Math isn't the art of answering mathematical questions, it is the art of asking the right questions, the questions that give you insight, the ones that lead you in interesting directions, the ones that connect with lots of other interesting questions – the ones with beautiful answers.

– G. Chaitin

Preface

What can a new book of problems in elementary mathematics possibly contribute to the vast existing collection of journals, articles, and books? This was our main concern when we decided to write this book. The inevitability of this question does not facilitate the answer, because after five years of writing and rewriting we still had something to add. It could be a new problem, a comment we considered pertinent, or a solution that escaped our rationale until this predictive moment, when we were supposed to bring it under the scrutiny of a specialist in the field.

A mere perusal of this book should be sufficient to identify its target audience: students and coaches preparing for mathematical Olympiads, national or international. It takes more effort to realize that these are not the only potential beneficiaries of this work. While the book is rife with problems collected from various mathematical competitions and journals, one cannot neglect the classical results of mathematics, which naturally exceed the level of time-constrained competitions. And no, classical does not mean easy! These mathematical beauties are more than just proof that elementary mathematics can produce jewels. They serve as an invitation to mathematics beyond competitions, regarded by many to be the "true mathematics". In this context, the audience is more diverse than one might think.

Even so, as it will be easily discovered, many of the problems in this book are very difficult. Thus, the theoretical portions are short, while the emphasis is squarely placed on the problems. Certainly, more subtle results like quadratic reciprocity and existence of primitive roots are related to the basic results in linear algebra or mathematical analysis. Whenever their proofs are particularly useful, they are provided. We will assume of the reader a certain familiarity with classical theorems of elementary mathematics, which we will use freely. The selection of problems was made by weighing the need for routine exercises that engender familiarity with the joy of the difficult problems in which we find the truly beautiful ideas. We strove to select only those problems, easy and hard, that best illustrate the ideas we wanted to exhibit.

Allow us to discuss in brief the structure of the book. What will most likely surprise the reader when browsing just the table of contents is the absence of any chapters on geometry. This book was not intended to be an exhaustive treatment of elementary mathematics; if ever such a book appears, it will be a sad day for mathematics. Rather, we tried to assemble problems that enchanted us in order to give a sense of techniques and ideas that become leitmotifs not just in problem solving but in all of mathematics.

Furthermore, there are excellent books on geometry, and it was not hard to realize that it would be beyond our ability to create something new to add to this area of study. Thus, we preferred to elaborate more on three important fields of elementary mathematics: algebra, number theory, and combinatorics. Even after this narrowing of focus there are many topics that are simply left out, either in consideration of the available space or else because of the fine existing literature on the subject. This is, for example, the fate of functional equations, a field which can spawn extremely difficult, intriguing problems, but one which does not have obvious recurring themes that tie everything together.

Hoping that you have not abandoned the book because of these omissions, which might be considered major by many who do not keep in mind the stated objectives, we continue by elaborating on the contents of the chapters. To start out, we ordered the chapters in ascending order of difficulty of the mathematical tools used. Thus, the exposition starts out lightly with some classical substitution techniques in algebra, emphasizing a large number of examples and applications. These are followed by a topic dear to us: the Cauchy-Schwarz inequality and its variations. A sizable chapter presents applications of the Lagrange interpolation formula, which is known by most only through rôte, straightforward applications. The interested reader will find some genuine pearls in this chapter, which should be enough to change his or her opinion about this useful mathematical tool. Two rather difficult chapters, in which mathematical analysis mixes with algebra, are given at the end of the book. One of them is quite original, showing how simple consideration of integral calculus can solve very difficult inequalities. The other discusses properties of equidistribution and dense numerical series. Too many books consider the Weyl equidistribution theorem to be “much too difficult” to include, and we cannot resist contradicting them by presenting an elementary proof. Furthermore, the reader will quickly realize that for elementary problems we have not shied away from presenting the so-called non-elementary solutions which use mathematical analysis or advanced algebra. It would be a crime to consider these two types of mathematics as two different entities, and it would be even worse to present laborious elementary solutions without admitting the possibility of generalization for problems that have conceptual and easy non-elementary solutions. In the end we devote a whole chapter to discussing criteria for polynomial irreducibility. We observe that some extremely efficient criteria (like those of Peron and Capelli) are virtually unknown, even though they are more efficient than the well-known Eisenstein criterion.

