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Chapter 1

Some Useful Substitutions

1.1 Theory and Examples

We know that in most inequalities with a constraint such as $abc = 1$ the substitution $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$ simplifies the solution (don't kid yourself, not all problems of this type become easier!). The use of substitutions is far from being specific to inequalities; there are many other similar substitutions that usually make life easier. For instance, have you ever thought of other conditions such as

$$xyz = x + y + z + 2, \quad xy + yz + zx + 2xyz = 1, \quad x^2 + y^2 + z^2 + 2xyz = 1$$

or $x^2 + y^2 + z^2 = xyz + 4$? The purpose of this chapter is to present some of the most classical substitutions of this kind and their applications.

You will be probably surprised (unless you already know it...) when finding out that the condition $xyz = x + y + z + 2$ together with $x, y, z > 0$ implies the existence of positive real numbers a, b, c such that

$$x = \frac{b+c}{a}, \quad y = \frac{c+a}{b}, \quad z = \frac{a+b}{c}.$$

Let us explain why. The condition $xyz = x + y + z + 2$ can be written in the

following equivalent way:

$$\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} = 1.$$

Proving this is just a matter of simple computations. Now take

$$a = \frac{1}{1+x}, \quad b = \frac{1}{1+y}, \quad c = \frac{1}{1+z}.$$

Then

$$a + b + c = 1 \text{ and } x = \frac{1-a}{a} = \frac{b+c}{a}.$$

Of course, in the same way we find $y = \frac{c+a}{b}$, $z = \frac{a+b}{c}$. The converse (that is, $\frac{b+c}{a}$, $\frac{c+a}{b}$, $\frac{a+b}{c}$ satisfy $xyz = x + y + z + 2$) is much easier and is settled again by basic computations. Now, what about the second set of conditions, that is $x, y, z > 0$ and $xy + yz + zx + 2xyz = 1$? If you look carefully, you will see that it is closely related to the first one. Indeed, $x, y, z > 0$ satisfy $xy + yz + zx + 2xyz = 1$ if and only if $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ verify

$$\frac{1}{xyz} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 2,$$

so the substitution here is

$$x = \frac{a}{b+c}, \quad y = \frac{b}{c+a}, \quad z = \frac{c}{a+b}.$$

Now, let us take a closer look at the other substitutions mentioned at the beginning of the chapter, namely

$$x^2 + y^2 + z^2 + 2xyz = 1 \quad \text{and} \quad x^2 + y^2 + z^2 = xyz + 4.$$

Let us begin with the following question, which can be considered an exercise, too: consider three real numbers a, b, c such that $abc = 1$ and let

$$x = a + \frac{1}{a}, \quad y = b + \frac{1}{b}, \quad z = c + \frac{1}{c} \tag{1.1}$$

The question is to find an algebraic relation between x, y, z , independent of a, b, c . An efficient way to answer this question (that is, without horrible computations that result from solving the quadratic equations) is to observe that

$$\begin{aligned} xyz &= \left(a + \frac{1}{a}\right) \left(b + \frac{1}{b}\right) \left(c + \frac{1}{c}\right) \\ &= \left(a^2 + \frac{1}{a^2}\right) + \left(b^2 + \frac{1}{b^2}\right) + \left(c^2 + \frac{1}{c^2}\right) + 2 \\ &= (x^2 - 2) + (y^2 - 2) + (z^2 - 2) + 2. \end{aligned}$$

Thus

$$x^2 + y^2 + z^2 - xyz = 4. \quad (1.2)$$

Because $\left|a + \frac{1}{a}\right| \geq 2$ for all real numbers a , it is clear that not every triple (x, y, z) satisfying (1.2) is of the form (1.1). However, with the extra-assumption $\min\{|x|, |y|, |z|\} \geq 2$ things get better and we do have the converse, that is if x, y, z are real numbers with $\min\{|x|, |y|, |z|\} \geq 2$ and satisfying (1.2), then there exist real numbers a, b, c with $abc = 1$ satisfying (1.1). Actually, it suffices to assume only that $\max\{|x|, |y|, |z|\} > 2$. Indeed, we may assume that $|x| > 2$. Thus there exists a nonzero real number u such that $x = u + \frac{1}{u}$. Now, let us regard (1.2) as a quadratic equation with respect to z . Because the discriminant is nonnegative, it follows that $(x^2 - 4)(y^2 - 4) \geq 0$. But since $|x| > 2$, we find that $y^2 \geq 4$ and so there exist a non-zero real number v for which $y = v + \frac{1}{v}$. How do we find the corresponding z ? Simply by solving the second degree equation. We find two solutions:

$$z_1 = uv + \frac{1}{uv}, \quad z_2 = \frac{u}{v} + \frac{v}{u}$$

and now we are almost done.

