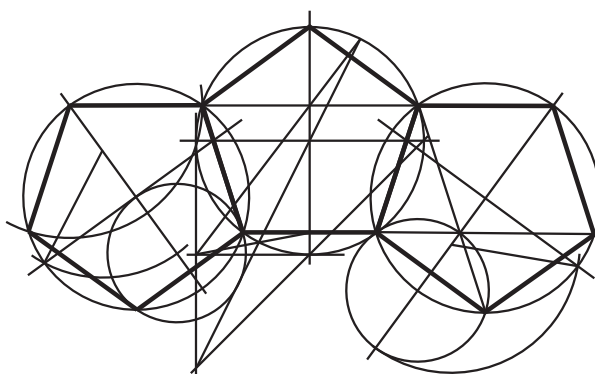


DJURA PAUNIĆ

REGULAR POLYGONS



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*Ich glaube, daß Sie mit Unrecht die Theorie der Polygone in meinen D A scheuen und daß es nur auf den Versuch ankommt, wo Sie gewiß alles sehr leicht finden werden.*¹

C. F. Gauß

Preface

Mathematics has a long tradition of problem solving. At that mathematicians are most persistent. They search for the solution until they find one, even if it takes tens, hundreds or even thousands of years. The most well known mathematical problem which was being solved for a long time, is squaring the circle. This problem, the construction of a square with the same area as a given circle, was already set in antiquity, very probably in the 5th or 6th century BC, only to be solved as late as 1882 after more than two and half thousand years. The problems of angle trisection (to construct one third of *any* given angle), duplication of the cube (to construct the edge of a cube that has twice the volume of a given cube), and the proof of the 5th postulate (proof of the existence of a unique straight line which passes through a given point and is parallel to a given straight line), all have only slightly shorter histories. These problems were also set in antiquity, and solved between 1826 and 1837.

This book presents the solution to one significant and fascinating mathematical, or more precisely, geometrical problem. Though this problem also has a long history, it was not worked on so intensively, so it is less well known. This is the problem of the construction of a simple regular polygon with n sides. It was set in antiquity and completely solved between 1796 and 1801. The precise definition of a simple regular polygon is a polygon which is equilateral and equiangular, and where no more than two neighbouring sides share a vertex. The vertices of such a polygon obviously lie on a circle, and consecutive vertices divide the circle in n equal arcs, so this problem is also known as cyclotomic (from the Greek words κύκλος – circle and τόμος – cut). This is a special case of the problem of dividing the arc of an arbitrary smooth curve into n equal parts.

Solving the construction of regular polygons began in antiquity, even though the problem was not explicitly formulated at the time. In antiquity the regular (equilateral) triangle, the square, and the pentagon were constructed, and it was clearly understood that for a given polygon, it is possible to construct a polygon with twice as many sides. It was also known that it is possible to construct a polygon with mn sides, if it is possible to construct polygons with m and n sides, under the condition that m and n are coprime. It seems that Archimedes analysed the possibility

¹I believe you are wrong to avoid the theory of [regular] polygons in my D[isquisitiones] A[rithmeticae], and if you would just try, you would surely find everything there very simple.

of constructing a regular heptagon, but his results on heptagons are only partially preserved, and solely in an Arabic translation. This Archimedes manuscript inspired medieval muslim mathematicians in the 10th and 11th centuries to find several constructions of a regular heptagon using conic sections.

The discovery of analytic geometry allowed simple transformation of geometric problems into algebraic ones. The existence problem of a geometric construction is reduced to the solvability of algebraic equations. In 1637, Descartes proved that every geometric construction using ruler and compass (by a ruler is understood a straight-edge without any markings) is equivalent to solving a sequence of linear and quadratic equations. The problem of construction of a regular n -sided polygon comes down to the analysis of the solution of $x^n - 1 = 0$. In the 17th century this equation was named cyclotomic.

The connection between regular polygons and the solution to the cyclotomic equation $x^n - 1 = 0$ was first made at the end of the 17th, and at the beginning of the 18th century, by Michel Rolle, Roger Cotes and Abraham de Moivre, but they were trying to solve this equation trigonometrically. The general method for the algebraic solution of the cyclotomic equation was found by Vandermond and Lagrange in the second half of the 18th century. However they did not analyse when it is possible to construct geometrically their algebraic solution. The essence of geometric construction of a regular polygon of n sides was first understood by the great German mathematician Gauss, before he got out of bed on the morning of the 29th March 1796. Gauss was not yet 19 years old at the time. What is fascinating about his approach to the algebraic solution to this problem is that it connects in an unexpected way geometry, algebra and number theory.

Unfortunately, in the mathematical literature, the results on regular polygon constructions are only discussed fragmentary. I do not know of a single book in which this problem is presented systematically. The overall solution to this problem requires rather deep algebraic analysis, with number theory playing a prominent role. Few geometry books even present the construction of a regular heptadecagon. In them the algebraic analysis is not given, or is replaced by geometrical reasoning, so that the construction itself seems unmotivated and unnatural. In algebra books the situation is not much better. It is algebraically proven that the heptadecagon is constructible, as an example of the application of Galois theory using excessively strong theorems. The geometric construction itself is either left to the reader, or a reference is given to another book featuring a sketch of Richmond's construction. The description of the construction is very sketchy and the motivation for it does not receive any attention.

This book systematically presents both the algebraic and geometric aspects of the construction of regular polygons. In doing so, I have tried to use a minimal amount of foreknowledge, to keep the proofs as simple as possible, so as to make this book accessible to advanced secondary school students. For this reason, the text is not short, but I hope it is clear. Above all, exact constructions are performed and

analysed, i.e. those which theoretically exactly divide a circle into n equal parts, regardless of whether the division itself may be performed with ruler and compass, or with some additional tools. There are many interesting constructions of circle division into n equal parts which can be made by ruler and compass only, which contain small errors. Approximate constructions will be presented only occasionally.

The fourth book of Euclid's Elements examines the relationships between polygons and circles. This is the only work from antiquity in which the construction of regular polygons is analysed. In the first chapter is presented the theory of regular polygons as known in the antiquity, and it ends with the easy constructions of regular polygons possible with either a compass alone or a ruler alone.

In the second chapter the constructions of regular pentagon are presented and analysed. The number of interesting constructions of regular pentagon is very surprising. When I began writing this book, I knew of only two or three constructions of the regular pentagon, and expected to find a few more in the literature. I never expected those "few more" to turn out to be more than *fifty* sufficiently different constructions for something as simple as the regular pentagon. I have tried to present the most diversified and the most elegant constructions of the regular pentagon, in an order which seemed logical to me. Some of the constructions are simplified, and some of the constructions and proofs of known constructions are original, which does not necessarily mean that they are new, because real novelty in this field is hard to establish. The constructions I believe to be new have been marked in the text.

In the third chapter are examined the constructions of the regular heptagon and nonagon made by muslim medieval mathematicians. The mathematical literature somewhat neglects the regular heptagon. Analysis of its the constructibility began as far back as the antiquity, and great interest in solving this problem was shown by medieval muslim mathematicians. It was not until 1927 that it was discovered that Archimedes was also interested in the construction of the regular heptagon. The regular heptagon constructions of the muslim mathematicians became generally accessible only in 1984, thanks to the great efforts of Dutch historian of mathematics Jan Hogendijk [43]. To construct a regular heptagon, apart from a ruler and compass, conic sections or some other more powerful tools must be used. This was not known at the time. The third chapter ends with constructions of the regular nonagon.

The content of the first three chapters is only partially chronological, as I have tried to keep the proofs and the constructions as simple and related as possible. The proofs given are mainly synthetic.

The fourth chapter presents the algebraic approach to constructing regular polygons. Both algebraic and geometric solutions to cubic equations are given in detail, since solution of many well known geometric problems require the solution of a cubic equation.

In the fifth chapter, regular polygons are constructed using algebraic analysis. For these constructions it is sufficient to know how to solve quadratic and cubic equations. The algebraic motivation of some previous constructions is explained,

and a few more, newer constructions of the regular heptagon are given, to bring the books total to more than 15. At the end, algebraic constructions of the regular nonagon and tridecagon are given.

The sixth chapter presents Gauss's proof that it is possible to construct the regular heptadecagon with ruler and compass. The chapter begins with a history of that fascinating discovery by Gauss. Why the heptadecagon? The task to answer that question has a great difficulty². In any case, for Pythagoreans seventeen was an unlucky number (maybe that is the reason for the name of Hitchcock's thriller "Number seventeen" from 1932). This was pretty much the only thing which distinguished 17 from other odd prime numbers up to that time. Gauss's proof that the regular heptadecagon is geometrically constructible is given in detail. His idea is then used to completely clarify the method for solving the cyclotomic equation for the tridecagon. At the end of the chapter are given derivations of some number theoretical propositions which are used in Gauss's proof.

In recent mathematical literature (published after World War II), if the construction of a regular heptadecagon is even mentioned, either Richmond's construction, or a reference to a book containing it, is given. One gets the impression that there is only Richmond's construction of it. This is a great shame. Although Richmond's construction is geometrically very elegant, there are many constructions of regular heptadecagon which are both algebraically and geometrically very interesting, and in a certain sense, more natural. In the seventh chapter are presented *fifteen* appreciably different and interesting constructions of a regular heptadecagon which can be made using Gaussian analysis. None of them is Richmond's.

The eighth chapter presents constructions of the regular heptadecagon which do not explicitly use Gauss's theory of solving the cyclotomic equation. Instead in them are used simple geometric analysis and algebraic and trigonometrical calculations. This approach shortens the exposition of the proof and reduces the amount of foreknowledge. However it fails to increase the knowledge and to understanding it contributes almost nothing. These proofs seem unmotivated and artificial. It remains unclear why these constructions work for heptadecagon. A detailed explanation of Richmond's construction of regular heptadecagon is given in this chapter.

In the ninth chapter is presented the algebraic solution of the general cyclotomic equation, and the complete solution to the problem of the constructibility of regular polygons. The basic concepts of field theory are given, the necessary properties of polynomials, and the algebraic solutions to the general cyclotomic equation are

²"He [Don Quixote] then begged the bachelor, if he were a poet, to do him the favour of composing some verses for him conveying the farewell he meant to take of his lady Dulcinea del Toboso, and to see that a letter of her name was placed at the beginning of each line, so that, at the end of the verses, "Dulcinea del Toboso" might be read by putting together the first letters. The bachelor replied that although he was not one of the famous poets of Spain, who were, they said, only three and a half, he would not fail to compose the required verses; though he saw a great difficulty in the task, as the letters which made up the name were seventeen;" M. de Cervantes "Don Quixote", part II, ch. iv

illustrated in detailed examples for $n = 5, 7$ and 11 . In the solution for $n = 11$ are corrected some historical mistakes. Algebraic criteria for solving geometrical problems with a ruler and compass are proven. It would be more mathematically correct to say constructible "with straight lines and circles" than "with a ruler and compass". The criteria for geometric constructions which use conic sections and a ruler and compass are given.

Proof of the constructibility of regular polygons can be derived from the algebraic solution to the cyclotomic equation. Hence, this proof becomes elementary. The algebraic solution of cyclotomic equation by Gauss and Lagrange are given, and so it is unnecessary to apply the general criteria of solvability for the n -th degree equation to prove its algebraic solvability. Therefore, the knowledge of Galois theory for the solution of this problem is not necessary. In the last section of the chapter, the division of arcs of curves related to the circle into equal parts is explained.

As is customary in recent mathematical literature, the set of integers is denoted by $\mathbb{Z}$, the set of rationals by $\mathbb{Q}$, the set of reals by $\mathbb{R}$, the set of complex numbers by $\mathbb{C}$, and the end of a proof by special sign $\odot$ (instead of more neutral $\square$). Brackets are used to number the books and articles referenced at the end of this book. Quotations in foreign languages for which I could not find accessible translation have been translated by myself, and I bear responsibility for the accuracy of it.

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Novi Sad, on New Year 2025.

Autor

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1

Classical polygons

People have always been fascinated by regular polygons. In the history of human civilisation, regular polygons appear as patterns very early on. The first regular polygons to appear were the equilateral triangle, square and regular hexagon. These three figures were found a very long time ago, and it is not possible to determine reliably where they first appeared. It is possible they were discovered many times over throughout prehistory and history. Their geometrical constructions are very simple and no one pretends to have discovered them.

Interestingly, in the the oldest saved pictures from ancient Egypt, only the division of a circle into four, eight etc. parts appears, whilst division into three, five, six, twelve and similar number of parts appears first in Mesopotamia (figure 1). This does not have to do with how long ago the figures themselves were found, or how culture influenced each other, since these figures are saved in images of gods, kings, sacral or royal chariots and so on. In images of kings or in images with religious subjects, every detail has a symbolic meaning, and patterns are not used at the artists whim. It is not clear whether such



Figure 1. Assyrian god

images include all the interesting and known patterns or figures of that civilisation. Upper and Lower Egypt were united to make the Ancient Egypt, and so duality is much more significant symbolically than trinity or some other multiplicity, which might explain the importance of doubling there.

However, it is one thing to notice a regular geometric pattern or figure, and quite another to study systematically the properties of a geometrical figure and in some way describe its regularity. The exposition of geometrically correct constructions of regular polygons, and their study can be made only in those civilisations where geometry has developed as an abstract art. In ancient Egypt and Mesopotamia mathematics, and thus geometry, was studied as a craft, practical skill, in a similar way to medicine, or working with clay, metal or wood. Those who knew mathematics were expected to know standard practical methods, empirical rules for solving certain classes of problems, and nothing more.

The oldest civilisation in which mathematics was studied abstractly, especially where abstract geometry has developed, was the ancient Greek civilisation.

1.1 Constructions of regular polygons in ancient Greece

Ancient Greek philosophers in Ionian towns started studying geometry in the 6th century BC. In about 300 years ancient Greek geometry achieved a very high level of development. The most famous antique work in Geometry is the *Elements* (gr. Στοιχεῖα) which consist of thirteen books written by Alexandrian mathematician Euclid (Εὐκλείδης, 3rd century BC). In the *Elements* is given foundation for all geometry, as well as a large part of antique geometrical achievements.

In the fourth book in Euclid's *Elements* is examined the relation between a polygon and a circle [36]. In this book, Euclid solves four problems: To inscribe a polygon in a circle, to circumscribe a polygon around a circle, to inscribe a circle into a polygon, and to circumscribe a circle around a polygon. These four problems can be easily solved for a triangle similar to a triangle of any shape. However, the solution of these problems for polygons with more than three sides is simple only if these polygons are regular. In *Elements*, these problems are further investigated only for the regular quadrilateral (a square), the regular pentagon, the regular hexagon and the regular pentadecagon (15 sided polygon). Theorems from Euclid's *Elements* are cited by the book number as Roman numeral followed by the theorem number as Arabic numeral.

The theorems of Book IV are briefly formulated as:

IV.1. In a given circle to fit a given line segment which is not greater than the diameter of the circle.

IV.2. In a given circle to inscribe a triangle similar to a given triangle.

- IV.3. About a given circle to circumscribe a triangle similar to a given triangle.
- IV.4. In a given triangle to inscribe a circle.
- IV.5. About a given triangle to circumscribe a circle.
- IV.6. In a given circle to inscribe a square.
- IV.7. About a given circle to circumscribe a square.
- IV.8. In a given square to inscribe a circle.
- IV.9. About a given square circumscribe a circle.
- IV.10. To construct an isosceles triangle having each of the angles at the base double of the remaining one.
- IV.11. In a given circle to inscribe a regular pentagon.
- IV.12. About a given circle to circumscribe a regular pentagon.
- IV.13. In a given regular pentagon to inscribe a circle.
- IV.14. About a given regular pentagon to circumscribe a circle.
- IV.15. In a given circle to inscribe a regular hexagon.
- IV.16. In a given circle to inscribe a regular pentadecagon.

In his book, Euclid always named regular polygons equilateral and equiangular figures, and the word regular appears much later.

It is easy to construct a square, a regular (equilateral) triangle and a hexagon inscribed in a given circle. The square is inscribed by constructing two perpendicular diameters, a side of a regular hexagon is equal to the radius of that circle, and an equilateral triangle is obtained by connecting every second vertex of a hexagon.

The most interesting problem is the construction of a regular polygon with an odd number of sides, especially when this number is an odd prime number. Polygons with an even number of sides can be simply obtained from polygons with half as many sides by halving the arc over each side. If the number of sides has an odd factor, then the process of halving begins from the regular polygon with that odd number of sides, and if it is of the form 2^k , $k > 2$, then the polygon is obtained by halving the arcs over the sides of a square. Of the polygons with odd numbers of sides Euclid only constructs the regular pentagon and the regular pentadecagon. We will describe the construction of the regular pentagon first.

Pythagoreans were fascinated by five cornered star, the stellated pentagon. This polygon is obtained when a circle is divided into five equal parts and every second division point is connected – a pentagram (gr. πεντάγραμμον five lines). They attached a magical significance to it, chose it as their secret sign, called it health (gr. ὑγίαια) and assiduously studied different geometrical relations in it. Some authors believe that incommensurable line segments were first discovered in the pentagram, and that only later it was found out the simpler incommensurability between the diagonal of a square and its side. The analysis of the constructibility of the regular pentagon and its construction by ruler and compass is also attributed to Pythagoreans.

The construction of a regular pentagon depends on a simple, but not very obvious theorem, which is that peripheral angles over the same arc, with vertices which

do not belong to that arc, are equal (III.21). We will use this proposition to derive and prove the construction of the regular pentagon.

Let us have a regular pentagon $ABCDE$ (figure 2). If the diagonals AC , AD and CE are drawn, and if CE and AD intersect at F , then the angles $\sphericalangle CAD$, $\sphericalangle ACE = \sphericalangle ACF$ and $\sphericalangle ECD = \sphericalangle FCD$ are equal. From these equalities it follows that a regular pentagon can be constructed if we know how to construct an isosceles triangle $\triangle ACD$ in which the angles at the base are twice the third angle. Therefore, we need to construct an isosceles triangle which has angles $2\pi/5 = 72^\circ$ on its base, and at the top, the angle $\pi/5 = 36^\circ$. Such an isosceles triangle is constructed algebraically. Let $\triangle ACD$ be one such triangle (figure 2). Let the angle bisector at the base AC at vertex C intersects the opposite side AD at point F . By assumption

$$\sphericalangle ACD = 2\sphericalangle CAD, \quad \text{and then} \quad \sphericalangle CAD = \sphericalangle FCD}.$$

Angle $\sphericalangle CDA = \sphericalangle CDF$ and so triangles $\triangle CAD$ and $\triangle DCF$ are similar and it follows $AD : CD = CD : DF$. From the equality of angles $\sphericalangle CAD = \sphericalangle DCF = \sphericalangle FCA$ we have that triangles $\triangle DCF$ and $\triangle CFA$ are isosceles and so $CD = CF = AF$. When equality $CD = AF$ is substituted into $AD : CD = CD : DF$ we obtain $AD : AF = AF : DF$.

In other words, the problem of the constructibility of the regular pentagon is reduced to the division of a line segment (AD) so that the ratio of the entire segment (AD) to its larger part (AF) is the same as the ratio of the larger part (AF) to the smaller part (DF). Such a division today is called the golden section. The expression golden section appeared at the beginning of the XIX century. In the middle ages, this division was called the Divine proportion, or the continuous proportion and the ancient Greeks called it division into extreme and mean ratio (gr. ἄκρον καὶ μέσον λόγον). If the entire segment AD is denoted by d , and its larger part AF denoted by x then we obtain that the proportion $AD : AF = AF : DF$ is equivalent to the equation

$$d : x = x : (d - x) \quad \text{or} \quad x^2 + dx - d^2 = 0.$$

The solution of this quadratic equation is

$$x_{1,2} = \frac{-d \pm \sqrt{5d^2}}{2} = -\frac{d}{2} \pm \sqrt{d^2 + \left(\frac{d}{2}\right)^2} = d \frac{-1 \pm \sqrt{5}}{2}.$$

The negative solution has no geometrical meaning, so we obtain that

$$x = \sqrt{d^2 + \left(\frac{d}{2}\right)^2} - \frac{d}{2} = d \frac{\sqrt{5} - 1}{2}$$

is the larger part when line segment d is divided by golden section. x is easy to construct from the formula. If we construct a right-angled triangle with legs d and $d/2$, and then subtract from its hypotenuse the smaller leg, the remainder is x .

It should be noted that $x = d(\sqrt{5} - 1)/2$ is s_{10} , the length of the side of a regular decagon inscribed in the circle with centre at A of radius d , since triangle $\triangle CAD$ in

figure 2 is an isosceles triangle with the angle $\pi/5 = 36^\circ$ at vertex A . The construction of the regular decagon is not mentioned in Euclid's Elements.

The ancient Greeks solved quadratic equations when they had positive real roots, using the geometrical method described in the second and sixth books of Euclid's Elements. In the second book, the quadratic equation with the coefficient one at x^2 is solved, and in the sixth book when that coefficient is different from one.

For geometrical solutions of quadratic equations the fifth and sixth theorem from book II of Euclid's Elements, which we denote II.5 and II.6, are used. We will explain the geometrical method for solving quadratic equations using these theorems.

Theorem II.5 states: "If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half".

For modern readers, the formulation of this theorem becomes much more understandable if a picture is drawn or if it is expressed algebraically. Let AB be a line segment, S its midpoint, and T the point of nonequal division (figure 3). If a is the smaller part of the line in the nonequal division, and b the line between the division points, then $a + b$ is half of the line. We will use Roman numerals to denote areas, as in the figure. Then the equality $(I + II) + IV = (III + 2II + IV)$ of areas has to be proved. By construction, $I = II + III$, so when it is substituted to the left hand side the desired equality obtained.

