Preface

Inspired by appreciative and constructive feedback from our faithful readers, we present Mathematical Reflections: the next two years, a compilation and revision of the 2010 and 2011 volumes from the online journal of the same name. Since its inception in January 2006, the journal has attracted readers and contributors from all over the world. It successfully brings together enthusiasts with different mathematical and cultural backgrounds for the common purpose of making mathematics even more elegant and exciting. The journal publishes six issues each year.

This book is aimed at high school students, participants in math competitions, undergraduates, and anyone who has a passion for mathematics. Many of the problems, solutions, and articles were submitted by passionate readers and they all require creativity, experience, and comprehensive mathematical knowledge. In publishing this volume, our efforts were especially geared towards correcting and improving on many of the solutions and articles, so that our audience can enjoy them even more.

The articles herein focus on a variety of interesting topics outside of the mainstream curriculum. Students can expand their mathematical horizons through material outside the scope of their regular classes. For instructors, the articles provide an intriguing opportunity to move away from a structured curriculum, motivate the problem discussed, and guide students through to the invaluable moments of discovery. All of the featured problems are original. In order to make the material more accessible to the readers, this book as well as the journal is organized by the mathematical ability required to solve the problems. The junior section presents introductory problems (though not necessarily easy ones). The senior and Olympiad sections are for students preparing for national or international contests such as the United States of America Mathematical Olympiad (USAMO) or the International Mathematical Olympiad (IMO). Last, but not least, the undergraduate section offers college students a unique opportunity to solve non-routine problems in areas such as linear algebra, calculus, or graph theory.

This book could not have seen the light of day without the loyal readers and collaborators of the online journal. We would like to thank them all and express my gratitude for their continuous support. We sincerely hope that others will follow in their footsteps and continue to carry the baton, so that the mission of the journal to offer the opportunity for mathematics enthusiasts to publish original and interesting work will be fulfilled in the future years as well.

We would also like to thank Maxim Ignatiuc for his help in putting together the material submitted by our contributors. Many thanks to Gabriel Dospinescu, Cosmin Pohoata, and Ivan Borsenco for their pertinent observations. And a special thank you to Richard Stong for the many improvements in the manuscript.

If you are interested in reading the journal, please visit its website: http://awesomemath.org/mathematical-reflections/. Readers may submit articles, problems, or solutions to reflections@awesomemath.org

Proceeds from the sale of this book will be used to sustain the journal in future years. Enjoy the problems and articles!

Dr. Titu Andreescu

Contents

PROBLEMS

1.1 Junior Problems

J145. Find all nine-digit numbers $aaaabbbbb$ that can be written as a sum of fifth powers of two positive integers.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J146. Let $A_1A_2A_3A_4A_5$ be a convex pentagon and let $X \in A_1A_2$, $Y \in A_2A_3$, $Z \in A_3A_4$, $U \in A_4A_5$, $V \in A_5A_1$ be points such that A_1Z, A_2U, A_3V, A_4X, A_5Y intersect at P. Prove that

$$\frac{A_1X}{A_2X} \cdot \frac{A_2Y}{A_3Y} \cdot \frac{A_3Z}{A_4Z} \cdot \frac{A_4U}{A_5U} \cdot \frac{A_5V}{A_1V} = 1.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J147. Let $a_0 = a_1 = 1$ and $a_{n+1} = 1 + \frac{a_1^2}{a_0} + \ldots + \frac{a_n^2}{a_{n-1}}$ for $n \geq 1$.
Find a_n in closed form.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J148. Find all n such that for each $\alpha_1, \ldots, \alpha_n \in (0, \pi)$ with $\alpha_1 + \ldots + \alpha_n = \pi$ and $\alpha_k \neq \frac{\pi}{2}$ for all k the following equality holds

$$\sum_{i=1}^{n} \tan \alpha_i = \frac{\sum_{i=1}^{n} \cot \alpha_i}{\prod_{i=1}^{n} \cot \alpha_i}.$$

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

J149. Let $ABCD$ be a quadrilateral with $\angle A > 60^\circ$. Prove that

$$AC^2 < 2(BC^2 + CD^2).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J150. Let n be an integer greater than 2. Find all real numbers x such that $\{x\} \le \{nx\}$, where $\{a\}$ denotes the fractional part of a.

