Preface

As a sequel to the previous *116 Algebraic Inequalities from the AwesomeMath Year-round Program* and *118 Inequalities for Mathematics Competitions*, this book delves into other elementary techniques but also powerful methods and generalizations for constrained optimization in the theory of inequalities. For this, we selected the most evocative examples from Mathematical Reflections, online math forums, and various math competitions. A vast amount of the problems were created by the authors of this book.

In the first section, the reader will encounter each of Abel's, Newton's, Maclaurin's, and Blundon's inequalities, the method of mathematical induction, the "mixing" and "stronger mixing" variable methods, and the method of Lagrange multipliers. We believe that the 100 examples are chosen in such a way as to contribute a thorough exposure and insightful analysis of these concepts for high school and college students, teachers, or anyone with a passion for mathematics.

The next sections are dedicated to the proposed problems, which are divided into introductory and advanced. Each problem is given at least one complete solution, and many problems have multiple solutions, useful in developing the necessary and broad array of mathematical strategies and techniques used in competitions.

We would like to thank Richard Stong for his pertinent suggestions and observations. Also, thanks to all Mathematical Reflections contributors and math enthusiasts who post problems on the AoPS website.

Enjoy the book!

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Some Classical and Some New Inequalities

1.1 Simple Techniques for Proving Inequalities

In this section, we present, through suggestive examples, some techniques for proving inequalities. Let's start with a few examples of applying the Cauchy-Schwarz Inequality.

Example 1. (Romanian NMO 2008) Let $a, b \in [0,1]$. Prove that

$$\frac{1}{1+a+b} \leq 1 - \frac{a+b}{2} + \frac{ab}{3}.$$

Solution. If $b = 0$ the inequality is reduced to $a(1-a) \geq 0$, obviously true. Suppose $a, b > 0$, with the substitutions $a = \frac{1}{x+1}$, $b = \frac{1}{y+1}$, $x, y \geq 0$ the inequality becomes

$$\frac{(x+1)(y+1)}{xy+2x+2y+3} \leq 1 - \frac{x+y+2}{2(x+1)(y+1)} + \frac{1}{3(x+1)(y+1)},$$

or

$$6(x+1)^2(y+1)^2 \leq (xy+2x+2y+3)(6xy+3x+3y+2),$$

or

$$(xy+2x+2y+3)\left(xy+\frac{x}{2}+\frac{y}{2}+\frac{1}{3}\right) \geq (xy+x+y+1)^2$$

which follows from the Cauchy-Schwarz Inequality. The equality holds when $x = y = 0$ so $a = b = 1$. $\square$

Remark. Using this we obtain the following inequality in three variables: (USAMO 1980) Prove that for numbers a, b, c in the interval $[0, 1]$,

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

Solution. We have

$$\frac{a}{1+b+c} \leq a - \frac{ab+ac}{2} + \frac{abc}{3},$$
$$\frac{b}{1+c+a} \leq b - \frac{bc+ab}{2} + \frac{abc}{3},$$
$$\frac{c}{1+a+b} \leq c - \frac{ca+bc}{2} + \frac{abc}{3}.$$

Summing up these three inequalities, we get

$$\sum_{cyc} \frac{a}{b+c+1} \leq a+b+c-ab-bc-ca+abc = 1-(1-a)(1-b)(1-c).$$

$\square$

Example 2. (Nguyen Phi Hung) Let a, b, c be nonnegative real numbers such that $a^2 + b^2 + c^2 = 8$. Prove that

$$4(a+b+c-4) \leq abc.$$

Solution. The inequality can be rewritten as

$$4(a+b) + c(4-ab) \leq 16.$$

By the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}[4(a+b)+c(4-ab)]^2 &\leq \left[(a+b)^2 + c^2\right]\left[16 + (4-ab)^2\right] \\ &= (8+2ab)(32-8ab+a^2b^2).\end{aligned}$$

If we denote $x = ab$, we need to show that

$$(x+4)(x^2 - 8x + 32) \leq 128$$

which is equivalent to

$$x^2(x-4) \leq 0.$$

But this is true because

$$x \leq \frac{a^2+b^2}{2} \leq \frac{a^2+b^2+c^2}{2} = 4.$$

The equality holds when $a = b = 2$, $c = 0$ or any cyclic permutation. □

Example 3. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{2a^3+3a+2} + \frac{1}{2b^3+3b+2} + \frac{1}{2c^3+3c+2} \geq \frac{3}{7}.$$

