem solving and competitions, as well as for their parents, teachers, and mentors. In doing this, we drew on our own experience working with young mathletes and on the collective wisdom of mathematics educators around the world. We aim to help parents and mentors challenge and teach their aspiring young math problem solvers.

Who is this book for? By their very nature, these units are not targeted at students of a particular age. Our experience shows that the topics contained in this book are best suited for advanced fourth- and fifth-graders, as well as for extremely talented third-graders. Also, we know of many students who discovered competitive mathematics later (in sixth grade or subsequently), and have benefited from the units in this book. Furthermore, the concepts and problems presented could be used as enrichment material by teachers in classrooms, parents teaching their kids at home, math team coaches, or in math clubs and circles.

What prior knowledge is needed? This work is an introductory book for mathematical problem solving and assumes very little prior knowledge. If a student shows interest in competitive math, then he or she most likely knows about integers, even and odd numbers, primes and composites, and solving simple equations. While not much is expected in terms of prior mathematical knowledge, a lot more is needed in order to follow the pace of the book. We expect the student to be highly motivated and to have support and guidance from an enthusiastic parent, teacher, or mentor. We would like to emphasize the importance of mentoring at every stage of mathematical education, especially at this early stage.

What does this book teach? This book will help you advance in several directions:

1. You will learn a lot more math: properties of integers and arithmetic operations (divisibility, primes, prime factorization), simple algebraic manipulations, methods to solve equations and systems of equations, numerical reasoning (fractions, percentages, proportions, averages, cryptarithmetic, magic squares), basic combinatorial reasoning (sequences, enumeration, pigeonhole principle, invariants), and puzzle-like math (tangrams, math and chess, toothpick math).
2. You will learn a variety of problem solving strategies and will be challenged to explain your solutions, write proofs, and explore connections with other problems.
3. You will learn about famous mathematicians, their discoveries, and about important mathematical constants.

In support of the learning process, each unit first discusses new concepts, illustrates them with examples, and then proceeds to exercises and problems. Detailed solutions to all exercises and problems are provided in the second part of the book. In order to teach students a variety of problem solving skills and to instill the importance of multiple solutions to a problem, we give more than one method of approach to numerous problems. The solutions we feature provide good examples of reasoning and proof-writing. These are invaluable skills for anyone who wants to pursue a career in mathematics, computer science, engineering, or science.

How were the exercises and problems chosen? There are more than 350 fully solved exercises and problems in this book and numerous examples preceding them. They were drawn from a vast mathematical literature and were inspired by various competitions, problem books, and journals from around the world. They were carefully selected so that they promote ingenuity, creativity, an open mind, and desire to tackle interesting and meaningful questions. This book is unique because it is a collection of topics and problems used in high quality programs for young gifted children. This is the first book containing such diverse ideas, examples, and challenges at this level.

How far will this book take me? We think that a good way to answer this question is to show a competition problem whose solution will be fully understandable to students who complete much of this book:

Find all positive integers x and y such that

$$1! + 2! + 3! + \ldots + x! = y^2$$

Take a minute to think about this problem, then continue reading. Spoiler alert: The full solution is given in the footnote on the next page!

If you are familiar with the concepts covered in this book then you will most likely find that $(1,1)$ and $(3,3)$ are solutions, and you will be able to understand why these are the only solutions[1]. Of course, understanding the solution is one thing, finding it yourself is another. This book offers a critical mass in terms of both the number and variety of related problems that will allow many readers to solve problems by themselves, not just to understand someone else's solution.

What comes after this book? This book will give you a big boost in your mathematical knowledge and problem solving skills, but there is much more to learn. For example, we talk about what Gauss discovered about the ancient Greek problem of constructing a regular heptagon and give his general rule to determine which regular polygons are constructible and which are not, but we do not really explain how Gauss figured it out. To understand that and many other interesting results in mathematics, and to solve math problems beyond the AMC 8 competition, students will need to continue to learn more math and acquire more problem solving skills.

To support further interest from students, parents, and teachers, we continue to work on making more interesting mathematical topics and problems available – this book is the first in a series. The following volumes will include more elaborate and complex algebraic and geometric concepts.

