Preface

Inspired by appreciative and constructive feedback from our faithful readers, we present *Mathematical Reflections: the first two years*, a compilation and revision of the 2006 and 2007 volumes from the online journal of the same name. Since its inception in January 2006, the journal has attracted readers and contributors from all over the world. It successfully brings together enthusiasts with different mathematical and cultural backgrounds for the common purpose of making mathematics even more beautiful and exciting. The journal publishes six issues each year.

This book is aimed at high school students, participants in math competitions, undergraduates, as well as anyone who has a passion for mathematics. Many of the problems and solutions were submitted by loyal and passionate readers. In publishing this volume, our efforts were especially geared towards organizing the problems and their solutions, as well as correcting and improving on many of the solutions and articles, so that our audience can enjoy them even more.

The articles herein focus on a variety of interesting topics outside of the mainstream curriculum. Students can expand their mathematical horizons through material outside the scope of their regular class. For instructors, the articles provide an intriguing opportunity to move away from a structured curriculum, motivate the problem discussed, and guide students through to the invaluable moments of discovery.

All of the featured problems are original. They require creativity, experience, and comprehensive mathematical knowledge. In order to make the material more accessible to the readers, this book – as well as the journal – is organized by the mathematical ability required to solve the problems. The junior section features introductory problems (though not necessarily easy ones). The senior and Olympiad sections are for students preparing for national or international contests such as the United States of America Mathematical Olympiad (USAMO) or the International Mathematical Olympiad (IMO). Last, but not least, the undergraduate section offers college students a unique opportunity to solve non-routine problems in areas such as linear algebra, calculus, or graph theory.

This book could not have seen the light of day without the loyal readers and collaborators of the online journal. We would like to thank them all and express our gratitude for their continuous support. We sincerely hope that others will follow in their footsteps and continue to carry the baton, so that the mission of the journal to offer the opportunity for mathematics enthusiasts to publish original and inter-

esting work will be fulfilled in future years as well. We would also like to thank Zac Cox, Ivan Borsenco, and Radu Sorici for their hard work in putting together the material submitted by our contributors. Many thanks to Dorin Andrica and Gabriel Dospinescu for their pertinent observations, as well as to Marius Craciunoiu for his website contributions.

If you are interested in reading the journal, please visit its website:
http://awesomemath.org/mathematical-reflections/.

Readers wishing to submit articles, problems, or solutions should send them to reflections@awesomemath.org

Proceeds from the sale of this book will be used to sustain the journal in future years. Enjoy the problems and articles!

Dr. Titu Andreescu

Contents

PROBLEMS

1.1 Junior Problems

J1. Solve in real numbers the system of equations

$$\begin{cases} x^4 + 2y^3 - x = -\dfrac{1}{4} + 3\sqrt{3} \\ y^4 + 2x^3 - y = -\dfrac{1}{4} - 3\sqrt{3}. \end{cases}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J2. Prove that for each integer a there is a nonzero integer b such that the equation

$$ax^2 - (a^2 + b)x + b = 0$$

has integer roots.

Proposed by Laurentiu Panaitopol, University of Bucharest, Romania

J3. Let

$$a_n = \sqrt{1 + \left(1 + \frac{1}{n}\right)^2} + \sqrt{1 + \left(1 - \frac{1}{n}\right)^2},\ n \geq 1.$$

Prove that $\dfrac{1}{a_1} + \dfrac{1}{a_2} + \ldots + \dfrac{1}{a_{20}}$ is an integer.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J4. The unit cells of an $n \times n$ array are colored white or black in such a way that each 2×2 square contains either exactly one white cell or exactly one black cell. Find all possible values of n for which such an array does not contain identical columns.

Proposed by Marius Ghergu, Bucharest, Romania

J5. Let x, y, z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{1}{(x+1)^2+y^2+1}+\frac{1}{(y+1)^2+z^2+1}+\frac{1}{(z+1)^2+x^2+1} \leq \frac{1}{2}.$$

Proposed by Cristinel Mortici, Valahia University, Targoviste, Romania

J6. Let $ABCD$ be a convex quadrilateral such that sides BC and CD have equal lengths and $2\angle A + \angle C = 180^\circ$. Let M be the midpoint of segment BD. Prove that $\angle MAD = \angle BAC$.

Proposed by Dinu Serbanescu, Sf. Sava National College, Romania

J7. Prove that $\displaystyle\sum_{n=1}^{9999} \frac{1}{(\sqrt{n}+\sqrt{n+1})(\sqrt[4]{n}+\sqrt[4]{n+1})} = 9.$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J8. Let a and b be distinct real numbers such that

$$|a-1|+|b-1| = |a|+|b| = |a+1|+|b+1|.$$

Find the least possible value of $|a-b|$.

Proposed by Bogdan Enescu, B.P. Hasdeu National College, Romania

J9. Solve in nonnegative integers the equation

$$(xy+z)^2+(x+yz)^2 = 2005^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J10. Let $A = 1!\cdot 2!\cdot \ldots \cdot 1002!$ and $B = 1004!\cdot 1005!\cdot \ldots \cdot 2006!$. Prove that $2AB$ is a square and that $A+B$ is not a square.

Proposed by Bogdan Enescu, B.P. Hasdeu National College, Romania

J11. Consider a parallelogram $ABCD$ with center O and a point P different from O that satisfies $PA \cdot PC = OA \cdot OC$ and $PB \cdot PD = OB \cdot OD$, Prove that the sum of lengths of two of the segments PA, PB, PC, PD equals the sum of lengths of the other two.

Proposed by Iurie Boreico, Harvard University, USA

J12. Let $ABCD$ be a convex quadrilateral. A square is called inscribed in $ABCD$ if its vertices lie on different sides of $ABCD$. If there are two different squares inscribed in $ABCD$, prove that there are infinitely many squares inscribed in $ABCD$.

Proposed by Iurie Boreico, Harvard University, USA

J13. Prove that the equation $x^3 + y^3 - 2z^3 = \frac{3}{2}(x + y + z)$ has infinitely many solutions in positive integers x, y, z.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J14. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$a(b^2 - \sqrt{b}) + b(c^2 - \sqrt{c}) + c(a^2 - \sqrt{a}) \geq 0.$$

Proposed by Zdravko Starc, Vrsac, Serbia

J15. Find the least positive real number α with the following property: for each triangle, we can choose two of its sides, say a and b, such that $\alpha > \frac{a}{b} \geq 1$.

Proposed by Bogdan Enescu, B.P. Hasdeu National College, Romania

J16. Consider a scalene triangle ABC and let $X \in (AB)$ and $Y \in (AC)$ be two variable points such that $BX = CY$. Prove that the circumcircle of triangle AXY passes through a fixed point (different from A).

Proposed by Liubomir Chiriac, Princeton University, USA

J17. Let a, b, c be positive numbers. Prove that

$$(ab + bc + ca)^3 \leq 3(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2).$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J18. Let n be a positive integer. Prove that $2^{2^{n+1}} + 2^{2^n} + 1$ is the product of at least $n+1$ distinct integers greater than 1.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J19. Let a and b be real numbers such that $3(a+b) \geq 2|ab+1|$. Prove that

$$9(a^3+b^3) \geq |a^3b^3+1|.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J20. Prove that
(a) There are infinitely many quadruples (a,b,c,d) of distinct positive integers such that $ab+cd=(a+b)(c+d)$.
(b) For each such quadruple, $\max(a,b,c,d) \geq \frac{\sqrt{3}+1}{4\sqrt{3}}(a+b+c+d)$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J21. A $(2m+1)\times(2n+1)$ grid is colored with two colors. An 1×1 square is called row-dominant if there are at least $n+1$ squares of its color in its row. Define column-dominant squares analogously. Prove that there are at least $m+n+1$ squares that are both column-dominant and row-dominant.

Proposed by Iurie Boreico, Harvard University, USA

J22. Starting with n 1s, we keep picking two of the numbers, say a and b, and replace them by $\sqrt[3]{\frac{a^2b^2}{a+b}}$. We continue this operation until there is only one number left. Prove that this number is not less than $\frac{1}{\sqrt[3]{n}}$.

Proposed by Liubomir Chiriac, Princeton University, USA

J23. Let $ABCDEF$ be a convex hexagon with parallel opposite sides and let $FC\cap AB = X_1$, $FC\cap ED = X_2$, $AD\cap EF = Y_1$, $AD\cap BC = Y_2$, $BE\cap CD = Z_1$, $BE\cap AF = Z_2$. Prove that if X_1, Y_1, Z_1 are collinear, then X_2, Y_2, Z_2 are also collinear and lines $X_1Y_1Z_1$ and $X_2Y_2Z_2$ are parallel.

Proposed by Santiago Cuellar, Spain

J24. Consider a triangle ABC and a point P in its interior. Denote by d_a, d_b, d_c the distances from P to the sides of the triangle. Prove that

$$2K\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{1}{R}\right) \geq d_a+d_b+d_c,$$

where K is the area of triangle ABC and R is its circumradius.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J25. Let k be a real number different from 1. Solve the system of equations

$$\begin{cases} (x+y+z)(kx+y+z) = k^3+2k^2 \\ (x+y+z)(x+ky+z) = 4k^2+8k \\ (x+y+z)(x+y+kz) = 4k+8. \end{cases}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J26. A line divides an equilateral triangle into two parts with the same perimeters and areas K_1 and K_2, respectively. Prove that $\frac{7}{9} \leq \frac{K_1}{K_2} \leq \frac{9}{7}$.

Proposed by Bogdan Enescu, B.P. Hasdeu National College, Romania

J27. Consider points M and N inside a triangle ABC such that $\angle BAM = \angle CAN$, $\angle MCA = \angle NCB$, $\angle MBC = \angle ABN$. Suppose points M and N are isogonal and $BMNC$ is a cyclic quadrilateral. If T is the circumcenter of $BMNC$, prove that $MN \perp AT$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J28. Let p be a prime such that $p \equiv 1 \pmod 3$ and let $q = \left\lfloor \frac{2p}{3} \right\rfloor$. If

$$\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{(q-1)q} = \frac{m}{n}$$

for some integers m and n, prove that $p \mid m$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA