Preface

Inspired by appreciative and constructive feedback from our faithful readers, we present Mathematical Reflections – two beautiful years, a compilation and revision of the 2018 and 2019 volumes from the online journal of the same name. Since its inception in January 2006, the journal has attracted readers and contributors from all over the world. It successfully brings together enthusiasts with different mathematical and cultural backgrounds for the common purpose of making mathematics even more elegant and exciting. The journal publishes six issues each year.

This book is aimed at high school students, participants in math competitions, undergraduates, and anyone who has a passion for mathematics. Many of the problems, solutions, and articles were submitted by passionate readers and they all require creativity, experience, and comprehensive mathematical knowledge. In publishing this volume, our efforts were especially geared towards correcting and improving on many of the solutions and articles, so that our audience can enjoy them even more.

The articles herein focus on a variety of interesting topics outside of the mainstream curriculum. Students can expand their mathematical horizons through material outside the scope of their regular classes. For instructors, the articles provide an intriguing opportunity to move away from a structured curriculum, motivate the problem discussed, and guide students through to the invaluable moments of discovery. All of the featured problems are original. In order to make the material more accessible to the readers, this book – as well as the journal – is organized by the mathematical ability required to solve the problems. The junior section presents introductory problems (though not necessarily easy ones). The senior and Olympiad sections are for students preparing for national or international contests such as the United States of America Mathematical Olympiad (USAMO) or the International Mathematical Olympiad (IMO). Last, but not least, the undergraduate section offers college students a unique opportunity to solve non-routine problems in areas such as linear algebra, calculus, or graph theory.

This book could not have seen the light of day without the loyal readers and collaborators of the online journal. We would like to thank them all and express our gratitude for their continuous support. We sincerely hope that others will follow in their footsteps and continue to carry the baton, so that the mission of the journal to offer the opportunity for mathematics enthusiasts to publish original and interesting work will be fulfilled in the future years as well.

Many thanks to Richard Stong and Li Zhou for the numerous improvements in the manuscript.

If you are interested in reading the journal, please visit its website: http://awesomemath.org/mathematical-reflections/. Readers may submit articles, problems, or solutions to reflections@awesomemath.org

Proceeds from the sale of this book will be used to sustain the journal in future years. Enjoy the problems and articles!

Dr. Titu Andreescu
Maxim Ignatiuc

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1.1 Junior Problems

J433. Let a, b, c, x, y, z be real numbers such that $a^2+b^2+c^2 = x^2+y^2+z^2 = 1$. Prove that

$$|a(y-z)+b(z-x)+c(x-y)| \leq \sqrt{6(1-ax-by-cz)}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J434. Solve in integers the equation

$$x^3+y^3 = 7\max(x,y)+7.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

J435. Let $a \geq b \geq c > 0$ be real numbers. Prove that

$$2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq 3.$$

Proposed by Mircea Becheanu, Montreal, Canada

J436. Let a, b, c be real numbers such that $a^4+b^4+c^4 = a+b+c$. Prove that

$$a^3+b^3+c^3 \leq abc+2.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

J437. Let a, b, c be real numbers such that $(a^2+2)(b^2+2)(c^2+2) = 512$. Prove that

$$|ab + bc + ca| \leq 18.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J438. (i) Find the greatest real number r such that

$$ab \geq r\left(1 - \frac{1}{a} - \frac{1}{b}\right)$$

for all positive real numbers a and b.

(ii) Find the maximum of $xyz(2-x-y-z)$ over all positive real numbers x, y, z.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J439. Solve in real numbers the system of equations:

$$\begin{cases} 2x^2 - 3xy + 2y^2 = 1 \\ y^2 - 3yz + 4z^2 = 2 \\ z^2 + 3zx - x^2 = 3. \end{cases}$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

J440. Let a, b, c, d be distinct nonnegative real numbers. Prove that

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-d)^2} + \frac{c^2}{(d-a)^2} + \frac{d^2}{(a-b)^2} > 2.$$

Proposed by An Zhenping, Xianyang Normal University, China

J441. Prove that for any positive real numbers a, b, c the following inequality holds

$$\frac{(a+b+c)^3}{3abc} + 1 \geq \left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2 + (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