Solution. In order to be able to apply the Cauchy-Schwarz Inequality, we first bring the inequality to an equivalent form that allows its application. For this, we homogenize the inequality with the substitutions

$$a = \frac{yz}{x^2},\ b = \frac{zx}{y^2},\ c = \frac{xy}{z^2},$$

the inequality becomes

$$\frac{x^6}{2x^6+3x^4yz+2y^3z^3} + \frac{y^6}{2y^6+3y^4zx+2z^3x^3} + \frac{z^6}{2z^6+3z^4xy+2x^3y^3} \geq \frac{3}{7}.$$

Now, we can apply the Cauchy-Schwarz Inequality, i.e.

$$LHS \geq \frac{(x^3+y^3+z^3)^2}{2(x^6+y^6+z^6)+3xyz(x^2+y^3+z^3)+2(y^3z^3+z^3x^3+x^3y^3)}.$$

Therefore, it remains to prove that

$$x^6+y^6+z^6+8(x^3y^3+y^3z^3+z^3x^3) \geq 9xyz(x^3+y^3+z^3).$$

But this results from *Example 18* of the book *116 Algebraic Inequalities from the AwesomeMath Year-round Program.* We have

$$x^6 + y^6 + 16x^3y^3 \geq 9x^2y^2(x^2 + y^2) \iff (x - y)^4(x^2 + 4xy + y^2) \geq 0,$$

and adding this with the other two similar ones, we get

$$\begin{aligned} 2\sum_{cyc} x^6 + 16\sum_{cyc} x^3y^3 &\geq 9x^4(y^2 + z^2) + 9y^4(z^2 + x^2) + 9z^4(x^2 + y^2) \\ &\geq 18x^4yz + 18y^4zx + 18z^4xy \\ &= 18xyz(x^3 + y^3 + z^3), \end{aligned}$$

as desired. □

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Next, a technique of homogenizing one inequality.

Example 4. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{1 + ab^2} + \frac{b^3}{1 + bc^2} + \frac{c^3}{1 + ca^2} \geq \frac{3abc}{1 + abc}.$$

Solution. Let $k > 0$ such that $1 = kabc$, so we need to prove that

$$\frac{a^3}{kabc + ab^2} + \frac{b^3}{kabc + bc^2} + \frac{c^3}{kabc + ca^2} \geq \frac{3abc}{kabc + abc} = \frac{3}{k + 1},$$

or

$$\frac{a^2}{kbc + b^2} + \frac{b^2}{kac + c^2} + \frac{c^2}{kab + a^2} \geq \frac{3}{k + 1}.$$

By the Cauchy-Schwarz Inequality, we have

$$\begin{aligned} LHS &= \frac{a^4}{ka^2bc + a^2b^2} + \frac{b^4}{kab^2c + b^2c^2} + \frac{c^4}{kabc^2 + c^2a^2} \\ &\geq \frac{(a^2 + b^2 + c^2)^2}{kabc(a + b + c) + a^2b^2 + b^2c^2 + c^2a^2}. \end{aligned}$$

Hence, we need to prove that

$$(k+1)(a^2+b^2+c^2)^2 \geq 3kabc(a+b+c)+3(a^2b^2+b^2c^2+c^2a^2),$$

or

$$(k+1)(a^4+b^4+c^4)+(2k-1)(a^2b^2+b^2c^2+c^2a^2) \geq 3kabc(a+b+c).$$

This inequality follows by adding the two inequalities

$$(k+1)(a^4+b^4+c^4) \geq (k+1)(a^2b^2+b^2c^2+c^2a^2)$$

and

$$3k(a^2b^2+b^2c^2+c^2a^2) \geq 3kabc(a+b+c),$$

both of which are special cases of the inequality $x^2+y^2+z^2 \geq xy+yz+zx$. The equality holds when $a=b=c$. □

Another common class of inequalities are those which involve products of the form

$$(a+b)(a+c) = a^2+ab+bc+ca = a(a+b+c)+bc,$$

respectively

$$(b+c)(b+a) = b^2+ab+bc+ca = b(a+b+c)+ca,$$
$$(c+a)(c+b) = c^2+ab+bc+ca = c(a+b+c)+ab,$$

for the numbers a, b, c satisfying a constraint $a+b+c=k$ or $ab+bc+ca=k$, where k is a constant. Here are some suggestive examples.

