Preface

Inspired by appreciative and constructive feedback from our faithful readers, we present Mathematical Reflections: two wonderful years, a compilation and revision of the 2016 and 2017 volumes from the online journal of the same name. Since its inception in January 2006, the journal has attracted readers and contributors from all over the world. It successfully brings together enthusiasts with different mathematical and cultural backgrounds for the common purpose of making mathematics even more elegant and exciting. The journal publishes six issues each year.

This book is aimed at high school students, participants in math competitions, undergraduates, and anyone who has a passion for mathematics. Many of the problems, solutions, and articles were submitted by passionate readers and they all require creativity, experience, and comprehensive mathematical knowledge. In publishing this volume, our efforts were especially geared towards correcting and improving on many of the solutions and articles, so that our audience can enjoy them even more.

The articles herein focus on a variety of interesting topics outside of the mainstream curriculum. Students can expand their mathematical horizons through material outside the scope of their regular classes. For instructors, the articles provide an intriguing opportunity to move away from a structured curriculum, motivate the problem discussed, and guide students through to the invaluable moments of discovery. All of the featured problems are original. In order to make the material more accessible to the readers, this book as well as the journal is organized by the mathematical ability required to solve the problems. The junior section presents introductory problems (though not necessarily easy ones). The senior and Olympiad sections are for students preparing for national or international contests such as the United States of America Mathematical Olympiad (USAMO) or the International Mathematical Olympiad (IMO). Last, but not least, the undergraduate section offers college students a unique opportunity to solve non-routine problems in areas such as linear algebra, calculus, or graph theory.

This book could not have seen the light of day without the loyal readers and collaborators of the online journal. We would like to thank them all and express my gratitude for their continuous support. We sincerely hope that others will follow in their footsteps and continue to carry the baton, so that the mission of the journal to offer the opportunity for mathematics enthusiasts to publish original and interesting work will be fulfilled in the future years as well.

A special thank you to Richard Stong and Alessandro Ventullo for the many improvements in the manuscript.

If your are interested in reading the journal, please visit its website:

http://awesomemath.org/mathematical-reflections/.

Readers may submit articles, problems, or solutions to

reflections@awesomemath.org

Proceeds from the sale of this book will be used to sustain the journal in future years. Enjoy the problems and articles!

Dr. Titu Andreescu
Maxim Ignatiuc

Contents

PROBLEMS

1.1 Junior Problems

J361. Solve in positive integers the equation

$$\frac{x^2 - y}{8x - y^2} = \frac{y}{x}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J362. Let a, b, c, d be real numbers such that $abcd = 1$. Prove that the following inequality holds:

$$ab + bc + cd + da \leq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}.$$

Proposed by Mircea Becheanu, University of Bucharest, Romania

J363. Solve in integers the system of equations

$$\begin{aligned} x^2 + y^2 - z(x + y) &= 10 \\ y^2 + z^2 - x(y + z) &= 6 \\ z^2 + x^2 - y(z + x) &= -2. \end{aligned}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J364. Consider a triangle ABC with circumcircle ω. Let O be the center of ω and let D, E, F be the midpoints of arcs BC, CA, AB, not containing

A, B, C, respectively. Let DO intersect ω again at a point A'. Define B' and C' similarly. Prove that

$$\frac{[ABC]}{[A'B'C']} \leq 1.$$

Note that $[X]$ denotes the area of figure X.

Proposed by Taimur Khalid, Coral Academy of Science, Las Vegas, USA

J365. Let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers such that

$$x_1 + x_2 + \ldots + x_n = 1.$$

Find the minimum possible value of

$$\sqrt{x_1 + 1} + \sqrt{2x_2 + 1} + \ldots + \sqrt{nx_n + 1}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J366. Prove that in any triangle ABC,

$$\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} \leq \sqrt{6 + \frac{r}{2R}} - 1.$$

Proposed by Florin Stănescu, Găeşti, Romania

J367. Let a and b be positive real numbers. Prove that

$$\frac{1}{4a} + \frac{3}{a+b} + \frac{1}{4b} \geq \frac{4}{3a+b} + \frac{4}{a+3b}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J368. Find the best constants α and β such that $\alpha < \frac{x}{2x+y} + \frac{y}{x+2y} \leq \beta$ for all $x, y \in (0, \infty)$.

Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

J369. Solve the equation

$$\sqrt{1+\frac{1}{x+1}}+\frac{1}{\sqrt{x+1}}=\sqrt{x}+\frac{1}{\sqrt{x}}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J370. Triangle ABC has sides lengths $BC=a$, $CA=b$, $AB=c$. If

$$(a^2+b^2+c^2)^2=4a^2b^2+b^2c^2+4c^2a^2,$$

find all possible values of $\angle A$.

Proposed by Adrian Andreescu, Dallas, TX, USA

J371. Prove that for all positive integers n,

$$\binom{n+3}{2}+6\binom{n+4}{4}+90\binom{n+5}{6}$$

is the sum of two perfect cubes.

Proposed by Alessandro Ventullo, Milan, Italy

J372. In triangle ABC, $\frac{\pi}{7}<A\le B\le C<\frac{5\pi}{7}$. Prove that

$$\sin\frac{7A}{4}-\sin\frac{7B}{4}+\sin\frac{7C}{4}>\cos\frac{7A}{4}-\cos\frac{7B}{4}+\cos\frac{7C}{4}.$$

Proposed by Adrian Andreescu, Dallas, TX, USA

J373. Let a,b,c be real numbers greater than -1. Prove that

$$(a^2+b^2+2)(b^2+c^2+2)(c^2+a^2+2)\ge(a+1)^2(b+1)^2(c+1)^2.$$

Proposed by Adrian Andreescu, Dallas, TX, USA

J374. Let a,b,c be positive real numbers such that $a+b+c\ge 3$. Prove that

$$abc+2\ge\frac{9}{a^3+b^3+c^3}.$$

Proposed by Mehmet Berke, Isler, Denizli, Turkey

J375. Solve in real numbers the equation

$$\sqrt[3]{x}+\sqrt[3]{y}=\frac{1}{2}+\sqrt{x+y+\frac{1}{4}}.$$

Proposed by Adrian Andreescu, Dallas, TX, USA

J376. Let α, β, γ be the angles of a triangle. Prove that

$$\frac{1}{5-4\cos\alpha}+\frac{1}{5-4\cos\beta}+\frac{1}{5-4\cos\gamma}\geq 1.$$

Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia

J377. Let ABC be a triangle with $\angle A \leq 90^\circ$. Prove that

$$\sin^2\frac{A}{2}\leq\frac{m_a}{2R}\leq\cos^2\frac{A}{2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J378. Let P be a point in the interior of the triangle ABC such that $\angle BAP = 105^\circ$, and let D, E, F be the intersections of BP, CP, DE with the sides AC, AB, BC, respectively. Assume that the point B lies between C and F and that $\angle BAF = \angle CAP$. Find $\angle BAC$.

Proposed by Marius Stănean, Zalău, Romania

J379. Prove that for any nonnegative real numbers a, b, c the following inequality holds:

$$(a-2b+4c)(-2a+4b+c)(4a+b-2c)\leq 27abc.$$

Proposed by Adrian Andreescu, Dallas, TX, USA

J380. Let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers such that

$$x_1+x_2+\ldots+x_n=1.$$

(a) Find the minimum value of

$$x_1\sqrt{1+x_1}+x_2\sqrt{1+x_2}+\ldots+x_n\sqrt{1+x_n}.$$

(b) Find the maximum value of

$$\frac{x_1}{1+x_2}+\frac{x_2}{1+x_3}+\ldots+\frac{x_n}{1+x_1}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J381. Let x, y, z be positive real numbers such that $x+y+z=3$. Prove that

$$\frac{xy}{4-y}+\frac{yz}{4-z}+\frac{zx}{4-x}\leq 1.$$

Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia

J382. Find all triples (x, y, z) of real numbers with $x, y, z > 1$ satisfying

$$\left(\frac{x}{2}+\frac{1}{x}-1\right)\left(\frac{y}{2}+\frac{1}{y}-1\right)\left(\frac{z}{2}+\frac{1}{z}-1\right)=\left(1-\frac{x}{yz}\right)\left(1-\frac{y}{zx}\right)\left(1-\frac{z}{xy}\right).$$

Proposed by Alessandro Ventullo, Milan, Italy

J383. Let ABC be a triangle with $AB=AC$ and $\angle BAC=72^\circ$. Let D and E be the points on sides AB and AC, respectively, such that $\angle ACD=12^\circ$ and $\angle ABE=30^\circ$. Prove that $DE=CE$.

Proposed by Marius Stănean, Zalău, Romania

J384. In triangle ABC, $A<B<C$. Prove that

$$\cos\frac{A}{2}\csc\frac{B-C}{2}+\cos\frac{B}{2}\csc\frac{C-A}{2}+\cos\frac{C}{2}\csc\frac{A-B}{2}<0.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J385. If the equalities

$$2(a+b) - 6c - 3(d+e) = 6$$
$$3(a+b) - 2c + 6(d+e) = 2$$
$$6(a+b) + 3c - 2(d+e) = -3$$

holds simultaneously, evaluate $a^2 - b^2 + c^2 - d^2 + e^2$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J386. Find all real solutions to the system of equations

$$x + yzt = y + ztx = z + txy = t + xyz = 2.$$

Proposed by Mohamad Kouroshi, Tehran, Iran

J387. Find all digits a, b, c, x, y, z for which $\overline{abc}$, $\overline{xyz}$, and $\overline{abcxyz}$ are all perfect squares (no leading zeros allowed).

Proposed by Adrian Andreescu, Dallas, TX, USA

J388. Let $ABCD$ be a cyclic quadrilateral with $AB = AD$. Points M and N are taken on sides CD and BC, respectively, such that $DM + BN = MN$. Prove that the circumcenter of triangle AMN lies on segment AC.

Proposed by Hayk Sedrakyan, Paris, France

J389. Solve in real numbers the system of equations

$$(x^2 - y + 1)(y^2 - x + 1) = 2[(x^2 - y)^2 + (y^2 - x)^2] = 4.$$

Proposed by Alessandro Ventullo, Milan, Italy

J390. Let ABC be a triangle. Points D, D' lie on side BC, points E, E' lie on side AC and points F, F' lie on side AB such that $AD = AD' = BE = BE' = CF = CF'$. Prove that if AD, BE, CF are concurrent, then so are AD', BE', CF'.

Proposed by Josef Tkadlec, Vienna, Austria