2.1 Junior Solutions

J433. Let a, b, c, x, y, z be real numbers such that $a^2+b^2+c^2 = x^2+y^2+z^2 = 1$. Prove that

$$|a(y-z)+b(z-x)+c(x-y)| \le \sqrt{6(1-ax-by-cz)}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution. By the Cauchy-Schwarz inequality,

$$\begin{aligned} & |a(y-z)+b(z-x)+c(x-y)| \\ =\ & |(a-x)(y-z)+(b-y)(z-x)+(c-z)(x-y)| \\ \le\ & \sqrt{[(a-x)^2+(b-y)^2+(c-z)^2][(y-z)^2+(z-x)^2+(x-y)^2]} \\ =\ & \sqrt{2(1-ax-by-cz)[3-(x+y+z)^2]} \\ \le\ & \sqrt{6(1-ax-by-cz)}. \end{aligned}$$

J434. Solve in integers the equation

$$x^3+y^3 = 7\max(x,y)+7.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

Solution. From the x, y symmetry, we may assume that $x \ge y$. Then we have

$$\begin{aligned} x^3+y^3 = 7\max(x,y)+7 &\Leftrightarrow x^3+y^3 = 7x+7 \\ &\Leftrightarrow x^3-7x-7 = (-y)^3. \end{aligned}$$

If $x \geq 4$, then it is easy to check that $(x-1)^3 < x^3 - 7x - 7 < x^3$. The upper bound is obvious and the lower bound reduces to

$$3x^2 \geq 12x \geq 10x + 8 > 10x + 6.$$

If $x \leq -3$, then we similarly find $x^3 < x^3 - 7x - 7 < (x+1)^3$.

Hence any solution to the given equation has $-2 \leq x \leq 3$. Checking these cases, we find that the only solutions to $x^3 + y^3 = 7x + 7$ are $(x, y) = (-2, 1), (-1, 1)$ or $(3, 1)$ and only the last has $x \geq y$ as required. Restoring generality, the only solutions are $(x, y) = (3, 1)$ or $(1, 3)$.

J435. Let $a \geq b \geq c > 0$ be real numbers. Prove that

$$2\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) - \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \geq 3.$$

Proposed by Mircea Becheanu, Montreal, Canada

Solution. By the AM-GM inequality we have

$$\frac{b}{a} + \frac{c}{b} + \frac{a}{c} \geq 3\sqrt[3]{\frac{b}{a} \cdot \frac{c}{b} \cdot \frac{a}{c}} = 3. \tag{1}$$

Since $a \geq b \geq c > 0$ we have that

$$(a-b)(b-c)(a-c) \geq 0,$$

which after expanding gives

$$b^2c + c^2a + a^2b \geq a^2c + b^2a + c^2b$$

and dividing by abc we get

$$\frac{b}{a} + \frac{c}{b} + \frac{a}{c} \geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a}. \tag{2}$$

Adding (1) and (2) we get

$$2\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) \geq 3 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$$

or

$$2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq 3.$$

Equality holds if and only if $a=b=c$.

J436. Let a,b,c be real numbers such that $a^4+b^4+c^4=a+b+c$. Prove that

$$a^3+b^3+c^3\leq abc+2.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

Solution. If $a+b+c=0$, then $a=b=c=0$ and we are done. Thus we may assume that $a+b+c>0$ and $a\geq b\geq c$. Then $a\geq |b|$, so

$$a^2(a-b)(a-c)\geq b^2(a-b)(b-c)$$

and

$$c^2(a-c)(b-c)\geq 0.$$

Therefore,

$$\begin{aligned}0&\leq a^2(a-b)(a-c)-b^2(a-b)(b-c)+c^2(a-c)(b-c)\\&=2(a^4+b^4+c^4)+abc(a+b+c)-(a^3+b^3+c^3)(a+b+c)\\&=(a+b+c)(2+abc-a^3-b^3-c^3).\end{aligned}$$

J437. Let a,b,c be real numbers such that $(a^2+2)(b^2+2)(c^2+2)=512$. Prove that

$$|ab+bc+ca|\leq 18.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution. Since

$$(a^2+2)(b^2+2)(c^2+2)=(abc-2(a+b+c))^2+2(ab+bc+ca-2)^2,$$

we have

$$2(ab+bc+ca-2)^2\leq 512\Leftrightarrow |ab+bc+ca-2|\leq 16.$$