Preface

Our success publishing, *Mathematical Reflections: The First Two Years*, serves to illustrate the need for problem-solving materials that allow depth and breadth of understanding in mathematics. We are pleased to provide the next book in this series, Mathematical *Mathematical Reflections: The Next Two Years* to continue to fill this need for problems that go beyond the general curriculum. The problems include a compilation and revision of the 2008 and 2009 volumes from the journal of the same name. Our problems are specifically designed to show the beauty of the concept being studied and our solutions will allow greater understanding to develop. How can we accomplish this lofty goal? Because this book is a collaboration of math enthusiasts from all over the world who want to share their excitement and joy of mathematics with you, the reader.

High school students, math competition participants, college undergraduates, and anyone who has a passion for the power and beauty of math can benefit from this book. Many of the problems and solutions included in this volume were submitted by individuals just like you, individuals with a high level of curiosity and desire to go beyond the typical problem. The problems have been revised and reviewed to ensure that every reader will learn a concept in a unique and elegant way leading to a whole new level of understanding.

Math instructors, professors, and teachers can move beyond the structured curriculum and engage their students on a new level guiding their class to invaluable moments of discovery.

All of the featured problems are original. They require creativity, experience, and comprehensive mathematical knowledge. To facilitate the readers progression through the book, the problems are all organized based on the mathematical ability necessary to solve them. The junior section features introductory problems (though not necessarily easy ones). The senior and Olympiad sections are for students preparing for national or international contests such as the United States of America Mathematical Olympiad (US- AMO) or the International Mathematical Olympiad (IMO). Last, but not least, the undergraduate section offers college students a rare opportunity to solve non-routine problems in areas such as linear algebra, calculus, or graph theory.

The loyal readers and collaborators of the online journal have made this second volume, like the first, possible. We would like to thank them all and express our gratitude for their continuous support. We sincerely hope that others will follow in their footsteps and continue to carry the baton, so that the mission of the journal to offer the opportunity for mathematics enthusiasts to publish original and interesting work will be fulfilled in future years as well.

We would also like to thank Ivan Borsenco, Radu Sorici, and Roberto Bosch Cabrera for their hard work in putting together the material submitted by our contributors. Many thanks to Dorin Andrica and Gabriel Dospinescu for their pertinent observations, as well as to Marius Craciunoiu for his website contributions.

If you are interested in reading the journal, please visit the website: http://awesomemath.org/mathematical-reflections/. Readers wishing to submit articles, problems, or solutions should send them to reflections@awesomemath.org.

Proceeds from the sale of this book will be used to sustain the journal in future years. Enjoy the problems and articles!

Dr. Titu Andreescu

Contents

Chapter 1

Problems

1.1 Junior Problems

J73. Let

$$a_n = \begin{cases} n^2 - n, & \text{if 4 divides } n^2 - n \\ n - n^2, & \text{otherwise.} \end{cases}$$

Evaluate $a_1 + a_2 + \cdots + a_{2008}$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J74. A triangle has altitudes h_a, h_b, h_c and inradius r. Prove that

$$\frac{3}{5} \leq \frac{h_a - 2r}{h_a + 2r} + \frac{h_b - 2r}{h_b + 2r} + \frac{h_c - 2r}{h_c + 2r} < \frac{3}{2}.$$

Proposed by Oleh Faynshteyn, Leipzig, Germany

J75. Jimmy has a box with n matches, not necessarily of equal length. He is able to use them to construct several cyclic n-gons. Prove that all of the n-gons have the same area.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J76. Let $a, b, c \geq 1$ be real numbers such that $a + b + c = 2abc$. Prove that

$$\sqrt[3]{(a+b+c)^2} \geq \sqrt[3]{ab-1} + \sqrt[3]{bc-1} + \sqrt[3]{ca-1}.$$

Proposed by Bruno de Lima Holanda, Fortaleza, Brazil

J77. Prove that in each triangle $\frac{1}{r}\left(\frac{b^2}{r_b}+\frac{c^2}{r_c}\right)-\frac{a^2}{r_b r_c}=4\left(\frac{R}{r_a}+1\right)$, where r_a, r_b, r_c are the exradii.

Proposed by Dorin Andrica, Babes-Bolyai University, Romania and K.L. Nguyen, Massachusetts Institute of Technology, USA

J78. Let p and q be odd primes. Prove that for any odd integer $d > 0$ there is an integer r such that the numerator of the rational number

$$\sum_{n=1}^{p-1} \frac{[n \equiv r (\mathrm{mod}\ q)]}{n^d}$$

is divisible by p, where $[Q]$ is a function such that $[Q]$ equals 1 if Q is true and $[Q]$ equals 0 otherwise.

Proposed by Robert Tauraso, Roma, Italy

J79. Find all integers that can be represented as $a^3+b^3+c^3-3abc$ for some positive integers a, b, and c.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J80. Characterize triangles such that their sidelengths form an arithmetical progression and their median lengths form an arithmetical progression.

Proposed by Daniel Lasaosa, Universidad Publica de Navarra, Spain

J81. Let a, b, c be positive real numbers such that

$$\frac{1}{a^2+b^2+1}+\frac{1}{b^2+c^2+1}+\frac{1}{c^2+a^2+1} \geq 1.$$

Prove that $ab+bc+ca \leq 3$.

Proposed by Alex Anderson, New Trier High School, Winnetka, USA

J82. Let $ABCD$ be a quadrilateral whose diagonals are perpendicular. Denote by $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ the centers of the nine-point circles of triangles ABC, BCD, CDA, DAB, respectively. Prove that the diagonals of $\Omega_1\Omega_2\Omega_3\Omega_4$ intersect at the centroid of $ABCD$.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J83. Find all positive integers n such that a divides n for all odd positive integers a not exceeding $\sqrt{n}$.