The section dedicated to number theory is the largest. Some introductory chapters related to prime numbers of the form $4k+3$ and to the order of an element are included to provide a better understanding of fundamental results which are used later in the book. A large chapter develops a tool which is as simple as it is useful: the exponent of a prime in the factorization of an integer. Some mathematical diamonds belonging to Paul Erdös and others appear within. And even though quadratic reciprocity is brought up in many books, we included an entire chapter on this topic because the problems available to us were too ingenious to exclude. Next come some difficult chapters concerning arithmetic properties of polynomials, the geometry of numbers (in which we present some arithmetic applications of the famous Minkowski's theorem), and the properties of algebraic numbers. A special chapter studies some applications of the extremely simple idea that a convergent series of integers is eventually stationary! The reader will have the chance to realize that in mathematics even simple ideas have great impact: consider, for example, the fundamental idea that in the interval $(-1,1)$ the only integer is 0. But how many fantastic results concerning irrational numbers follow simply from that!

Another chapter dear to us concerns the sum of digits, a subject that always yields unexpected and fascinating problems, but for which we could not find a unique approach.

Finally, some words about the combinatorics section. The reader will immediately observe that our presentation of this topic takes an algebraic slant, which was, in fact, our intention. In this way we tried to present some unexpected applications of complex numbers in combinatorics, and a whole chapter is dedicated to useful formal series. Another chapter shows how useful linear algebra can be when solving problems on set combinatorics. Of course, we are traditional in presenting applications of Turan's theorem and of graph theory in general, and the pigeonhole principle could not be omitted. We faced difficulties here, because this topic is covered extensively in other books, though rarely in a satisfying way. For this reason, we tried to present lesser-known problems, because this topic is so dear to elementary mathematics lovers. At the end, we included a chapter on special applications of polynomials in number theory and combinatorics, emphasizing the Combinatorial Nullstellensatz, a recent and extremely useful theorem by Noga Alon.

We end our description with some remarks on the structure of the chapters. In general, the main theoretical results are stated, and if they are sufficiently profound or obscure, a proof is given. Following the theoretical part, we present between ten and fifteen examples, most from mathematical contests or from journals such as Kvant, Kömal, and American Mathematical Monthly. Others are new problems or classical results. Each chapter ends with a series of problems, the majority of which stem from the theoretical results. Finally, a change that will please some and scare others: the end-of-chapter problems do not have solutions! We had several reasons for this. The first and most practical consideration was minimizing the mass of the book. But the second and more important factor was this: we consider solving problems to necessarily include the inevitably lengthy process of trial and research to which the inclusion of solutions provides perhaps too tempting of a shortcut. Keeping this in mind, the selection of the problems was made with the goal that the diligent reader could solve about a third of them, make some progress in the second third and have at least the satisfaction of looking for a solution in the remainder.

We come now to the most delicate moment, the one of saying thank you. First and foremost, we thank Marian Tetiva and Paul Stanford, whose close reading of the manuscript uncovered many errors that we would not have liked in this final version. We thank them for the great effort they put into reviewing the book. All of the remaining mistakes are the responsibility of the authors, who would be grateful for reports of errors so that in a future edition they will disappear. Many thanks to Radu Sorici for giving the book the look it has now and for the numerous suggestions for improvement. We thank Adrian Zahariuc for his help in writing the sections on the sums of digits and graph theory. Several solutions are either his own or the fruit of his experience. Special thanks are due to Valentin Vornicu for creating Mathlinks, which has generated many of the problems we have included. His website, mathlinks.ro, hosts a treasure trove of problems, and we invite every passionate mathematician to avail themselves of this fact. We would also like to thank Ravi Boppana, Vesselin Dimitrov, and Richard Stong for the excellent problems, solutions, and comments they provided. Lastly, we have surely forgotten many others who helped throughout the writing process; our thanks and apologies go out to them.

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Chapter 1

Some Useful Substitutions

1.1 Theory and Examples

We know that in most inequalities with a constraint such as $abc = 1$ the substitution $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$ simplifies the solution (don't kid yourself, not all problems of this type become easier!). The use of substitutions is far from being specific to inequalities; there are many other similar substitutions that usually make life easier. For instance, have you ever thought of other conditions such as

$$xyz = x + y + z + 2, \quad xy + yz + zx + 2xyz = 1, \quad x^2 + y^2 + z^2 + 2xyz = 1$$

or $x^2 + y^2 + z^2 = xyz + 4$? The purpose of this chapter is to present some of the most classical substitutions of this kind and their applications.

You will be probably surprised (unless you already know it...) when finding out that the condition $xyz = x + y + z + 2$ together with $x, y, z > 0$ implies the existence of positive real numbers a, b, c such that

$$x = \frac{b+c}{a}, \quad y = \frac{c+a}{b}, \quad z = \frac{a+b}{c}.$$

Let us explain why. The condition $xyz = x + y + z + 2$ can be written in the following equivalent way:

$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}=1.$$

Proving this is just a matter of simple computations. Now take

$$a=\frac{1}{1+x},\quad b=\frac{1}{1+y},\quad c=\frac{1}{1+z}.$$

Then

$$a+b+c=1 \text{ and } x=\frac{1-a}{a}=\frac{b+c}{a}.$$

Of course, in the same way we find $y=\frac{c+a}{b}$, $z=\frac{a+b}{c}$. The converse (that is, $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ satisfy $xyz=x+y+z+2$) is much easier and is settled again by basic computations. Now, what about the second set of conditions, that is $x,y,z>0$ and $xy+yz+zx+2xyz=1$? If you look carefully, you will see that it is closely related to the first one. Indeed, $x,y,z>0$ satisfy $xy+yz+zx+2xyz=1$ if and only if $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ verify

$$\frac{1}{xyz}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+2,$$

so the substitution here is

$$x=\frac{a}{b+c},\quad y=\frac{b}{c+a},\quad z=\frac{c}{a+b}.$$

Now, let us take a closer look at the other substitutions mentioned at the beginning of the chapter, namely

$$x^2+y^2+z^2+2xyz=1 \quad \text{and} \quad x^2+y^2+z^2=xyz+4.$$

Let us begin with the following question, which can be considered an exercise, too: consider three real numbers a,b,c such that $abc=1$ and let

$$x=a+\frac{1}{a},\quad y=b+\frac{1}{b},\quad z=c+\frac{1}{c} \tag{1.1}$$

The question is to find an algebraic relation between x, y, z, independent of a, b, c. An efficient way to answer this question (that is, without horrible computations that result from solving the quadratic equations) is to observe that

$$\begin{aligned} xyz &= \left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right) \\ &= \left(a^2+\frac{1}{a^2}\right)+\left(b^2+\frac{1}{b^2}\right)+\left(c^2+\frac{1}{c^2}\right)+2 \\ &= (x^2-2)+(y^2-2)+(z^2-2)+2. \end{aligned}$$

Thus

$$x^2+y^2+z^2-xyz=4. \tag{1.2}$$

Because $\left|a+\frac{1}{a}\right| \geq 2$ for all real numbers a, it is clear that not every triple (x, y, z) satisfying (1.2) is of the form (1.1). However, with the extra-assumption $\min\{|x|,|y|,|z|\} \geq 2$ things get better and we do have the converse, that is if x, y, z are real numbers with $\min\{|x|,|y|,|z|\} \geq 2$ and satisfying (1.2), then there exist real numbers a, b, c with $abc = 1$ satisfying (1.1). Actually, it suffices to assume only that $\max\{|x|,|y|,|z|\} > 2$. Indeed, we may assume that $|x| > 2$. Thus there exists a nonzero real number u such that $x = u + \frac{1}{u}$. Now, let us regard (1.2) as a quadratic equation with respect to z. Because the discriminant is nonnegative, it follows that $(x^2-4)(y^2-4) \geq 0$. But since $|x| > 2$, we find that $y^2 \geq 4$ and so there exist a non-zero real number v for which $y = v + \frac{1}{v}$. How do we find the corresponding z? Simply by solving the second degree equation. We find two solutions:

$$z_1 = uv + \frac{1}{uv}, \quad z_2 = \frac{u}{v} + \frac{v}{u}$$

and now we are almost done.

If $z = uv + \frac{1}{uv}$ we take $(a, b, c) = \left(u, v, \frac{1}{uv}\right)$ and if $z = \frac{u}{v} + \frac{v}{u}$, then we take $(a, b, c) = \left(\frac{1}{u}, v, \frac{u}{v}\right)$.

Inspired by the previous equation, let us consider another one,

$$x^2 + y^2 + z^2 + xyz = 4 \tag{1.3}$$

where $x, y, z > 0$. We will prove that the set of solutions of this equation is the set of triples $(2\cos A, 2\cos B, 2\cos C)$, where A, B, C are the angles of an acute triangle. First, let us prove that all these triples are solutions. This reduces to the identity

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1.$$

This identity can be proved readily by using the sum-to-product formulas.
For the converse, we see first that $0 < x, y, z < 2$, hence there are numbers $A, B \in \left(0, \frac{\pi}{2}\right)$ such that $x = 2\cos A$, $y = 2\cos B$.
Solving the equation with respect to z and taking into account that $z \in (0, 2)$ we obtain $z = -2\cos(A + B)$. Thus we can take $C = \pi - A - B$ and we will have

$$(x, y, z) = (2\cos A, 2\cos B, 2\cos C).$$

Let us summarize: we have seen some nice substitutions, with even nicer proofs, but we still have not seen any applications. We will see them in a moment... and there are quite a few problems that can be solved by using these "tricks". First, an easy and classical problem, due to Nesbitt. It has so many extensions and generalizations that we must discuss it first.

Example 1.1. *Prove that*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

for all $a, b, c > 0$.

Solution. With the "magical" substitution, it suffices to prove that if $x, y, z > 0$ satisfy $xy + yz + zx + 2xyz = 1$, then $x + y + z \geq \frac{3}{2}$. Let us suppose that this is not the case, i.e. $x + y + z < \frac{3}{2}$. Because $xy + yz + zx \leq \frac{(x+y+z)^2}{3}$, we must have $xy+yz+zx < \frac{3}{4}$ and since $xyz \le \left(\frac{x+y+z}{3}\right)^3$, we also have $2xyz < \frac{1}{4}$. It follows that $1 = xy+yz+zx+2xyz < \frac{3}{4}+\frac{1}{4} = 1$, a contradiction, so we are done.

Let us now increase the level of difficulty and make an experiment: imagine that you did not know about these substitutions and try to solve the following problem. Then look at the solution provided and you will see that sometimes a good substitution can solve a problem almost alone.

Example 1.2. *Let $x, y, z > 0$ be such that $xy+yz+zx+2xyz = 1$. Prove that*

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \ge 4(x+y+z).$$

(Mircea Lascu)

Solution. With our substitution the inequality becomes

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c} \ge 4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right).$$

But this follows from

$$\frac{4a}{b+c} \le \frac{a}{b}+\frac{a}{c}, \quad \frac{4b}{c+a} \le \frac{b}{c}+\frac{b}{a}, \quad \frac{4c}{a+b} \le \frac{c}{a}+\frac{c}{b}.$$

Simple and efficient, these are the words that characterize this substitution. Here is a geometric application of the previous problem.

Example 1.3. *Prove that in any acute triangle ABC the following inequality holds*

$$\cos^2 A\cos^2 B+\cos^2 B\cos^2 C+\cos^2 C\cos^2 A \le \frac{1}{4}(\cos^2 A+\cos^2 B+\cos^2 C).$$

(Titu Andreescu)

Solution. We observe that the desired inequality is equivalent to

$$\frac{\cos A\cos B}{\cos C}+\frac{\cos B\cos C}{\cos A}+\frac{\cos A\cos C}{\cos B}$$
$$\leq\frac{1}{4}\left(\frac{\cos A}{\cos B\cos C}+\frac{\cos B}{\cos C\cos A}+\frac{\cos C}{\cos A\cos B}\right).$$

Setting

$$x=\frac{\cos B\cos C}{\cos A},\quad y=\frac{\cos A\cos C}{\cos B},\quad z=\frac{\cos A\cos B}{\cos C},$$

the inequality reduces to

$$4(x+y+z)\leq\frac{1}{x}+\frac{1}{y}+\frac{1}{z}.$$

But this is precisely the inequality in the previous example. All that remains is to show that $xy+yz+zx+2xyz=1$. This is equivalent to

$$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1,$$

which we have already discussed.