Example 5. (Dan Moldovan, Romanian NMO 2018) Let $a, b, c \geq 0$, so that $ab+bc+ca=3$. Prove that

$$\frac{a}{a^2+7}+\frac{b}{b^2+7}+\frac{c}{c^2+7} \leq \frac{3}{8}.$$

Solution. Rewrite the inequality as follows

$$\frac{a}{a^2+ab+bc+ca+4}+\frac{b}{b^2+ab+bc+ca+4}+\frac{c}{c^2+ab+bc+ca+4}\le\frac{3}{8},$$

$$\frac{a}{(a+b)(a+c)+4}+\frac{b}{(b+c)(b+a)+4}+\frac{c}{(c+a)(c+b)+4}\le\frac{3}{8}.$$

By the AM-GM Inequality, we have

$$(a+b)(a+c)+4\ge 4\sqrt{(a+b)(a+c)}$$

and similarly for any other two variables. Hence, it suffices to show that

$$\frac{a}{\sqrt{(a+b)(a+c)}}+\frac{b}{\sqrt{(b+c)(b+a)}}+\frac{c}{\sqrt{(c+a)(c+b)}}\le\frac{3}{2}.$$

By the AM-GM Inequality,

$$\frac{a}{\sqrt{(a+b)(a+c)}}=\sqrt{\frac{a}{a+b}\cdot\frac{a}{a+c}}\le\frac{1}{2}\left(\frac{a}{a+b}+\frac{a}{a+c}\right),$$

and similarly

$$\frac{b}{\sqrt{(b+a)(b+c)}}\le\frac{1}{2}\left(\frac{b}{a+b}+\frac{b}{b+c}\right),$$

$$\frac{c}{\sqrt{(c+a)(c+b)}}\le\frac{1}{2}\left(\frac{c}{c+a}+\frac{c}{b+c}\right).$$

Summing up the above inequalities we get the desired inequality. □

Example 6. Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{bc}{a^2+3}+\frac{ca}{b^2+3}+\frac{ab}{c^2+3}\le\frac{9}{4(ab+bc+ca)}.$$

Solution. We have

$$3 = \frac{(a+b+c)^2}{3} \geq ab + ac + bc.$$

Then

$$LHS \leq \sum \frac{bc}{a^2+ab+ac+bc} = \sum \frac{bc}{(a+b)(a+c)} = 1 - \frac{2abc}{(a+b)(a+c)(b+c)}.$$

Hence, it suffices to show that

$$1 - \frac{2abc}{(a+b)(a+c)(b+c)} \leq \frac{(a+b+c)^2}{4(ab+bc+ca)},$$

or

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{8abc}{(a+b)(a+c)(b+c)} \geq 2.$$

Without loss of generality, we may assume that $a \geq b \geq c$, then

$$\begin{aligned}\frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{8abc}{(a+b)(a+c)(b+c)} &\geq \frac{a^2+b^2+2c^2}{(c+a)(c+b)} + \frac{8abc}{(a+b)(a+c)(b+c)} \\ &= \frac{(a+b-2c)(a-b)^2}{(a+b)(b+c)(c+a)} + 2 \geq 2.\end{aligned}$$

□

Example 7. (Dan Moldovan, Romanian NMO 2019) If $a, b, c \in (0, \infty)$ such that $a+b+c=3$, then

$$\frac{a}{3a+bc+12} + \frac{b}{3b+ca+12} + \frac{c}{3c+ab+12} \leq \frac{3}{16}.$$

Solution. By the Cauchy-Schwarz Inequality, we have

$$\frac{16}{3a+bc+12} \leq \frac{1}{3a+bc} + \frac{9}{12} = \frac{1}{3a+bc} + \frac{3}{4}$$

so

$$\sum_{cyc} \frac{a}{3a+bc+12} \leq \frac{1}{16}\left(\sum_{cyc} \frac{a}{3a+bc} + \frac{9}{4}\right).$$

But

$$\sum_{cyc} \frac{a}{3a+bc} = \sum_{cyc} \frac{a}{(a+b)(a+c)} = \frac{2(ab+bc+ca)}{(a+b)(b+c)(c+a)}$$
$$= \frac{2(a+b+c)(ab+bc+ca)}{3(a+b)(b+c)(c+a)} \le \frac{3}{4}$$

since

$$9(a+b)(b+c)(c+a) \ge 8(a+b+c)(ab+bc+ca) \Longleftrightarrow \sum_{cyc} a(b-c)^2 \ge 0,$$

so we are done. $\square$

Example 8. (Marius Stănean) Let $a, b, c > 0$ such that $a+b+c=1$. Prove that

$$\frac{20a}{a+bc} + \frac{15b}{b+ca} + \frac{12c}{c+ab} \le 36.$$

Solution. Rewrite the inequality as follows

$$\frac{20a}{a(a+b+c)+bc} + \frac{15b}{b(a+b+c)+ca} + \frac{12c}{c(a+b+c)+ab} \le 36,$$

$$\frac{20a}{(a+b)(a+c)} + \frac{15b}{(a+b)(b+c)} + \frac{12c}{(b+c)(c+a)} \le 36,$$

$$20a(b+c) + 15b(c+a) + 12c(a+b) \le 36(a+b)(b+c)(c+a),$$

$$20a(b+c) + 15b(c+a) + 12c(a+b) \le 36(a+b+c)(ab+bc+ca) - 36abc,$$

$$ab + 9bc + 4ac \ge 36abc,$$

$$\frac{9}{a} + \frac{4}{b} + \frac{1}{c} \ge 36.$$

This inequality follows easily by the Cauchy-Schwarz Inequality,

$$\frac{9}{a} + \frac{4}{b} + \frac{1}{c} \ge \frac{(3+2+1)^2}{a+b+c} = 36.$$

The equality holds when $a = \frac{1}{2}$, $b = \frac{1}{3}$, $c = \frac{1}{6}$. $\square$

Example 9. Let a, b, c be positive real numbers such that $a+b+c=1$. Find maximum value of the following expression

$$E = \frac{a}{a+bc} + \frac{b}{b+ca} + \frac{\sqrt{abc}}{c+ab}.$$

First Solution. Considering the first two terms on the expression E, we have

$$\begin{aligned}
\frac{a}{a+bc} + \frac{b}{b+ca} &= \frac{a}{(a+b)(a+c)} + \frac{b}{(a+b)(b+c)} = \frac{2ab+bc+ca}{(a+b)(b+c)(c+a)} \\
&= \frac{(2ab+bc+ca)(a+b+c)}{(a+b)(b+c)(c+a)} \\
&= \frac{(a+b)(b+c)(c+a)+abc+ab(a+b+c)}{(a+b)(b+c)(c+a)} \\
&= 1 + \frac{ab(a+b+2c)}{(a+b)(b+c)(c+a)}.
\end{aligned}$$

But

$$\frac{(a+b+2c)^2}{(c+a)(b+c)} - 4 = \frac{(a-b)^2}{(c+a)(b+c)} \le \frac{(a-b)^2}{ab} = \frac{(a+b)^2}{ab} - 4$$

so

$$\frac{ab(a+b+2c)}{(a+b)(b+c)(c+a)} \le \sqrt{\frac{ab}{(c+a)(b+c)}}.$$

Therefore

$$\begin{aligned}
E &\le 1 + \sqrt{\frac{ab}{(c+a)(b+c)}} + \frac{\sqrt{abc}}{(c+a)(b+c)} \\
&= 1 + \sqrt{\frac{ab}{(c+a)(b+c)}}\left(1 + \sqrt{\frac{c(a+b+c)}{(c+a)(b+c)}}\right) \\
&= 1 + \sqrt{\frac{ab}{(c+a)(b+c)}}\left(1 + \sqrt{\frac{(c+a)(b+c)-ab}{(c+a)(b+c)}}\right) \\
&= 1 + \sqrt{\frac{ab}{(c+a)(b+c)}}\left(1 + \sqrt{1 - \frac{ab}{(c+a)(b+c)}}\right).
\end{aligned}$$

If we denote $t = \sqrt{1 - \dfrac{ab}{(c+a)(b+c)}}$ and using the AM-GM Inequality, we deduce that

$$E \le 1 + (1+t)\sqrt{1-t^2} = 1 + \sqrt{(1-t)(1+t)^3} = 1 + \sqrt{\frac{(3-3t)(1+t)^3}{3}}$$
$$\le 1 + \sqrt{\frac{1}{3}\left(\frac{3-3t+1+t+1+t+1+t}{4}\right)^4} = 1 + \frac{3\sqrt{3}}{4}.$$

The equality holds when $a = b = 2\sqrt{3} - 3$, $c = 7 - 4\sqrt{3}$. □

Second Solution. Denote $x^2 = \dfrac{bc}{a}$, $y^2 = \dfrac{ca}{b}$, $z^2 = \dfrac{ab}{c}$, $x, y, z > 0$, then $a = yz$, $b = zx$, $c = xy$ and $xy + yz + zx = 1$. The expression E written in terms of x, y, z is

$$E = \frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{z}{1+z^2}.$$

The condition $xy + yz + zx = 1$ is equivalent to the existence of a triangle ABC such that $x = \tan\dfrac{A}{2}$, $y = \tan\dfrac{B}{2}$, $z = \tan\dfrac{C}{2}$. Everything is now reduced to finding the maximum value of the expression

$$\begin{aligned}
E &= \cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \sin\frac{C}{2}\cos\frac{C}{2} \\
&= 1 + \frac{1}{2}(\cos A + \cos B) + \sin\frac{C}{2}\cos\frac{C}{2} \\
&= 1 + \cos\frac{A+B}{2}\cos\frac{A-B}{2} + \sin\frac{C}{2}\cos\frac{C}{2} \\
&= 1 + 2\sin\frac{C}{2}\cos\left(\frac{\pi}{4} - \frac{B}{2}\right)\cos\left(\frac{\pi}{4} - \frac{A}{2}\right) \\
&= 1 + 2\sin(\alpha+\beta)\cos\alpha\cos\beta,
\end{aligned}$$

where $\alpha, \beta \in \left(-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$ and $\alpha + \beta \ge 0$.

By the Cauchy-Schwarz Inequality and the AM-GM Inequality, we have

$$\begin{aligned}
(\sin(\alpha+\beta)\cos\alpha\cos\beta)^2 &= (\sin\alpha\cos\beta+\cos\alpha\sin\beta)^2\cos^2\alpha\cos^2\beta \\
&\le (\sin^2\alpha+\sin^2\beta)(\cos^2\beta+\cos^2\alpha)\cos^2\alpha\cos^2\beta \\
&= \frac{4}{3}\cos^2\alpha\cos^2\beta\left(\frac{3\sin^2\alpha+3\sin^2\beta}{2}\right)\left(\frac{\cos^2\alpha+\cos^2\beta}{2}\right) \\
&\le \frac{4}{3}\left(\frac{\cos^2\alpha+\cos^2\beta+\frac{3\sin^2\alpha+3\sin^2\beta}{2}+\frac{\cos^2\alpha+\cos^2\beta}{2}}{4}\right)^4 \\
&= \frac{27}{64}.
\end{aligned}$$

Therefore

$$E \le 1+\frac{3\sqrt{3}}{4}.$$

We get the maximum value for

$$\alpha=\beta=\frac{\pi}{6} \iff A=B=\frac{\pi}{6}, C=\frac{2\pi}{3}$$

so $x=y=2-\sqrt{3}, z=\sqrt{3}$ which means $a=b=2\sqrt{3}-3$, $c=7-4\sqrt{3}$. From this solution we can deduce that $\inf(E)=1$ when $C\to 0$, $A=B\to\dfrac{\pi}{2}$ which means $x=y\to 1, z\to 0$. □

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One method that can lead to success in proving symmetric inequalities is to establish an order of the variables, as already seen in the solution of Example 6. In other cases, the inequality can be reduced to an inequality in one variable.

Example 10. (IMO 1984) Prove that

$$0 \le ab+bc+ca-2abc \le \frac{7}{27},$$

where a, b and c are nonnegative real numbers satisfying $a+b+c=1$.

Solution. Let $c = \min\{a, b, c\}$ so that $c \leq \dfrac{1}{3}$. Denoting $I = ab+ac+bc-2abc$, we have

$$I = ab(1-2c) + c - c^2 \geq 0.$$

The equality holds when $c = 0$ and $ab = 0$.
For the right hand side of inequality applying the AM-GM Inequality, we have

$$\begin{aligned} I &\leq (1-2c)\frac{(a+b)^2}{4} + c - c^2 \\ &= (1-2c)\frac{(1-c)^2}{4} + c - c^2 \\ &= \frac{1+c^2-2c^3}{4} = \frac{1}{4} + \frac{c^2(1-2c)}{4} \\ &\leq \frac{1}{4} + \frac{1}{4}\left(\frac{c+c+1-2c}{3}\right)^3 = \frac{7}{27}. \end{aligned}$$

The equality holds when $a = b = c = \dfrac{1}{3}$. □

Example 11. Let a, b, c be real numbers such that

$$a + b + c = 6 \text{ and } a^2 + b^2 + c^2 = 14.$$

Prove that

$$(a-b)(b-c)(c-a) \leq 2.$$

Solution. The inequality being cyclical, we have two cases:
Case 1. $a \geq b \geq c$, and the inequality is obvious.
Case 2. $a \leq b \leq c$, then we have

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = 3(a^2+b^2+c^2) - (a+b+c)^2 = 42 - 36 = 6.$$

Also, by the Cauchy-Schwarz Inequality,

$$12 - 2(c-a)^2 = 2(a-b)^2 + 2(b-c)^2 \geq (a-b+b-c)^2 = (c-a)^2 \Longrightarrow (c-a)^2 \leq 4.$$

Finally, by the AM-GM Inequality,

$$(a-b)(b-c)(c-a) \le (c-a)\frac{(a-b+b-c)^2}{4} = \frac{(c-a)^3}{4} \le 2.$$

The equality holds if and only if $(a,b,c) = (1,2,3)$ or one of its cyclic permutations. □

Example 12. (2020 Caucasus Mathematical Olympiad) Let a, b, c be real numbers such that

$$abc + a + b + c = ab + bc + ca + 5.$$

Find the least possible value of $a^2 + b^2 + c^2$.

Solution. The given condition can be rewritten as

$$(a-1)(b-1)(c-1) = 4.$$

Without loss of generality, we can assume that $a \ge b \ge c$. It follows that $a - 1 > 0$, otherwise $(a-1)(b-1)(c-1) < 0$. Using the Cauchy-Schwarz Inequality and the AM-GM Inequality, we have

$$\begin{aligned}
a^2+b^2+c^2 &= a^2 - 2 + (b^2 + c^2 + 1 + 1) \\
&\ge a^2 - 2 + \frac{(b+c-1-1)^2}{4} \\
&\ge a^2 - 2 + (b-1)(c-1) = a^2 - 2 + \frac{4}{a-1} \\
&= (a-1)^2 + 2(a-1) + \frac{4}{a-1} - 1 \\
&\ge 7\sqrt[7]{(a-1)^2(a-1)(a-1)\cdot\frac{1}{a-1}\cdot\frac{1}{a-1}\cdot\frac{1}{a-1}\cdot\frac{1}{a-1}} - 1 = 6.
\end{aligned}$$

The equality holds if and only if $a = 2$, $b = c = -1$, and its cyclic permutations. □

Example 13. Let a, b, c be nonnegative real numbers such that $a+b+c=1$. Prove that
$$6(a^3+b^3+c^3)+1 \geq 5(a^2+b^2+c^2).$$

Solution. Let $a=\min\{a,b,c\}$, which implies $a \leq \dfrac{1}{3}$ and let
$$t=bc \leq \frac{(b+c)^2}{4}=\frac{(1-a)^2}{4}.$$

Then the inequality becomes
$$5a^2-6a^3-1+5(b+c)^2-10bc-6(b+c)((b+c)^2-3bc) \leq 0,$$

then
$$2(4-9a)bc+5a^2-6a^3-1+5(1-a)^2-6(1-a)^3 \leq 0,$$
and finally
$$(4-9a)t \leq (2a-1)^2.$$
Since $4-9a \geq 0$ it is enough to show that
$$(1-a)^2(4-9a) \leq 4(2a-1)^2 \Longleftrightarrow a(3a-1)^2 \geq 0,$$

which is clearly true. The equality holds when $a=b=c=\dfrac{1}{3}$ or $a=0$, $b=c=\dfrac{1}{2}$. □

Example 14. Let a, b, c be nonnegative real numbers such that $a+b+c=1$. Prove that
$$3(a^3+b^3+c^3)+1 \leq 4(a^2+b^2+c^2).$$

Solution. Let $a=\max\{a,b,c\}$ which implies $a \geq \dfrac{1}{3}$ and let
$$t=bc \leq \frac{(b+c)^2}{4}=\frac{(1-a)^2}{4}.$$

Then the inequality becomes

$$3a^3 - 4a^2 + 1 + 3(b+c)((b+c)^2 - 3bc) - 4(b+c)^2 + 8bc \le 0$$

then

$$(9a-1)bc + 3a^3 - 4a^2 + 1 + 3(1-a)^3 - 4(1-a)^2 \le 0,$$

and finally

$$(9a-1)t \le a - a^2.$$

Since $9a - 1 \ge 0$ it is enough to show that

$$(9a-1)(1-a)^2 \le 4a - 4a^2 \iff (a-1)(3a-1)^2 \le 0,$$

which is clearly true. The equality holds when $a = b = c = \dfrac{1}{3}$ or $a = 1$, $b = c = 0$. □

Example 15. (Ukrainian Mathematical Olympiad 2019) Let x, y, z be positive reals such that $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 3$. Prove that

$$(x-1)(y-1)(z-1) \le \frac{1}{4}(xyz - 1).$$

Solution. Denoting $a = \dfrac{1}{x}$, $b = \dfrac{1}{y}$, $c = \dfrac{1}{z}$, then $a+b+c = 3$ and the inequality is equivalent to the following

$$4(1-a)(1-b)(1-c) \le (1 - abc),$$

or

$$-9 + 4(ab + bc + ca) - 3abc \le 0,$$

or

$$-3 + \frac{4}{3}(ab + bc + ca) - abc \le 0,$$

or

$$\left(\frac{4}{3} - a\right)\left(\frac{4}{3} - b\right)\left(\frac{4}{3} - c\right) \le \frac{1}{27}$$

Suppose $c \le b \le a$. The inequality is trivial if the left hand side is negative. In the other two cases the AM-GM Inequality gives:

1. If $a \le \frac{4}{3}$, then

$$LHS \le \left(\frac{\frac{4}{3} - a + \frac{4}{3} - b + \frac{4}{3} - c}{3}\right)^3 = \frac{1}{27};$$

2. If $\frac{4}{3} \le b$, then $c \le \frac{1}{3}$ and

$$LHS \le \left(\frac{4}{3} - c\right)\left(\frac{a - \frac{4}{3} + b - \frac{4}{3}}{2}\right)^2 = \frac{(4-3c)(1-3c)^2}{4 \cdot 27} < \frac{1}{27}.$$

The equality holds when $a = b = c = 1$ which means $x = y = z = 1$. $\square$

Example 16. Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1) \ge 1.$$

Solution. Let $c = \min\{a, b, c\}$ which implies $c \le 1$. From the AM-GM Inequality, for $x \ge 0$,

$$x^2 - x + 1 \ge x^2 - \frac{x^2 + 1}{2} + 1 = \frac{x^2 + 1}{2}.$$

Thus, by the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1) &\ge \left(\frac{a^2 + 1}{2}\right)\left(\frac{b^2 + 1}{2}\right)(c^2 - c + 1)\\ &\ge \left(\frac{a + b}{2}\right)^2 (c^2 - c + 1)\\ &= \left(\frac{3 - c}{2}\right)^2 (c^2 - c + 1).\end{aligned}$$

It remains to prove that

$$(3 - c)^2(c^2 - c + 1) \ge 4,$$

or after factoring

$$(c-1)^2(c^2-5c+5) \geq 0,$$

which is clearly true for $c \leq 1$. The equality holds when $a = b = c = 1$. $\square$

Example 17. Let $1 \leq a, b, c, d \leq 9$ be real numbers. Prove that

$$abcd \geq \left(\frac{a+b+c+d}{4}\right)^2.$$

Solution. Note that the inequality is symmetric, so without loss of generality, we may assume $1 \leq a \leq b \leq c \leq d \leq 9$ and let

$$f(a,b,c,d) = 4\sqrt{abcd} - a - b - c - d.$$

We have

$$f(a,b,c,d) \geq f(1,b,c,d) \geq f(1,1,c,d) \geq f(1,1,1,d) \geq 0.$$

Indeed

$$\begin{aligned} f(a,b,c,d) - f(1,b,c,d) &= 4\sqrt{bcd}(\sqrt{a}-1) - a + 1 \\ &= (\sqrt{a}-1)\left(4\sqrt{bcd} - \sqrt{a} - 1\right) \geq 0. \\ f(1,b,c,d) - f(1,1,c,d) &= 4\sqrt{cd}(\sqrt{b}-1) - b + 1 \\ &= (\sqrt{b}-1)\left(4\sqrt{cd} - \sqrt{b} - 1\right) \geq 0. \\ f(1,1,c,d) - f(1,1,1,d) &= 4\sqrt{d}(\sqrt{c}-1) - c + 1 \\ &= (\sqrt{c}-1)\left(4\sqrt{d} - \sqrt{c} - 1\right) \geq 0. \end{aligned}$$

$$f(1,1,1,d) = 4\sqrt{d} - 3 - d = \left(\sqrt{d}-1\right)\left(3-\sqrt{d}\right) \geq 0.$$

The equality holds when $(a,b,c,d) = (1,1,1,1)$, $(1,1,1,9)$, or permutations. $\square$

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After we isolate the smallest (or largest) variable, we can apply a number of other techniques. For example, if the inequality is homogeneous, we can normalize by setting the sum of the other variables equal to 1. This can simplify the calculations as will be seen in the solutions of the following examples.