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Titu Andreescu Branislav Kisačanin

[1]The proof is as follows: For $x \geq 5$ the last digit of the expression on the left-hand side is 3. Since squares of integers can end only in 0, 1, 4, 5, 6, or 9, and never in 3, there are no solutions for $x \geq 5$. Therefore, solutions are possible only when $x < 5$ and they are $(1,1)$ and $(3,3)$.

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Part 1

Concepts, Exercises, and Problems

Let's Get Started !

What is the greatest number? In fact, there is no such thing as the greatest number. Why? If someone tells us that the greatest number is, for example, a quintillion, we can easily construct a slightly bigger number by simply adding 1 to our friend's number.

An even bigger number is a quintillion multiplied by itself, in other words, quintillion squared. Here is another way, which yields a much, much bigger number than what our friend came up with: a quintillion is a number that we write as 1 followed by 18 zeros. Now, try to imagine a number written as 1 followed by a quintillion zeros!

Googolplex – one of the greatest numbers with a name. Let us first try to imagine a number written as 1 followed by 100 zeros. That number is called *googol.* Its name probably sounds familiar, and that is because the Internet search website Google was named after this number. So, googol is

$$\text{googol} = 1\underbrace{0000\ldots0000}_{100 \text{ zeros}} = \underbrace{10\cdot10\cdot10\cdot\ldots\cdot10\cdot10\cdot10}_{100 \text{ tens}} = 10^{100}.$$

Then, make a big leap and try to imagine a number that has googol zeros! This incredibly large number is called *googolplex.*

$$\text{googolplex} = 1\underbrace{0000\ldots0000}_{\text{googol zeros}} = 10^{\text{googol}} = 10^{10^{100}}.$$

According to the famous astronomer and writer Carl Sagan, googol is greater than the number of all elementary particles (electrons, protons, neutrons) in our Universe. Furthermore, even if we had enough paper and ink to write googolplex on, with all of its googol zeros, the paper needed to do this would not fit into the known Universe! Yet, googolplex is not the greatest named number, because mathematicians name ever greater numbers when they need them in their work: Skewes', Moser's, and Graham's numbers are much, much greater then googolplex.

Numbers with names. The following table lists several large numbers with special names. In different languages these numbers have similar names, with some confusing twists. The scientific notation, like 10^3 for one thousand or 10^6 for one million, is universal.

decimal	scientific	SI prefix	English	French	German
1,000	10^3	kilo (k)	thousand	mille	Tausend
1,000,000	10^6	mega (M)	million	million	Million
1,000,000,000	10^9	giga (G)	billion	milliard	Milliarde
1,000,000,000,000	10^{12}	tera (T)	trillion	billion	Billion
1,000,000,000,000,000	10^{15}	peta (P)	quadrillion	billiard	Billiarde
1,000,000,000,000,000,000	10^{18}	exa (E)	quintillion	trillion	Trillion
$1\underbrace{0\ldots0}_{100}$	10^{100}		googol	gogol	Googol
$1\underbrace{0\ldots0}_{googol}$	$10^{10^{100}}$		googolplex	gogolplex	Googolplex

1.1 Integers and Divisibility

Integers (also called whole numbers) are the numbers $\ldots, -2, -1, 0, 1, 2, \ldots$ For the set of integers Z we write $Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

The set of natural numbers N is a subset of the set of integers. We write this as $N \subset Z$ (read: set N is a subset of set Z). Some mathematicians consider natural numbers to be positive integers, that is, $N = \{1, 2, 3, \ldots\}$, while others consider zero to be among natural numbers. In this book we will be specific about this and say, for example, "find positive integers such that…" in order to avoid any confusion. Either way, we will explore many interesting problems and properties related to integers.

Divisibility of integers

You certainly know that some integers are divisible by 3, while others are not. For example, among numbers $1, 2, 3, \ldots, 12$, only $3, 6, 9, 12$ are divisible by 3.

If a and n stand for any integers, we say that n is divisible by a if n is a product of a and another integer. In that case, we also say that n is a multiple of a and that a divides n. In mathematical notation this is written as $a|n$ (read: a divides n).

In this book we use $\cdot$ to denote multiplication, so, for example, $15 = 3 \cdot 5$. We say that 15 is divisible by 3. We also say that 15 is a multiple of 3 and that 3 divides 15. This is written $3|15$.

If n is not a multiple of a, we say that n is not divisible by a. Another way to say this is that a does not divide n.

If n is a two-digit number, we can write it as $n = \overline{xy}$. This tells us that its first digit is x and second digit is y. We can also write $n = 10x + y$. More generally, if n has digits $d_k, \ldots, d_1, d_0$, we write

$$n = \overline{d_k \ldots d_1 d_0} = 10^k d_k + \ldots + 10 d_1 + d_0.$$

Rules for divisibility

Many interesting math problems can be solved using the following simple rules. When we state these rules, it is with *if and only if* to indicate that these rules have no exceptions (the *if* part) and that they cover all possible cases (the *only if* part).

- **Divisibility by 2:** An integer is divisible by 2 if and only if its last digit is even.

  Example: The numbers 100, 102, 104, 106, and 108 are divisible by 2, while numbers 101, 103, 105, 107, and 109 are not.

- **Divisibility by 3:** An integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

  Example: The number 77777 is not divisible by 3 because $7+7+7+7+7=35$ and 35 is not divisible by 3.

- **Divisibility by 4:** An integer is divisible by 4 if and only if the number formed by its last two digits is divisible by 4.

  Example: The number 10032 is divisible by 4 because 32 is divisible by 4.

- **Divisibility by 5:** An integer is divisible by 5 if and only if its last digit is 0 or 5.

  Example: The number 50505051 is not divisible by 5 because its last digit is neither 0 nor 5.

- **Divisibility by 6:** An integer is divisible by 6 if and only if it is divisible by both 2 and 3.

  Example: The number 123458 is divisible by 2 (it ends with an even digit) but not by 3 (the sum of its digits is not divisible by 3), so it is not divisible by 6.

- **Divisibility by 7:** The number with digits $a_k, a_{k-1}, \ldots, a_1, a_0$ is divisible by 7 if and only if

  $$7 \mid (\overline{a_2a_1a_0} - \overline{a_5a_4a_3} + \overline{a_8a_7a_6} - \ldots).$$

  Example: The number 1112444333555444666 is divisible by 7 because $666-444+555-333+444-112+1=777$, which is divisible by 7.

- **Divisibility by 8:** An integer is divisible by 8 if and only if the number formed by its last three digits is divisible by 8.

  Example: The number 777888 is divisible by 8 because 888 is divisible by 8.

- **Divisibility by 9:** A number is divisible by 9 if and only if the sum of its digits is divisible by 9.

  Example: The number 12345678 is divisible by 9, because $1+2+3+4+5+6+7+8=(1+8)+(2+7)+(3+6)+(4+5)=4\cdot 9=36$ is divisible by 9.

- **Divisibility by 10:** An integer is divisible by 10 if and only if it is divisible by both 2 and 5, that is, if and only if its last digit is 0.

  Example: The number $1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot 63 \cdot 64$ is divisible by 10 because 2 and 5 are among its factors. (In fact, it ends in 14 zeros, because there are 63 twos and 14 fives among its prime factors.)

- **Divisibility by 11:** We can use any of the following criteria for 11: Integer a with digits $a_k, a_{k-1}, \ldots, a_1, a_0$ is divisible by 11 if and only if

$$
\begin{array}{lll}
(1) & 11 \mid (a_0 - a_1 + a_2 - a_3 + a_4 - \ldots) & \text{or} \\
(2) & 11 \mid (\overline{a_1 a_0} + \overline{a_3 a_2} + \overline{a_5 a_4} + \overline{a_7 a_6} + \ldots) & \text{or} \\
(3) & 11 \mid (\overline{a_2 a_1 a_0} - \overline{a_5 a_4 a_3} + \overline{a_8 a_7 a_6} - \ldots). &
\end{array}
$$

  Example: We can use the first criterion to show that 132121 is divisible by 11 because $1 - 2 + 1 - 2 + 3 - 1 = 0$ is divisible by 11. We can also use the second criterion, which gives us $21 + 21 + 13 = 55$, which is divisible by 11. The third criterion also shows this number is divisible by 11 because $121 - 132 = -11$ is divisible by 11.

- **Divisibility by 12:** An integer is divisible by 12 if and only if it is divisible by 4 and 3.

  Example: The number 23456565432 is divisible by 4 and by 3, therefore it is divisible by 12.

- **Divisibility by 13:** An integer a with digits $a_k, a_{k-1}, \ldots, a_1, a_0$ is divisible by 13 if and only if

$$13 \mid (\overline{a_2 a_1 a_0} - \overline{a_5 a_4 a_3} + \overline{a_8 a_7 a_6} - \ldots).$$

  Example: The number 1112444333555444666 is not divisible by 13 because $666 - 444 + 555 - 333 + 444 - 112 + 1 = 777$ and $777 = 3 \cdot 7 \cdot 37$ is not divisible by 13.

Note the similarity in the criteria for 2, 4, and 8, the criteria for 3 and 9, and the criteria for 7, 11, and 13. While the numbers in the first two groups are related as powers of 2 and 3, respectively, the connection between the numbers in the third group may not be as evident. What connects them is the fact that $7 \cdot 11 \cdot 13 = 1001$.

For 11, we can also use the fact that it has remainder 1 when divided by 10 (that is the basis for the first divisibility criterion for 11), and also the fact $99 = 9 \cdot 11$ has remainder -1 when divided by 100 (that is the basis for the second divisibility criterion for 11). This gives us a hint that there is a criterion for divisibility by 9 similar to the second criterion for 11.

Note also that the criterion for divisibility by $3^3 = 27$ is not analogous to the criteria for 3 and 9, with the first counterexample being the number 27 itself: it is certainly divisible by 27, but its digits do not sum up to a number divisible by 27. We can create a criterion for divisibility by 27 (and 37) based on the fact that $27 \cdot 37 = 999$.

All these criteria are consequences of a general criterion due to Pascal:

Pascal's Criterion. If

$$a \quad = \quad 10^k a_k + 10^{k-1} a_{k-1} + \ldots + 10a_1 + a_0 \quad = \quad \overline{a_k a_{k-1} \ldots a_1 a_0},$$

then $m \mid a$ if and only if $m \mid (r_k a_k + r_{k-1} a_{k-1} + \ldots + r_1 a_1 + r_0 a_0)$, where r_i $(i = 0, 1, 2, \ldots, k)$ are the remainders of 10^i when divided by m.

Exercises and Problems

1. Is googolplex divisible by 3?

2. Which of the following numbers are not divisible by 4: 99998, 100000, 100002, 100004?

3. Determine the 2013th digit after the decimal point in the decimal representation of $\frac{7}{27}$. Hint: It is useful to know that $999 = 27 \cdot 37$ and that $\frac{1}{999} = 0.001001001001\ldots$

4. It is useful to know how to determine a prime factorization of a number. For example, $120 = 2^3 \cdot 3 \cdot 5$. Also, we talk about the product of digits of a number, for example, the product of digits of 172 is $1 \cdot 7 \cdot 2 = 14$.

   (a) Give examples of a few numbers whose product of digits is 216. Determine the smallest such number.

   (b) Is there any number whose product of digits is 140? If there is, provide at least one example; if not, explain why not.

   (c) Is there any number whose product of digits is 220? If there is, provide at least one example; if not, explain why not.

5. A number has as digits 2000 ones, 2000 twos, and all other digits are zeros. Can that number be a perfect square (that is, a square of another integer)? Hint: Use criteria for divisibility by 3 and by 9.

6. Find all 3-digit numbers $\overline{abc}$ such that $\overline{ab} + \overline{bc} + \overline{ca} = \overline{abc}$. *Note:* In problems like this, a, b, c are digits. Leading 0 is not allowed, so $1 \leq a, b, c \leq 9$.

7. What is the remainder when $20132013\ldots2013$ (consisting of the number 2013 repeated 2013 times) is divided by $333,333$?