

Contents

Preface	v
1 Problems	1
1.1 Junior Problems	3
1.2 Senior Problems	18
1.3 Undergraduate Problems	33
1.4 Olympiad Problems	49
2 Solutions	65
2.1 Junior Solutions	67
2.2 Senior Solutions	133
2.3 Undergraduate Solutions	224
2.4 Olympiad Solutions	330
3 Articles	441
3.1. Concurrency, Collinearity, and Cyclicity using Homotheties . . .	443
3.2. More on an Extension of Fagnano's Problem	471
3.3. Three Reflections	483
3.4. A New Approach to the Interior Goat and Bird Problems	498
3.5. The Missing Dominant Terms in a Fourier Series	505
3.6. Solving Functional Equations in $\mathbb{R}^+$ Using Inequalities and Sequences	524
3.7. Local Monotonicity and Continuity	542
3.8. Formulas for Diagonals of any Quadrilateral	579

3.9. Sperner's Lemma and Brouwer's Fixed Point Theorem	585
3.10. An Interesting Generator of Parabolic Systems of Coaxial Circles Using Apollonius' Problem	591
3.11. A Few Generalizations of the Euler's Totient Function	602
3.12. Solutions for the Frobenius Number in Three Variables	617
3.13. On a Combinatorial Identity	635
3.14. A Triangle Ratio Theorem and its Generalization to Higher Dimensions	644
3.15. Newton's Quadrilateral Theorem	661
3.16. On a Class of Functional Equations	675
3.17. Projective Plane, Extended Complex Plane	686
3.18. On a Number Theory Problem: Chasing a Red Herring	702
3.19. A New Method to Solve and Refine Algebraic Inequalities	710
3.20. Distance Formula for a Point Inside a Triangle	735
3.21. The Best Constant for an Inequality from Mathematical Reflections	749
Problem Author Index	757
Article Author Index	759



1.1 Junior Problems

J577. Let a and b be positive real numbers such that $a + b = 1$. Prove that

$$\left(\frac{1}{a^2} - b - 1\right)\left(\frac{1}{b^2} - a - 1\right) \geq \frac{25}{4}.$$

Proposed by An Zhenping, Xianyang Normal University, China

J578. Let a be a positive real number and let $f(x) = a^x + \frac{1}{a^x}$.

Given that $f\left(\frac{2}{3}\right) = 1 + 2\sqrt{2}$, find $f\left(\frac{3}{2}\right)$.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J579. Find all pairs (x, y) of integers such that

$$3x^2 + 10x + 5 = 9 \cdot 2^y.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J580. Solve in real numbers the equation

$$\sqrt[3]{x^2 + 4} = \sqrt{x^3 - 4}.$$

Proposed by Alessandro Ventullo, Milan, Italy

J581. Let a, b, c, d be real numbers such that

$$ab + ac + ad + bd + cd - abcd = 11.$$

Prove that

$$(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1) \geq 100.$$

Proposed by An Zhenping, Xianyang Normal University, China

J582. Let $ABCD$ be a quadrilateral inscribed in a circle Γ with center O . Lines AC and BD intersect at X and Γ_1 is the circumcircle of triangle XAB . Lines AD and BC intersect Γ_1 at N and M , respectively. We know that the area of the concave polygon $ONDXCM$ is $DC \cdot AB$. Find the measure of $\angle AXB$.

Proposed by Mihaela Berindeanu, Bucharest, Romania

J583. Let m and n be positive integers. Prove that

$$N = 27^{2m+n+1} + 27^{m+2n+1} - 27^{m+n+1} + 1$$

has a factor greater than $6 \cdot 27^{\min(m,n)}$ and smaller than $\sqrt{N}$.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J584. Let x, y, z be rational numbers such that

$$3x^2 + 2022yz - 2016zx, 3y^2 + 2022xz - 2016xy, 672yz - z^2 - 674xy$$

are all squares of rational numbers. Prove that $x = y = z = 0$.

*Proposed by Navid Safaei, Sharif University of Technology,
Tehran, Iran*

J585. Let $a, b, c > 1$ be real numbers such that

$$\frac{1}{a-1} + \frac{1}{b-1} + \frac{1}{c-1} = 1.$$

Prove that

$$abc + 44 \geq 9(a + b + c).$$

Proposed by Marius Stănean, Zalău, Romania

- J586. Let ABC be a triangle with $AC = BC$ and altitudes AD , BE , CF . The circle with diameter BD cuts AB in M and BE in N . Line MN cuts AC in Q and CF in P . Let S denote the midpoint of segment DC . Show that SQP is an isosceles triangle.

Proposed by Mihaela Berindeanu, Bucharest, Romania

- J587. Let a, b, c be nonnegative real numbers, no two of which are zero.

Prove that

$$\frac{a^2}{a^2 + bc} + \frac{b^2}{b^2 + ca} + \frac{c^2}{c^2 + ab} + \frac{a^3 + b^3 + c^3 + 9abc}{(a+b)(b+a)(c+a)} \geq 3.$$

When does the equality occur?

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

- J588. Find all nonnegative integers x, y, z such that

$$4^x + 3^y = z^2.$$

Proposed by Mihaela Berindeanu, Bucharest, Romania

- J589. Let $a, b, c \in [0, 1]$ such that $a + b + c = 2$. Prove that

$$a^3 + b^3 + c^3 + 2abc \leq 2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

- J590. Let p be a positive integer. Evaluate

$$S_p = \sum_{m=1}^p \sum_{n=1}^m \sum_{k=1}^n \frac{k^2}{2k^2 - 2nk + n^2}.$$

Proposed by Florică Anastase, Lehliu-Gară, Romania

- J591. Let D_A , D_B , D_C be disks in the plane with centers O_A , O_B , O_C , respectively. Consider points $A \in D_A$, $B \in D_B$, $C \in D_C$ such that the area of triangle ABC is maximal. Prove that lines AO_A , BO_B , CO_C are concurrent.

Proposed by Josef Tkadlec, Czech Republic

J592. Let M be a point inside triangle ABC . Let D, E, F be the orthogonal projections of M onto sides BC, CA, AB , respectively. Prove that

$$MA \sin \frac{A}{2} + MB \sin \frac{B}{2} + MC \sin \frac{C}{2} \geq MD + ME + MF.$$

When does equality hold?

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J593. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{(1+2a)^3} + \frac{1}{(1+2b)^3} + \frac{1}{(1+2c)^3} \geq \frac{1}{3(1+2abc)}.$$

Proposed by An Zhenping, Xianyang Normal University, China

J594. Let a be a positive real number other than 1 and let c, d be real numbers such that

$$a^c + a^d = (a+1)a^{\frac{c+d-1}{2}}.$$

Prove that for all positive real numbers $b \neq 1$,

$$b^c + b^d = (b+a)b^{\frac{c+d-1}{2}}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J595. Solve the equation

$$\sqrt[3]{(x-1)^2} - \sqrt[3]{2(x-5)^2} + \sqrt[3]{(x-7)^2} = \sqrt[3]{4x}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J596. Let x and y be positive real numbers. Prove that

$$\frac{1}{2x+y} + \frac{x}{y+2} + \frac{y}{x+y+1} \geq 1.$$

Proposed by An Zhenping, Xianyang Normal University, China