The following problem is a nice characterization of the equation (1.2) by polynomials and also teaches us some things about polynomials, in two or three variables.

Example 1.4. *Find all polynomials $f(x,y,z)$ with real coefficients such that*

$$f\left(a+\frac{1}{a},b+\frac{1}{b}+c+\frac{1}{c}\right)=0$$

whenever $abc=1$.

(Gabriel Dospinescu)

Solution. From the introduction, it is now clear that the polynomials divisible by $x^2+y^2+z^2-xyz-4$ are solutions to the problem. But it is not obvious why any desired polynomial should be of this form. To show this, we use the classical polynomial long division. There are polynomials $g(x, y, z)$, $h(y, z)$, $k(y, z)$ with real coefficients such that

$$f(x, y, z) = (x^2 + y^2 + z^2 - xyz - 4)g(x, y, z) + xh(y, z) + k(y, z).$$

Using the hypothesis, we deduce that

$$0 = \left(a + \frac{1}{a}\right) h\left(b + \frac{1}{b}, c + \frac{1}{c}\right) + k\left(b + \frac{1}{b}, c + \frac{1}{c}\right)$$

whenever $abc = 1$. Well, it seems that this is a dead end. Not exactly. Now we take two numbers x, y such that $\min\{|x|, |y|\} > 2$ and we write

$$x = b + \frac{1}{b}, \quad y = c + \frac{1}{c}$$

with $b = \dfrac{x + \sqrt{x^2 - 4}}{2}$, $c = \dfrac{y + \sqrt{y^2 - 4}}{2}$.

Then it is easy to compute $a + \dfrac{1}{a}$. It is exactly $xy + \sqrt{(x^2 - 4)(y^2 - 4)}$. So, we have found that

$$\left(xy + \sqrt{(x^2 - 4)(y^2 - 4)}\right) h(x, y) + k(x, y) = 0$$

whenever $\min\{|x|, |y|\} > 2$. And now? The last relation suggests that we should prove that for each y with $|y| > 2$, the function $x \to \sqrt{x^2 - 4}$ is not rational, that is, there are not polynomials p, q such that

$$\sqrt{x^2 - 4} = \frac{p(x)}{q(x)}.$$

But this is easy because if such polynomials existed, than each zero of $x^2 - 4$ should have even multiplicity, which is not the case. Consequently, for each y with $|y| > 2$ we have $h(x, y) = k(x, y) = 0$ for all x. But this means that $h(x, y) = k(x, y) = 0$ for all x, y, that is our polynomial is divisible by $x^2 + y^2 + z^2 - xyz - 4$.

The level of difficulty continues to increase. When we say this, we refer again to the proposed experiment. The reader who will try first to solve the problems discussed without using the above substitutions will certainly understand why we consider these problems hard.

Example 1.5. *Prove that if $x, y, z > 0$ and $xyz = x + y + z + 2$, then*

$$2\left(\sqrt{xy} + \sqrt{yz} + \sqrt{zx}\right) \leq x + y + z + 6.$$

Solution. This is tricky, even with the substitution. There are two main ideas: using some identities that transform the inequality into an easier one and then using the substitution. Let us see. What does $2\left(\sqrt{xy} + \sqrt{yz} + \sqrt{zx}\right)$ suggest? Clearly, it is related to

$$\left(\sqrt{x} + \sqrt{y} + \sqrt{z}\right)^2 - (x + y + z).$$

Consequently, our inequality can be written as

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \leq \sqrt{2(x + y + z + 3)}.$$

The first idea that comes to mind (that is using the Cauchy-Schwarz inequality in the form $\sqrt{x} + \sqrt{y} + \sqrt{z} \leq \sqrt{3(x + y + z)} \leq \sqrt{2(x + y + z + 3)}$) does not lead to a solution. Indeed, the last inequality is not true: setting $x + y + z = s$, we have $3s \leq 2(s + 3)$ This is because the AM-GM inequality implies that

$$xyz \leq \frac{s^3}{27}, \quad \text{so} \quad \frac{s^3}{27} \geq s + 2,$$

which is equivalent to $(s - 6)(s + 3)^2 \geq 0$, implying $s \geq 6$.
Let us see how the substitution helps. The inequality becomes

$$\sqrt{\frac{b + c}{a}} + \sqrt{\frac{c + a}{b}} + \sqrt{\frac{a + b}{c}} \leq \sqrt{2\left(\frac{b + c}{a} + \frac{c + a}{b} + \frac{a + b}{c} + 3\right)}.$$