Proposed by Dorin Andrica, Babeş-Bolyai University, Romania and Mihai Piticari, Dragoş-Vodă National College, Romania

J151. Let $a \ge b \ge c > 0$. Prove that

$$(a-b+c)\left(\frac{1}{a}-\frac{1}{b}+\frac{1}{c}\right) \ge 1.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J152. Let $a, b, c > 0$. Prove that the following inequality holds

$$\frac{a+b}{a+b+2c}+\frac{b+c}{b+c+2a}+\frac{c+a}{c+a+2b}+\frac{2(ab+bc+ca)}{3(a^2+b^2+c^2)} \le \frac{13}{6}.$$

Proposed by Andrei Răzvan Băleanu, George Coşbuc College, Motru, Romania

J153. Find all integers n such that $n^2 + 2010n$ is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J154. Let ABC be a triangle and let $MNPQ$ be a rectangle inscribed in the triangle such that $M, N \in BC$, $P \in AC$, $Q \in AB$. Prove that

$$\operatorname{area} MNPQ \le \frac{1}{2} \operatorname{area} ABC.$$

Proposed by Dorin Andrica, Babeş-Bolyai Univ., Cluj-Napoca, Romania

J155. Find all n for which there are n consecutive integers whose sum of squares is a prime.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J156. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) + f(x + y)$ is a rational number for all real numbers x and all $y > 0$. Prove that $f(x)$ is a rational number for all real numbers x.

Proposed by Bogdan Enescu, B.P. Haşdeu National College, Buzău, Romania

J157. Evaluate

$$1^2 + 2^2 + 3^2 - 4^2 - 5^2 + 6^2 + 7^2 + 8^2 - 9^2 - 10^2 + \ldots - 2010^2,$$

where each three consecutive signs $+$ are follows by two signs $-$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J158. Let n be a positive integer relatively prime with 10. Prove that the hundreds digit of n^{20} is even.

Proposed by Badar Al-Ghamdi, Saudi Arabia

J159. Find all integers n for which $9n+16$ and $16n+9$ are both perfect squares.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J160. Let ABC be a triangle with $\angle A = 90°$ and let d be a line passing through the incenter of the triangle and intersecting sides AB and AC in P and Q, respectively. Find the minimun of $AP \cdot AQ$.

Proposed by Dorin Andrica, Babeş-Bolyai Univ., Cluj-Napoca, Romania

J161. Let a, b, c be positive real numbers such that $a + b + c + 2 = abc$. Find the minimum of

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Proposed by Abdulmajeed Al-Gasem, Saudi Arabia

J162. Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove that

$$\frac{a_1}{(1+a_1)^2}+\frac{a_2}{(1+a_1+a_2)^2}+\ldots+\frac{a_n}{(1+a_1+\ldots+a_n)^2} \leq \frac{a_1+\ldots+a_n}{1+a_1+\ldots+a_n}.$$

Proposed by Neculai Stanciu, Buzău, Romania

J163. Let a, b, c be nonzero real numbers such that $ab+bc+ca \geq 0$. Prove that

$$\frac{ab}{a^2+b^2}+\frac{bc}{b^2+c^2}+\frac{ca}{c^2+a^2} \geq -\frac{1}{2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J164. If x and y are positive real numbers such that

$$\left(x+\sqrt{x^2+1}\right)\left(y+\sqrt{y^2+1}\right)=2011,$$

find the minimum possible value of $x+y$.