If $z = uv + \frac{1}{uv}$ we take $(a, b, c) = \left(u, v, \frac{1}{uv}\right)$ and if $z = \frac{u}{v} + \frac{v}{u}$, then we take $(a, b, c) = \left(\frac{1}{u}, v, \frac{u}{v}\right)$.

Inspired by the previous equation, let us consider another one,

$$x^2 + y^2 + z^2 + xyz = 4 \quad (1.3)$$

where $x, y, z > 0$. We will prove that the set of solutions of this equation is the set of triples $(2 \cos A, 2 \cos B, 2 \cos C)$, where A, B, C are the angles of an acute triangle. First, let us prove that all these triples are solutions. This reduces to the identity

$$\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1.$$

This identity can be proved readily by using the sum-to-product formulas.

For the converse, we see first that $0 < x, y, z < 2$, hence there are numbers $A, B \in (0, \frac{\pi}{2})$ such that $x = 2 \cos A$, $y = 2 \cos B$.

Solving the equation with respect to z and taking into account that $z \in (0, 2)$ we obtain $z = -2 \cos(A + B)$. Thus we can take $C = \pi - A - B$ and we will have

$$(x, y, z) = (2 \cos A, 2 \cos B, 2 \cos C).$$

Let us summarize: we have seen some nice substitutions, with even nicer proofs, but we still have not seen any applications. We will see them in a moment... and there are quite a few problems that can be solved by using these “tricks”. First, an easy and classical problem, due to Nesbitt. It has so many extensions and generalizations that we must discuss it first.

Example 1.1. *Prove that*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

for all $a, b, c > 0$.

Solution. With the “magical” substitution, it suffices to prove that if $x, y, z > 0$ satisfy $xy + yz + zx + 2xyz = 1$, then $x + y + z \geq \frac{3}{2}$. Let us suppose that this is not the case, i.e. $x + y + z < \frac{3}{2}$. Because $xy + yz + zx \leq \frac{(x + y + z)^2}{3}$, we must

have $xy + yz + zx < \frac{3}{4}$ and since $xyz \leq \left(\frac{x+y+z}{3}\right)^3$, we also have $2xyz < \frac{1}{4}$.

It follows that $1 = xy + yz + zx + 2xyz < \frac{3}{4} + \frac{1}{4} = 1$, a contradiction, so we are done.

Let us now increase the level of difficulty and make an experiment: imagine that you did not know about these substitutions and try to solve the following problem. Then look at the solution provided and you will see that sometimes a good substitution can solve a problem almost alone.

Example 1.2. *Let $x, y, z > 0$ be such that $xy + yz + zx + 2xyz = 1$. Prove that*

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 4(x + y + z).$$

(Mircea Lascu)

Solution. With our substitution the inequality becomes

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 4 \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \right).$$

But this follows from

$$\frac{4a}{b+c} \leq \frac{a}{b} + \frac{a}{c}, \quad \frac{4b}{c+a} \leq \frac{b}{c} + \frac{b}{a}, \quad \frac{4c}{a+b} \leq \frac{c}{a} + \frac{c}{b}.$$

Simple and efficient, these are the words that characterize this substitution. Here is a geometric application of the previous problem.

Example 1.3. *Prove that in any acute triangle ABC the following inequality holds*

$$\cos^2 A \cos^2 B + \cos^2 B \cos^2 C + \cos^2 C \cos^2 A \leq \frac{1}{4}(\cos^2 A + \cos^2 B + \cos^2 C).$$

(Titu Andreescu)

Solution. We observe that the desired inequality is equivalent to

$$\begin{aligned} & \frac{\cos A \cos B}{\cos C} + \frac{\cos B \cos C}{\cos A} + \frac{\cos A \cos C}{\cos B} \\ & \leq \frac{1}{4} \left(\frac{\cos A}{\cos B \cos C} + \frac{\cos B}{\cos C \cos A} + \frac{\cos C}{\cos A \cos B} \right). \end{aligned}$$

Setting

$$x = \frac{\cos B \cos C}{\cos A}, \quad y = \frac{\cos A \cos C}{\cos B}, \quad z = \frac{\cos A \cos B}{\cos C},$$

the inequality reduces to

$$4(x + y + z) \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

But this is precisely the inequality in the previous example. All that remains is to show that $xy + yz + zx + 2xyz = 1$. This is equivalent to

$$\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1,$$

which we have already discussed.

The following problem is a nice characterization of the equation (1.2) by polynomials and also teaches us some things about polynomials, in two or three variables.

Example 1.4. Find all polynomials $f(x, y, z)$ with real coefficients such that

$$f\left(a + \frac{1}{a}, b + \frac{1}{b}, c + \frac{1}{c}\right) = 0$$

whenever $abc = 1$.

(Gabriel Dospinescu)

Solution. From the introduction, it is now clear that the polynomials divisible by $x^2 + y^2 + z^2 - xyz - 4$ are solutions to the problem. But it is not obvious why any desired polynomial should be of this form. To show this, we use the