The last step is probably the most important.
We have to change the expression $\frac{b + c}{a} + \frac{c + a}{b} + \frac{a + b}{c} + 3$ a little bit.
We see that if we add 1 to each fraction, then $a + b + c$ will appear as a common factor, so in fact

$$\frac{b + c}{a} + \frac{c + a}{b} + \frac{a + b}{c} + 3 = (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

And now we have finally solved the problem, amusingly, by employing again the Cauchy-Schwarz inequality:

$$\sqrt{\frac{b+c}{a}}+\sqrt{\frac{c+a}{b}}+\sqrt{\frac{a+b}{c}} \leq \sqrt{(b+c+c+a+a+b)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}.$$

We continue with a difficult 2003 USAMO problem. There are numerous proofs for this inequality, none of them easy. The following solution is again not simple, but seems natural for someone familiar with such a substitution.

Example 1.6. *Prove that for any positive real numbers a, b, c the following*

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2} \leq 8.$$

(Titu Andreescu, Zuming Feng, USAMO 2003)

Solution. The desired inequality is equivalent to

$$\frac{\left(2+\dfrac{b+c}{a}\right)^2}{2+\left(\dfrac{b+c}{a}\right)^2}+\frac{\left(2+\dfrac{c+a}{b}\right)^2}{2+\left(\dfrac{c+a}{b}\right)^2}+\frac{\left(2+\dfrac{a+b}{c}\right)^2}{2+\left(\dfrac{a+b}{c}\right)^2} \leq 8.$$

Taking our substitution into account, it suffices to prove that if

$$xyz = x+y+z+2,$$

then

$$\frac{(2+x)^2}{2+x^2}+\frac{(2+y)^2}{2+y^2}+\frac{(2+z)^2}{2+z^2} \leq 8.$$

This is in fact the same as

$$\frac{2x+1}{x^2+2}+\frac{2y+1}{y^2+2}+\frac{2z+1}{z^2+2} \leq \frac{5}{2}.$$

Now, we transform this inequality into

$$\frac{(x-1)^2}{x^2+2} + \frac{(y-1)^2}{y^2+2} + \frac{(z-1)^2}{z^2+2} \geq \frac{1}{2}.$$

This last form suggests using the Cauchy-Schwarz inequality to prove that

$$\frac{(x-1)^2}{x^2+2} + \frac{(y-1)^2}{y^2+2} + \frac{(z-1)^2}{z^2+2} \geq \frac{(x+y+z-3)^2}{x^2+y^2+z^2+6}.$$

So, we are left with proving that $2(x+y+z-3)^2 \geq x^2+y^2+z^2+6$. But this is not difficult. Indeed, this inequality is equivalent to

$$2(x+y+z-3)^2 \geq (x+y+z)^2 - 2(xy+yz+zx) + 6.$$

Now, from $xyz \geq 8$ (recall who x, y, z are and use the AM-GM inequality three times), we find that $xy+yz+zx \geq 2$ and $x+y+z \geq 6$ (by the same AM-GM inequality). This shows that it suffices to prove that $2(s-3)^2 \geq s^2 - 18$ for all $s \geq 6$, which is equivalent to $(s-3)(s-6) \geq 0$, clearly true. And this difficult problem is solved!

The following problem is also hard. Yet there is an easy solution using the substitutions described in this chapter.

Example 1.7. *Prove that if $x, y, z > 0$ satisfy $xy + yz + zx + xyz = 4$ then*

$$x + y + z \geq xy + yz + zx.$$

(India 1998)

Solution. Let us write the given condition as

$$\frac{x}{2} \cdot \frac{y}{2} + \frac{y}{2} \cdot \frac{z}{2} + \frac{z}{2} \cdot \frac{x}{2} + 2 \cdot \frac{x}{2} \cdot \frac{y}{2} \cdot \frac{z}{2} = 1.$$

Hence there are positive real numbers a, b, c such that

$$x = \frac{2a}{b+c}, \quad y = \frac{2b}{c+a}, \quad z = \frac{2c}{a+b}.$$