Proposed by Dorin Andrica, Babes-Bolyai University, Romania

J84. Al and Bo play a game: there are 22 cards labeled 1 through 22. Al chooses one of them and places it on a table. Bo then places one of the remaining cards at the right of the one placed by Al such that the sum of the two numbers on the cards is a perfect square. Al then places one of the remaining cards such that the sum of the numbers on the last two cards played is a perfect square, and so on. The game ends when all the cards were played or no more cards can be placed on the table. The winner is the one who played the last card. Does Al have a winning strategy?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J85. Let a and b be positive real numbers. Prove that

$$\sqrt[3]{\frac{(a+b)(a^2+b^2)}{4}} \geq \sqrt{\frac{a^2+ab+b^2}{3}}.$$

Proposed by Arkady Alt, San Jose, California, USA

J86. A triangle is called α-angular if none of its angles exceeds α degrees. Find the least α for which each non α-angular triangle can be dissected into some α-angular triangles.

Proposed by Titu Andreescu, University of Texas at Dallas and Gregory Galperin, Eastern Illinois University, USA

J87. Prove that for any acute triangle ABC, the following inequality holds:

$$\frac{1}{-a^2+b^2+c^2}+\frac{1}{a^2-b^2+c^2}+\frac{1}{a^2+b^2-c^2}\geq\frac{1}{2Rr}.$$

Proposed by Mircea Becheanu, Bucharest, Romania

J88. Find the greatest n for which there are points $P_1, P_2, \ldots, P_n$ in the plane such that each triangle whose vertices are among $P_1, P_2, \ldots, P_n$, has a side less than 1 and a side greater than 1.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J89. Let A and B lie on circle $\mathcal{C}$ of center O and let C be the point on the small arc AB such that OA is the external angle bisector of $\angle BOC$. Denote by M the midpoint of BC and by N the intersection of AM and OC. Prove that the intersection of the angle bisector of $\angle BOC$ with the circle of center O and radius ON is the center of the circle tangent to lines OB and OC, and also internally tangent to $\mathcal{C}$.

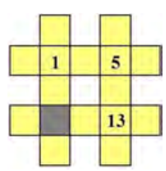
Proposed by Francisco Javier Garcia Capitan, Spain

J90. For a positive integer n let $a_k = 2^{2^{k-n}} + k$, $k = 0, 1, \ldots, n$. Prove that

$$(a_1 - a_0)\cdots(a_n - a_{n-1}) = \frac{7}{a_1 + a_0}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J91. The squares in the figure below are labeled 1 through 16 such that the sum of the numbers in each row and each column is the same. The positions of $1, 5$, and 13 are given.



Find the number in the darkened square.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J92. Find all primes $q_1, q_2, \ldots, q_6$ such that $q_1^2 = q_2^2 + \cdots + q_6^2$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J93. Let a and b be positive real numbers. Prove that

$$\frac{a^6+b^6}{a^4+b^4} \geq \frac{a^4+b^4}{a^3+b^3} \cdot \frac{a^2+b^2}{a+b}.$$

Proposed by Arkady Alt, San Jose, California, USA

J94. Prove that the equation $x^3 + y^3 + z^3 + w^3 = 2008$ has infinitely many solutions in integers.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J95. Let ABC be a triangle and let I_a, I_b, I_c be its excenters. Denote by O_a, O_b, O_c the circumcenters of triangles I_aBC, I_bAC, I_cAB. Prove that the area of triangle $I_aI_bI_c$ is twice the area of hexagon $O_aCO_bAO_cB$.

Proposed by Mehmet Sahin, Ankara, Turkey

J96. Let n be an integer. Find all integers m such that $a^m + b^m \geq a^n + b^n$ for all positive real numbers a and b with $a + b = 2$.

Proposed by Oleg Mushkarov, Bulgarian Academy of Sciences, Bulgaria

J97. Let a, b, c, d be integers such that $a + b + c + d = 0$. Prove that $30 \mid a^5 + b^5 + c^5 + d^5$.

Proposed by Johan Gunardi, Jakarta, Indonesia

J98. Find all primes p and q such that 24 does not divide $q+1$ and p^2q+1 is a perfect square.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J99. In a triangle ABC, let ϕ_a, ϕ_b, ϕ_c be the angles between medians and altitudes emerging from the same vertex. Prove that one of the numbers $\tan\phi_a, \tan\phi_b, \tan\phi_c$ is the sum of the other two.

Proposed by Oleh Faynshteyn, Leipzig, Germany

J100. Consider the set of points from the plane such that the distance between any two points is a real number from the interval $[a,b]$. Prove that the number of these points is finite.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J101. Consider triangle ABC with circumcenter O and orthocenter H. Let A_1 be the projection of A onto BC and let D be the intersection of AO with BC. Denote by A_2 the midpoint of AD. Similarly, we define B_1, B_2 and C_1, C_2. Prove that A_1A_2, B_1B_2, C_1C_2 are concurrent.

Proposed by Andrea Munaro, Italy and Ivan Borsenco, USA

J102. Evaluate

$$\binom{2008}{3}-2\binom{2008}{4}+3\binom{2008}{5}-4\binom{2008}{6}+\cdots-2004\binom{2008}{2006}+2005\binom{2008}{2007}$$

Proposed by Zuming Feng, Phillips Exeter Academy, USA

J103. The numbers $1, 2, \ldots, 9$ are randomly arranged on a circle. Prove that there are three adjacent numbers whose sum is at least 16.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA