Foreword

Mathematical induction is a method of proof that has been known to mathematicians for hundreds of years, with one of the earliest proofs using Induction dating back to Francesco Maurolico (1575), when he showed that the sum of first n odd positive integers is n^2. Nevertheless, it was only formalized as an axiom of the nonnegative integers in the 19-th century due to the work of Peano and other mathematicians on Set Theory. Nowadays, it has become one of the basic tools that any mathematician is familiar with. Induction is an important technique used in competitions and its applications permeate almost every area of mathematics. Due to its elegance and popularity, we have decided to write this book whose main goal is to offer a detailed exposition of the method, its subtleties and beautiful applications. The book is designed as follows:

First chapter, entitled *A brief overview of Induction* offers an introduction to the subject. We begin by describing Induction in the context of Set Theory, as one of Peano's axioms and present a few examples of how different properties of the nonnegative integers can be derived from it. We then look at some classical examples which are solved using Induction and illustrate in detail the approach one should follow when writing their proof. After discussing some variants of Induction, we move on to presenting one of the most intriguing aspects of the method, called the *Paradox of Induction*. There we look at various examples, fully motivating all the steps that lie behind their elegant proofs, where we strengthen the original statement to make the solution easier. The chapter ends with a section dedicated to Transfinite Induction.

Second chapter, *Sums, products, and identities* is mainly aimed at those who want to familiarize themselves with the basics of applying Induction. The nature of the questions presented is similar to the ones that originally motivated the use of Induction as an algebraic tool. The chapter also illustrates the power of the method, showing how easy it is to use Induction to prove identities, rather than using other techniques.

From the third chapter onwards, we follow a problem solving based approach by discussing Induction in various areas of mathematics. Each chapter is divided into two sections: Theory with examples and Proposed problems. The book is designed so that it is as self-contained as possible. Therefore, each chapter starts by introducing the reader to all the notions that are required for understanding the examples and tackling the proposed problems. Various beautiful examples are then discussed in full detail, explaining the main themes that occur in each specific field (see for example Chapters 6 and 7 on Number theory and Combinatorics). With the aim of being as comprehensive as possible, we explore a total of 10 different areas of mathematics, including topics that are not usually discussed in an Olympiad-oriented book on Induction (such as Chapter 3 on Functions and functional equations). The second part of each chapter consists of a carefully chosen list of proposed problems. These add up to more than 200 elegant questions from over 20 worldwide renowned Olympiads and mathematical magazines, as well as original problems designed by the authors of this book and their collaborators. Fully detailed solutions are provided for each problem at the end of the book.

We truly believe that the book can serve as a very good resource and teaching material for anyone who wants to explore the beauty of Induction and its applications, from novice mathematicians to Olympiad-driven students and professors teaching undergraduate courses. We hope that at the end of the journey, every reader will agree with our belief formulated in the title of the book, and find that Induction is indeed a powerful and elegant method of proof.

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Chapter 1

A Brief Overview of Induction

1.1 Theory and Examples

1.1.1 Introductory Notions

What is induction? Is it an axiom or a theorem? What kind of questions does it address? What is the largest context where it holds and can be applied? These are some of the questions that we will address throughout this chapter and in general, throughout this book.

Let us begin with the following informal discussion. Assume that a certain country decides at some point to become a kingdom and they choose their first royal family, call it generation 0. For each generation, the first offspring of the current royal family will be the successor to the throne. Unfortunately, all members of generation 0 suffer from a degenerative disease which is certain to be passed to the next generation. How can we prove that all the royal families that will rule that country will suffer from the same degenerative disease?

This seems like an intuitively clear result. Let us denote by $P(n)$ the proposition that the n-th generation suffers from the degenerative disease. We know that $P(0)$ is true and also that if $P(n)$ is true, then so is $P(n+1)$. We want to prove that $P(n)$ holds for any non-negative integer n. In fact, this is what the Principle of Induction says in general:

Principle of Induction. If $P(n)$ is a proposition that depends on a non-negative integer n such that $P(0)$ is true and $P(n)$ holds implies that $P(n+1)$ holds, then $P(n)$ is true for all non-negative integers n.

From now on, $\mathbb{N}$ denotes the set of natural numbers, i.e. non-negative integers. Here is one way we could try to prove the Principle of Induction:

Assume by contradiction that there is a non-negative integer for which the proposition P is false. Let us denote by A the non-empty set of all non-negative integers for which the proposition does not hold and let a be the smallest element of A, i.e. $P(a)$ is false and $P(0)$, $P(1)$, $\ldots$, $P(a-1)$ are all true. This implies in particular that $a > 0$ and $P(a-1)$ is true; but then $P(a)$ is true, which gives the desired contradiction. The conclusion follows.

However, the above argument has a gap. The reason is the following: in the proof we constructed a non-empty subset A of $\mathbb{N}$. Then we picked a to be the smallest element of A. But how do we know that such an a exists? Given that there may be very complicated and hard-to-understand subsets of $\mathbb{N}$, how do we know that **any** non-empty subset of $\mathbb{N}$ has a least element? This might seem like a silly question given the facts that we know about the natural numbers. But the convention in mathematics is that we should take only the simplest of these "facts"as our axioms. We should use this small set of axioms for constructing $\mathbb{N}$. Anything more complicated should be theorems, i.e. facts that can be deduced from our axioms.

The Principle of Induction is usually regarded as the simplest way to say something about all natural numbers. Thus it has been chosen as an axiom itself and it cannot be proved from other axioms. The fact that any non-empty subset of the natural numbers has a least element is seen as a little more complicated. We will prove that it follows from the Principle of Induction and we will think about the implication in this order and not the other way round. The Italian mathematician Giuseppe Peano was the first one to postulate rigorously a set of axioms on the natural numbers $\mathbb{N}$. We present them at the end of this section as an appendix, together with the proof that the Principle of Induction implies that every non-empty subset of the natural numbers has a least element.

Let us now look at a typical example which uses the Principle of Induction:

Show that for all positive integers n, the following identity holds:

$$1+2+\ldots+n=\frac{n(n+1)}{2}.$$

Proof. Let $P(n)$ be the statement

$$P(n): 1+2+\ldots+n=\frac{n(n+1)}{2}.$$

We are required to prove that $P(n)$ is true for all positive integers n. We start by proving that $P(1)$ holds. This is the *base case*, i.e. the smallest value for which we have to show that our property holds.

$P(1)$ simply says that $1=\frac{1\cdot 2}{2}$, which is clear.

We now prove that assuming $P(n)$ is true for some $n \geq 1$, we obtain that $P(n+1)$ is true. This is called the *induction step* and the assumption that $P(n)$ is true is called the *induction hypothesis*.

From our induction hypothesis we know that

$$1+2+\ldots+n=\frac{n(n+1)}{2}.$$

So to prove $1+2+\ldots+n+n+1=\frac{(n+1)(n+2)}{2}$, it suffices to show that

$$\frac{n(n+1)}{2}+(n+1)=\frac{(n+1)(n+2)}{2},$$

which follows easily after bringing the left hand side to a common denominator. This shows that $P(n) \Rightarrow P(n+1)$, completing our proof.

Remark. Notice that the Principle of Induction as originally stated above uses the base case $n=0$, while we started from $n=1$ in the above proof. Nevertheless, it follows from the Principle of Induction that if $P(n_0)$ holds for some $n_0 \in \mathbb{N}$ and $P(n) \Rightarrow P(n+1)$, then $P(n)$ holds for any $n \geq n_0$ (hint: let $Q(n)=P(n+n_0)$). This observation regarding the base case will apply to all the variants we are going to present later.

The solution we have given to the above example illustrates the outline that a proof by induction should have. The crucial aspect one has to keep in mind is to always check both the base case(s) and the induction step. To see how important this is, consider the following two examples, both of which are false:

a) For any non-negative integer n, $n^2 + n + 41$ is prime.

b) For any non-negative integer n, $3n + 1$ is divisible by 3.

For statement a), one can show that the base cases are true, as in fact $n^2 + n + 41$ is prime for all $n \in \{0, 1, \ldots, 39\}$. However, we cannot prove that $P(n) \Rightarrow P(n + 1)$.

In the example b), assuming $P(n)$ was true i.e. $3 \mid (3n + 1)$ then

$$3(n + 1) + 1 = (3n + 1) + 3,$$

so $P(n + 1)$ is also true. But in this situation, we cannot prove that the base case holds, as in fact $P(0)$ is false.

Appendix: Peano's axioms for $\mathbb{N}$

Before we state the axioms introduced by Giuseppe Peano, we need to recall the following definitions: a **function** f between two sets A and B (written $f : A \to B$) is a correspondence which associates to each element $a \in A$ precisely one element $f(a) \in B$; the **image** of f, denoted $Im(f)$, is the set $Im(f) = \{f(a) : a \in A\}$ (note that $Im(f) \subseteq B$ and this inclusion can be strict, depending on the function f); a function is called **injective** if no two distinct elements of A are sent to the same element of B. For more details and examples, the reader can refer to Section 3.1 of this book.

The axioms introduced by Peano are the following:

1. There exists a special element $0 \in \mathbb{N}$;

2. There exists an injective function (called the *successor function*) $S : \mathbb{N} \to \mathbb{N}$ such that 0 does not lie in the image of S;

3. If $K \subseteq \mathbb{N}$ is a set such that $0 \in K$ and for every natural number $n \in K$ implies $S(n) \in K$, then $K = \mathbb{N}$ (Axiom of Induction).

At first glance, the Axiom of Induction as stated here does not look quite the same as the version we stated informally earlier. Before, we referred to a family of propositions $P(n)$. Now we have just a set K. However, they are easily seen to be equivalent if we interpret K as being the set of all n such that $P(n)$ is true.

Peano originally included several other axioms which are now commonly regarded as first-order logic axioms and in modern treatments are not included as axioms of the natural numbers. It is remarkable how complex a system can become even when it is founded only on a handful of axioms! The universe we live in seems to be governed by just four laws, namely the gravitational force, the electromagnetic force, the strong and the weak nuclear forces. And just look what is out there! Look at the complexity, the intricacy, the splendour! Also, the above three laws for the set of natural numbers which were postulated by Peano govern the whole universe of Number Theory and explain anything from operations such as addition and multiplication to modern results regarding bounded gaps between primes and arbitrary long arithmetic progressions of primes!

To get a small flavour of these, let us see how one can define the usual addition on $\mathbb{N}$ from Peano's axioms and how we can prove some standard properties of it. We define a function $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ inductively by

a) For all $a \in \mathbb{N}$, we define $a + 0 = a$;

b) Having defined $a + b$ for some $b \in \mathbb{N}$, we define $a + S(b) = S(a + b)$.

Lemma. The operation $+$ defined above is commutative.

Proof. We will prove the result in three steps. We start by showing that $n + 0 = 0 + n$ for any $n \in \mathbb{N}$. We do this by induction on n. Let $P(n)$ be the proposition

$$P(n) : n + 0 = 0 + n.$$

By a), we have $0 + 0 = 0$, so $P(0)$ holds.

Assume now that $P(n)$ is true for some $n \in \mathbb{N}$ and we deduce that $P(S(n))$ is true:

$$0 + S(n) \overset{b)}{=} S(0 + n) \overset{P(n)}{=} S(n + 0) \overset{a)}{=} S(n) \overset{a)}{=} S(n) + 0.$$

Therefore, by the Axiom of Induction, $P(n)$ holds for all $n \in \mathbb{N}$.

We now prove that $S(a) + n = S(a + n)$, for any $a, n \in \mathbb{N}$. For a fixed $a \in \mathbb{N}$, let $P(n)$ be the proposition

$$P(n) : S(a) + n = S(a + n).$$

When $n = 0$, we have $S(a) + 0 \overset{a)}{=} S(a) \overset{a)}{=} S(a + 0)$, so $P(0)$ is true.

Assume now that $P(n)$ holds for some $n \in \mathbb{N}$, and we prove that $P(S(n))$ also holds. We have

$$S(a) + S(n) \overset{b)}{=} S(S(a) + n) \overset{P(n)}{=} S(S(a + n)) \overset{b)}{=} S(a + S(n)).$$

Thus $P(S(n))$ is true. Hence, by the Axiom of Induction, we have that $P(n)$ is true for all $n \in \mathbb{N}$.

As a was arbitrary, the property $S(a) + n = S(a + n)$ holds for any a and n in $\mathbb{N}$.

Finally, we prove that $a + n = n + a$ for any $a, n \in \mathbb{N}$. We do so by induction on a, for some fixed n. As before, we let $P(a)$ be the corresponding Proposition.

We have $0 + n = n + 0$ from the first step, so $P(0)$ holds.

Assuming $P(a)$, we have from the second step

$$S(a) + n = S(a + n) \overset{P(a)}{=} S(n + a) \overset{b)}{=} n + S(a).$$

This completes the proof of our last step. Since n was arbitrary, we obtain that $a + n = n + a$, for any $a, n \in \mathbb{N}$, showing that $+$ is commutative, as we wanted.

From now on, we write $S(0) = 1$ and $S(n) = n + 1 = 1 + n$. The other standard properties of addition are proved in a similar manner. With the aid of addition, we can define the usual ordering "$\leq$" between two elements of $\mathbb{N}$ as: $a \leq b$ if there exists $c \in \mathbb{N}$ such that $a + c = b$. The properties of $\leq$ are now deduced from those of addition.

We have established so far what the Principle of Induction is and that it is an axiom. Before we move on to discussing other applications, variations and generalizations, we give the promised proof that the Principle of Induction implies that every non-empty subset $A \subset \mathbb{N}$ has a least element with respect to $\leq$:

Let $A \subset \mathbb{N}$ be a subset that has no minimal element with respect to $\leq$. We would like to prove that A is empty in this case. We set $P(n)$ to be the proposition:

$$P(n): \quad k \notin A, \quad \text{for all} \quad 0 \leq k \leq n.$$

Certainly $P(0)$ is true, as otherwise we would have $0 \in A$ and this would be the minimal element of A (as there is no non-negative integer smaller than 0). Let us now show that $P(n)$ true for some $n \geq 0$ implies $P(n+1)$ true as well:

If $P(n)$ is true and $P(n+1)$ is false, then $0, 1, 2, \dots n \notin A$, but $n+1 \in A$, and then $n+1$ is the least element of A. This is a contradiction and we conclude that $P(n+1)$ is true.

By the Principle of Induction, $P(n)$ is true for every $n \in \mathbb{N}$ and hence A is empty.

The above argument shows that the only subset of $\mathbb{N}$ with no least element is the empty set. Therefore, every non-empty subset of $\mathbb{N}$ has a least element, which proves what we wanted.

1.1.2 Variants of Induction

We have seen above what the axiom of Induction is and how a typical proof by induction looks. We have also seen that in some applications we could have the base case bigger than 0. The next topic to look at is what happens when we modify our induction step. We prove the following:

Theorem. Let k be some fixed positive integer and $P(n)$ be a mathematical statement that satisfies the following properties:

1. All of $P(0), P(1), \dots, P(k-1)$ are true;
2. $P(n) \Rightarrow P(n+k)$, for any $n \geq 0$.

Then $P(n)$ is true for every $n \in \mathbb{N}$.

Proof. When $k = 1$, the above statement is the Axiom of Induction, so we have nothing to prove. So let $k \geq 2$ and assume by contradiction that there is some $n \in \mathbb{N}$ for which $P(n)$ does not hold. Then the set $S = \{n \in \mathbb{N} : P(n) \text{ does not hold}\}$ is non-empty and so it has a least element, call it m.

Let r be the remainder when we divide m by k, so that we have $m = q \cdot k + r$, for some $q \in \mathbb{N}$ and $0 \leq r \leq k-1$. From the given hypotheses we know that $P(r)$ holds, thus in particular we cannot have $q = 0$. This implies $q \geq 1$, hence $0 \leq (q-1) \cdot k + r < m$. Because m is the least element of S, we have that $P((q-1) \cdot k + r)$ is true and then using the second hypothesis, $P((q-1) \cdot k + r + k) = P(m)$ is true, giving a contradiction.

Remark. What is important to note when using this variant is that if we prove $P(n) \Rightarrow P(n+k)$, we need to check k base cases instead of just one; for example, if $k = 2$, then checking just $P(0)$, together with $P(n) \Rightarrow P(n+2)$ would only imply that $P(n)$ holds when n is an even positive integer! The above variant (and hence also the Axiom of Induction obtained for $k = 1$) bears the name of the *Weak Principle of Induction*.

Example 1.1. Prove that any square can be divided in n (not necessarily congruent) squares ($n \geq 6$).

Solution. The key observation is that given any square, we can partition it into four smaller congruent squares, which shows that $P(n) \Rightarrow P(n+3)$. Hence all we need is to check that the statement holds for $n = 6, 7, 8$.

For $n = 6$, we can divide the square into 9 congruent squares and then join four of them into a larger one.

For $n = 7$, we first divide the square into 4 equal squares and then divide one of those further into 4 equal squares.

Finally, for $n = 8$, divide the square into 16 congruent squares and then join 9 of them into a bigger one.

The next variant is known as the *Strong Principle of Induction*:

Theorem. Let $P(n)$ be a statement about $n \in \mathbb{N}$. Suppose that

1. $P(0)$ is true;
2. $\forall n \in \mathbb{N}$, if $P(k)$ is true $\forall k < n$ then $P(n)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Suppose that $P(0)$ is true and for all $n \in \mathbb{N}$, if $P(0)$, $P(1)$, …, $P(n-1)$ are all true, then $P(n)$ is true. We want to show that $P(n)$ is true for all n using the weak principle.

Let $Q(n)$ be the statement "$P(k)$ is true $\forall k \leq n$". Then $Q(0)$ is true from the hypothesis. Suppose that $Q(n)$ is true. Then all of $P(0), P(1), \ldots, P(n)$ are true. So $P(n+1)$ is true. Hence $Q(n+1)$ is true. By the Weak Principle of Induction, $Q(n)$ is true for all n. So $P(n)$ is true for all n.

Example 1.2. Show that if $x + \dfrac{1}{x} \in \mathbb{Z}$, then $x^n + \dfrac{1}{x^n} \in \mathbb{Z}$ for all positive integers n.