Example 18. Let x, y, z be positive real numbers. Prove that

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y} \geq \frac{(x+y+z)\left(x^2+y^2+z^2\right)}{2(xy+xz+yz)}.$$

Solution. Since the inequality is symmetrical and homogeneous, we can consider $z \leq y \leq x$ and $x+y=1$. Denoting $t=xy$, by the AM-GM Inequality it follows that $t \leq \frac{1}{4}$. Obviously we also have $z \leq \frac{1}{2}$. Hence, it remains to prove that

$$\frac{x^3+y^3+z(x^2+y^2)}{z^2+z(x+y)+xy}+\frac{z^2}{x+y} \geq \frac{((x+y)+z)((x+y)^2-2xy+z^2)}{2((x+y)z+xy)},$$

or

$$\frac{1-3t+z-2zt}{z^2+z+t}+z^2 \geq \frac{(1+z)(1-2t+z^2)}{2(z+t)}.$$

After clearing denominators, canceling a factor of $z+1$, and expanding, this becomes

$$-2(2-z)t^2+(2z^3+3z^2-4z+1)t+z(z-1)^2(z+1) \geq 0.$$

Denote the left hand side by $f(t)$. Since $f(t)$ is a quadratic function in t with negative leading coefficient, it follows that the minimum of $f(t)$ for $0 \leq t \leq \frac{1}{4}$ is attained at one of the endpoints. Since

$$f(0)=z(z-1)^2(z+1) \geq 0,$$
$$f\left(\frac{1}{4}\right)=\frac{z(2z-1)^2(2z+1)}{8} \geq 0,$$

it follows that $f(t) \geq 0$ for $t \in \left[0, \frac{1}{4}\right]$.

The equality holds when $z=0$, $x=y=\frac{1}{2}$ or $x=y=z=\frac{1}{2}$. □

Example 19. Let x, y, z be positive real numbers. Prove that

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \leq \frac{(x+y+z)\left(2x^2+2y^2+2z^2-xy-yz-zx\right)}{2(xy+xz+yz)}.$$

Solution. Since the inequality is symmetrical and homogeneous, we can consider $z \leq y \leq x$ and $x+y=1$. Denoting $t = xy$, by the AM-GM Inequality it follows that $t \leq \frac{1}{4}$. Obviously we also have $z \leq \frac{1}{2}$. Hence, it remains to prove that

$$\frac{x^3+y^3+z(x^2+y^2)}{z^2+z(x+y)+xy} + \frac{z^2}{x+y} \leq \frac{((x+y)+z)(2(x+y)^2-5xy+2z^2-(x+y)z)}{2((x+y)z+xy)},$$

or

$$\frac{1-3t+z-2zt}{z^2+z+t} + z^2 \leq \frac{(1+z)(2-5t+2z^2-z)}{2(z+t)}.$$

After clearing denominators, canceling a factor of $z+1$, and expanding, this becomes

$$(1-2z)t^2 - z^2(2z+5)t + z^2(1+z) \geq 0.$$

Denote the left hand side by $f(t)$. Notice that $f(t)$ is a quadratic function in t with nonnegative leading coefficient. If $0 < z \leq \frac{1}{4}$, then the discriminant of this quadratic is

$$-z^2(1+2z)(4-12z-9z^2-2z^3) \leq -z^2(1+2z)\left(4-3-\frac{9}{16}-\frac{1}{32}\right) < 0.$$

Since the discriminant is negative and the leading coefficient is nonnegative, we have $f(t) \geq 0$ for all t. If $\frac{1}{4} \leq z \leq \frac{1}{2}$, then writing $t = \frac{1}{4} - s$ we have

$$f\left(\frac{1}{4}-s\right) = (1-2z)s^2 + \frac{1}{2}(2z+1)(2z^2+4z-1)s + \frac{1}{16}(2z-1)^2(2z+1),$$

which is nonnegative for $s \geq 0$ since each coefficient is nonnegative.
The equality holds when $x = y = z = \frac{1}{2}$. □