Proposed by Neculai Stanciu, Buzău, Romania

J165. Find all triples (x, y, z) of integers satisfying the system of equations

$$\begin{cases} (x^2+1)(y^2+1)+\dfrac{z^2}{10}=2010 \\ (x+y)(xy-1)+14z=1985. \end{cases}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J166. Let P be a point inside triangle ABC and let d_a, d_b, d_c be the distances from point P to the sides of the triangle. Prove that

$$\frac{K}{d_a d_b d_c} \geq \frac{s}{Rr}$$

where K is the area of the pedal triangle of P and s, R, r are the semiperimeter, circumradius, and inradius of triangle ABC.

Proposed by Andrei Răzvan Băleanu, Motru, Romania

J167. Let a, b, c be real numbers greater than 1 such that

$$\frac{b+c}{a^2-1} + \frac{c+a}{b^2-1} + \frac{a+b}{c^2-1} \geq 1.$$

Prove that

$$\left(\frac{bc+1}{a^2-1}\right)^2 + \left(\frac{ca+1}{b^2-1}\right)^2 + \left(\frac{ab+1}{c^2-1}\right)^2 \geq \frac{10}{3}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J168. Let n be a positive integer. Find the least positive integer a such that the system

$$\begin{cases} x_1 + x_2 + \ldots + x_n = a \\ x_1^2 + x_2^2 + \ldots + x_n^2 = a \end{cases}$$

has no integer solutions.

Proposed by Dorin Andrica, Babeş-Bolyai Univ., Cluj-Napoca, Romania

J169. If $x, y, z > 0$ and $x + y + z = 1$, find the maximum of

$$E(x, y, z) = \frac{xy}{x+y} + \frac{yz}{y+z} + \frac{zx}{z+x}.$$

Proposed by Dorin Andrica, Babeş-Bolyai Univ., Cluj-Napoca, Romania

J170. In the interior of a regular pentagon $ABCDE$ consider the point M such that the triangle MDE is equilateral. Find the angles of triangle AMB.

Proposed by Cătălin Barbu, V. Alecsandri College, Bacău, Romania

J171. If different letters represent different digits, could the addition

$$\begin{array}{cccccc} & A & X & X & X & U \\ & & B & X & X & V \\ & & C & X & X & Y \\ + & D & E & X & X & Z \\ \hline & X & X & X & X & X \end{array}$$

be correct?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J172. Let P be a point situated in the interior of an equilateral triangle ABC and let A', B', C' be the intersections of AP, BP, CP with sides BC, CA, AB, respectively. Find all P such that

$$A'B^2 + B'C^2 + C'A^2 = AB'^2 + BC'^2 + CA'^2.$$

Proposed by Cătălin Barbu, V. Alecsandri College, Bacău, Romania

J173. Let a and b be rational numbers such that

$$|a| \le \frac{47}{|a^2 - 3b^2|} \quad \text{and} \quad |b| \le \frac{52}{|b^2 - 3a^2|}.$$

Prove that $a^2 + b^2 \le 17$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J174. The incircle of triangle ABC touches sides BC, CA, AB at D, E, F, respectively. Let K be a point on side BC and let M be the point on the line segment AK such that $AM = AE = AF$. Denote by L and N the incenters of triangles ABK and ACK, respectively. Prove that K is the foot of the altitude from A if and only if $DLMN$ is a square.

Proposed by Bogdan Enescu, B.P. Haşdeu National College, Buzău, Romania

J175. Let $a, b \in \left(0, \frac{\pi}{2}\right)$ such that

$$\sin^2 a + \cos 2b \ge \frac{1}{2}\sec a \quad \text{and} \quad \sin^2 b + \cos 2a \ge \frac{1}{2}\sec b.$$

Prove that $\cos^6 a + \cos^6 b \ge \dfrac{1}{2}$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J176. Solve in positive real numbers the system of equations

$$\begin{cases} x_1 + x_2 + \ldots + x_n = 1 \\ \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} + \frac{1}{x_1 x_2 \ldots x_n} = n^3 + 1. \end{cases}$$

Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania