2.1 2003 Middle and High School

Problem 1. In eight years Henry will be three times the age that Sally was last year. Twenty five years ago their ages added to 83. How old is Henry now?

Answer: 97

Let h represent Henry's current age, and s represent Sally's current age. In eight years Henry will have age $h + 8$. Last year Sally had age $s - 1$. So

$$h + 8 = 3(s - 1).$$

Also, 25 years ago their ages were $h - 25$ and $s - 25$, so the sum of their ages 25 years ago was

$$(h - 25) + (s - 25) = h + s - 50 = 83.$$

These equations simplify to

$$h - 3s = -11 \quad \text{and} \quad h + s = 133.$$

Adding three time the second equation to the first equation yields

$$h - 3s + 3(h + s) = -11 + 3 \cdot 133 \quad \text{or} \quad 4h = 388 \quad \text{or} \quad h = 97.$$

So Henry is currently 97 years old.

Problem 2. What is the least number that could be the date of the first Saturday after the second Monday following the second Thursday of a month?

Answer: 1

If the month in question is a 28-day February starting on a Saturday, the the first Thursday is the 6th. The second Thursday is the 13th. The first Monday after that is the 17th. The second Monday after the second Thursday is then the 24th. The following Saturday is then March 1. Since the date must be a positive integer, the smallest possible value for the date is 1.

Note: This is the only problem in the history of the Purple Comet*!* Math Meet for which no team submitted a correct answer.

Problem 3. What is the greatest integer whose prime factors add to 14?

Answer: 162

Note that $2 \times 2 \times 2 = 8$ and $3 \times 3 = 9$, so you get a larger product by multiplying two threes than by multiplying three twos even though $2+2+2 = 3+3 = 6$. Taking products of larger numbers is less economical. The product of two fives is $5 \times 5 = 25$ whereas the product of five twos $2 \times 2 \times 2 \times 2 \times 2 = 32$ is larger. So, to get the largest possible product, you would want to take the product of as many threes as possible to get $3 \times 3 \times 3 \times 3 \times 2 = 162$.

Problem 4. The lengths in inches of the diagonals of a rhombus are two consecutive integers. The area of the rhombus is 210 square inches. Find its perimeter, in inches.

Answer: 58

Let x and $x+1$ be the lengths of the two diagonals. Then

$$\frac{x(x+1)}{2} = 210,$$

which reduces to

$$x^2 + x - 420 = 0 \quad \text{or} \quad (x-20)(x+21) = 0.$$

The only acceptable solution is $x = 20$. The side length s of the rhombus is the side length of the hypotenuse in a right-angled triangle whose legs are 10 and 10.5. The Pythagorean Theorem yields $s = 14.5$, so the perimeter of the rhombus is $4 \times 14.5 = 58$.

Problem 5. Let a, b, c be nonzero real numbers such that $a + \dfrac{1}{b} = 5$, $b + \dfrac{1}{c} = 12$, and $c + \dfrac{1}{a} = 13$. Find $abc + \dfrac{1}{abc}$.

Answer: 750

We have $\left(a + \dfrac{1}{b}\right)\left(b + \dfrac{1}{c}\right)\left(c + \dfrac{1}{a}\right) = 5 \cdot 12 \cdot 13$, yielding

$$abc + \left(a + \frac{1}{b}\right) + \left(b + \frac{1}{c}\right) + \left(c + \frac{1}{a}\right) + \frac{1}{abc} = 780.$$

Then

$$abc + \frac{1}{abc} = 780 - 5 - 12 - 13 = 750.$$

Problem 6. Evaluate

$$\frac{1}{\log_2\left(\frac{1}{6}\right)} - \frac{1}{\log_3\left(\frac{1}{6}\right)} - \frac{1}{\log_4\left(\frac{1}{6}\right)}.$$

Answer: 1

The given expression is equal to

$$\log_{\frac{1}{6}} 2 - \log_{\frac{1}{6}} 3 - \log_{\frac{1}{6}} 4 = \log_{\frac{1}{6}} \frac{2}{3 \cdot 4} = \log_{\frac{1}{6}} \frac{1}{6} = 1.$$

Problem 7. Find the least n such that every subset of $\{1, 2, 3, \ldots, 2004\}$ with n elements contains at least two elements that are relatively prime.

Answer: 1003

The answer must be larger than 1002 since the set

$$\{2, 4, 6, 8, \ldots, 2004\}$$

has 1002 elements yet all of the elements have a factor of 2, so no two elements are relatively prime. Consider the collection A of 1002 sets

$$\{1, 2\},\ \{3, 4\},\ \{5, 6\}, \ldots, \{2003, 2004\}.$$

If you have 1003 numbers from the set $1, 2, 3, 4, \ldots, 2004$, then by the pigeon hole principal, at least two of the elements must fall into one of the sets in the collection A. Thus, there is a k so that the 1003 numbers include both k and $k+1$. Since these two numbers are relatively prime, all sets of size 1003 must contain two numbers which are relatively prime. Thus, the answer is 1003.

Problem 8. Let $ABCDEFGHIJKL$ be a regular dodecagon. Find

$$\frac{AB}{AF} + \frac{AF}{AB}.$$

Answer: 4

Let R be the radius of the circle circumscribed about the regular dodecagon. From the Law of Sines derive that

$$AB = 2R\sin\frac{\pi}{12} \quad \text{and} \quad AF = 2R\sin\frac{5\pi}{12}.$$

Then

$$\begin{aligned}\frac{AB}{AF}+\frac{AF}{AB} &= \frac{\sin\frac{\pi}{12}}{\sin\frac{5\pi}{12}}+\frac{\sin\frac{5\pi}{12}}{\sin\frac{\pi}{12}} = \frac{\sin\frac{\pi}{12}}{\cos\frac{\pi}{12}}+\frac{\cos\frac{\pi}{12}}{\sin\frac{\pi}{12}}\\ &= \frac{\sin^2\frac{\pi}{12}+\cos^2\frac{\pi}{12}}{\sin\frac{\pi}{12}\cos\frac{\pi}{12}} = \frac{1}{\sin\frac{\pi}{12}\cos\frac{\pi}{12}}\\ &= \frac{2}{2\sin\frac{\pi}{12}\cos\frac{\pi}{12}} = \frac{2}{\sin\frac{\pi}{6}} = \frac{2}{\frac{1}{2}} = 4.\end{aligned}$$

Problem 9. Let f be a real-valued function of real and positive argument such that

$$f(x)+3xf\left(\frac{1}{x}\right)=2(x+1)$$

for all real numbers $x>0$. Find $f(2003)$.

Answer: 1002

We have

$$f(x)+3xf\left(\frac{1}{x}\right)=2(x+1) \text{ for all } x>0,$$

so

$$f\left(\frac{1}{x}\right)+3\cdot\frac{1}{x}\cdot f(x)=2\left(\frac{1}{x}+1\right) \text{ for all } x>0$$

as well. Multiplying the second equality by $3x$ yields

$$3xf\left(\frac{1}{x}\right)+9f(x)=6+6x,$$

and subtracting the first equality now gives

$$8f(x)=4x+4, \text{ and } f(x)=\frac{x+1}{2}.$$

It follows that

$$f(2003)=\frac{2003+1}{2}=1002.$$

Problem 10. How many gallons of a solution which is 15% alcohol do we have to mix with a solution that is 35% alcohol to make 250 gallons of a solution that is 21% alcohol?

Answer: 175

Let x represent the number of gallons of 15% alcohol solution that we need to mix. To make 250 gallons, we need to mix the x gallons of 15% alcohol solution with $(250-x)$ gallons of the 35% alcohol solution. The mix will then contain $.15x + .35(250-x)$ gallons of alcohol which should be 21% of the total 250 gallons or .21 (250). This gives

$$\begin{aligned} .15x + .35(250-x) &= .21(250) \\ 15x + 35(250) - 35x &= 21(250) \\ 14(250) &= 20x \\ 7(25) &= 175 = x. \end{aligned}$$

So we need to use 175 gallons of 15% alcohol solution.

Problem 11. If

$$\frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+\cdots+20} = \frac{m}{n},$$

where m and n are positive integers with no common divisor, find $m+n$.

Answer: 40

Because

$$1+2+\cdots+k = \frac{k(k+1)}{2},$$

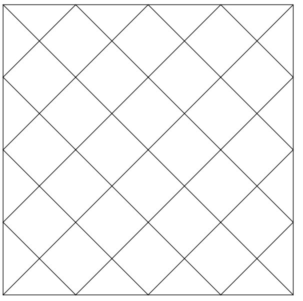
we have

$$\frac{1}{1+2+\cdots+k} = \frac{2}{k(k+1)} = \frac{2}{k} - \frac{2}{k+1}.$$

$$\begin{aligned} \frac{1}{1+2} + \cdots + \frac{1}{1+2+\ldots+20} &= \frac{2}{2} - \frac{2}{3} + \cdots + \frac{2}{20} - \frac{2}{21} \\ &= 1 - \frac{2}{21} = \frac{19}{21} \end{aligned}$$

and so $m+n = 40$.

Problem 12. How many triangles appear in the diagram below?



Answer: 56

To count the triangles notice that all triangles are right triangles. There are 10 triangles with a right angle at the intersection of two diagonal lines and a hypotenuse along the left vertical side of the rectangle. By symmetry, there are just as many triangles with a hypotenuse along the top, right, or bottom side of the rectangle. This accounts for 40 triangles. There are 4 triangles with right angle at the lower right corner of the rectangle. Again by symmetry, there are four times this many triangles, or 16. Thus, the diagram contains $40 + 16 = 56$ triangles.

Problem 13. Let $P(x)$ be a polynomial such that, when divided by $x - 2$, the remainder is 3 and, when divided by $x - 3$, the remainder is 2. If, when divided by $(x - 2)(x - 3)$, the remainder is $ax + b$, find $a^2 + b^2$.

Answer: 26

By the remainder theorem, $P(2) = 3$ and $P(3) = 2$. The division algorithm would give us a polynomial $K(x)$ so that

$$P(x) = (x - 2)(x - 3)K(x) + ax + b.$$

Letting $x = 2$ gives $3 = 2a + b$. Letting $x = 3$ gives $2 = 3a + b$. Now solve for a and b.

Solving

$$3a + b = 2 \quad \text{and} \quad 2a + b = 3$$

gives $a = -1$ and $b = 5$. So $a^2 + b^2 = (-1)^2 + (5)^2 = 26$.

Problem 14. Let a, b, c be real numbers such that $a^2-2 = 3b-c$, $b^2+4 = 3c+a$, and $c^2 + 4 = 3a - b$. Find $a^4 + b^4 + c^4$.

Answer: 18

Adding up these three equalities gives $a^2+b^2+c^2+6 = 4a+2b+2c$ or $(a - 2)^2 + (b - 1)^2 + (c - 1)^2 = 0$, yielding $a = 2$, $b = 1$, and $c = 1$. Thus $a^4 + b^4 + c^4 = 16 + 1 + 1 = 18$.

Problem 15. Let r be a real number such that $\sqrt[3]{r} - \dfrac{1}{\sqrt[3]{r}} = 2$. Find

$$r^3 - \frac{1}{r^3}.$$

Answer: 2786

The identity $x^3 - y^3 = (x-y)^3 + 3xy(x-y)$ yields

$$r - \frac{1}{r} = 8 + 6 = 14.$$

Applying the above again gives

$$r^3 - \frac{1}{r^3} = 14^3 + 42 = 2786.$$

Problem 16. Find the greatest real number x such that

$$\left(\frac{x}{x-1}\right)^2 + \left(\frac{x}{x+1}\right)^2 = \frac{325}{144}.$$

Answer: 5

Adding $\dfrac{2x^2}{x^2-1}$ to both sides gives

$$\left(\frac{x}{x-1} + \frac{x}{x+1}\right)^2 = \frac{325}{144} + \frac{2x^2}{x^2-1}$$

or

$$\left(\frac{2x^2}{x^2-1}\right)^2 - \frac{2x^2}{x^2-1} = \frac{325}{144}.$$

Setting $\dfrac{2x^2}{x^2-1} = t$ yields $t^2 - t - \dfrac{325}{144} = 0$, whose solutions are

$$\frac{1}{2}\left(1 \pm \sqrt{1 + \frac{325}{36}}\right) = \frac{1}{2}\left(1 \pm \frac{19}{6}\right).$$

Since $\dfrac{2x^2}{x^2-1} > 2$, only $\dfrac{1}{2}\left(1 + \dfrac{19}{6}\right) = \dfrac{25}{12}$ qualifies, hence $\dfrac{2x^2}{x^2-1} = \dfrac{25}{12}$ yielding $x^2 = 25$. The largest x with this property is 5.

Problem 17. Given that $3\sin x + 4\cos x = 5$, where x is in $\left(0, \frac{\pi}{2}\right)$, find

$$2\sin x + \cos x + 4\tan x.$$

Answer: 5

Setting $\tan \frac{x}{2} = t$, we have

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}.$$

Then

$$\frac{6t}{1+t^2} + \frac{4(1-t^2)}{1+t^2} = 5$$

which reduces to $9t^2 - 6t + 1 = 0$ or, equivalently, $(3t-1)^2 = 0$. It follows that $t = \frac{1}{3}$ and

$$\tan x = \frac{2t}{1-t^2} = \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{3}{4}.$$

Then

$$2\sin x + \cos x + 4\tan x = \frac{\frac{4}{3}}{\frac{10}{9}} + \frac{\frac{8}{9}}{\frac{10}{9}} + 3 = 2 + 3 = 5.$$

Alternatively, divide the given equation by 5 to get

$$\frac{3}{5}\sin x + \frac{4}{5}\cos x = 1.$$

There is an angle θ so that $\sin(x+\theta) = 1$. Thus, x and θ are complements, implying

$$\sin\theta = \frac{3}{5}, \ \cos\theta = \frac{4}{5}, \text{ and } \tan\theta = \frac{3}{4}$$

as above.

Problem 18. A circle of radius 320 is tangent to the inside of a circle of radius 1000. The smaller circle is tangent to a diameter of the larger circle at a point P. How far is P from the outside of the larger circle?