Theorem II.5 can be reduced to a couple of equivalent algebraic identities depending on how the line segment and its parts are denoted. If the line segments are denoted as in the figure: $I + II$ is $(a + 2b)a$, IV is b^2 , and $III + 2II + IV$ is $(a + b)^2$, then we obtain the identity

$$(a + 2b)a + b^2 = (a + b)^2.$$

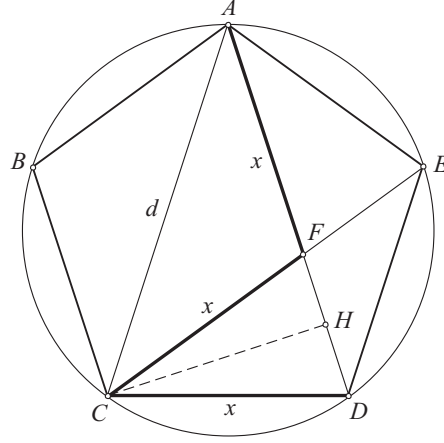


Figure 2. Analysis of the construction of the regular pentagon

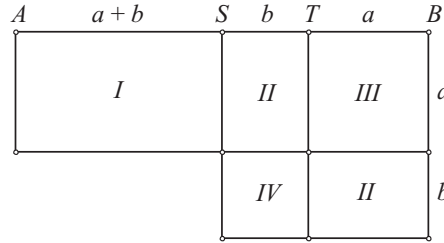
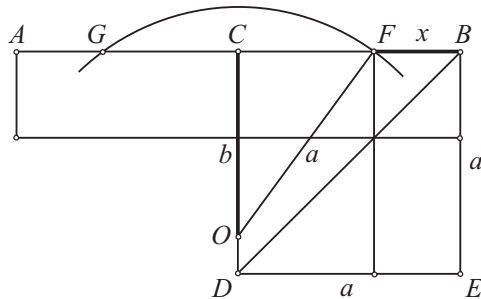


Figure 3. Proof of theorem II.5

$$ab + \left(\frac{a-b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2, \quad \text{or} \quad \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab.$$
$$(a+b)(a-b) + b^2 = a^2, \quad \text{or} \quad a^2 - b^2 = (a+b)(a-b).$$


If we have to divide a given line segment $2a$ into two parts x , and $2a - x$, so that their product $x(2a - x)$ has to be equal to a given square b^2 , then we have to solve the quadratic equation $2ax - x^2 = b^2$. Its solutions are $x_{1,2} = a \pm \sqrt{a^2 - b^2}$. In order for the solutions to be real and positive, it has to be $a > b$, and if x is the smaller segment then $x = a - \sqrt{a^2 - b^2}$.

$$(2a - x)x + (a^2 - b^2) = a^2.$$

If this formulation is translated into algebraic language, one interpretation is

$$(2a+x)x+a^2=(a+x)^2,$$

when the whole segment is denoted by $2a$, and the extension x . The algebraic proof of this theorem is very simple, and the geometric proof of this theorem is clear from figure 5. We just need to note that rectangle $AFPO$ is congruent to rectangle $GHDE$. These rectangles are denoted with I .

Theorem II.6 makes it possible to easily solve quadratic equation

$$x^2 + 2ax = b^2,$$

as $OS = OC$ (figure 5) and so

$$x = OC - OB = \sqrt{a^2 + b^2} - a, \iff x^2 + 2ax = b^2.$$

For the construction of the regular pentagon, we need to solve $x^2 + ax = a^2$, a special case of the previous quadratic equation (coefficient of x is a and $b = a$). Euclid states and solves this special case in a separate theorem II.11.

"To cut a given straight line so that rectangle contained by the whole and one of the segments is equal to the square on the remaining segment."

If the line segment is denoted by a and the other segment by x then we get

$$a(a - x) = x^2 \quad \text{or} \quad a : x = x : (a - x) \\ \text{or} \quad x^2 + ax = a^2.$$

In Euclid's Elements the regular pentagon is inscribed in a given circle in the following manner. In theorem IV.10, based on the formula for the length of the side of a pentagon (a is the diagonal)

$$x = a \frac{\sqrt{5} - 1}{2} = \sqrt{a^2 + \left(\frac{a}{2}\right)^2} - \frac{a}{2},$$

we construct an arbitrary isosceles triangle, in which the angles in the base are twice the third angle. Using theorem IV.2 a triangle similar to this isosceles triangle is inscribed in the circle. This construction is shown in figure 6.

Let $\triangle ABC$ be the given triangle. We construct tangent pp' at an arbitrary point P on the circle. We construct angles $\angle pPQ = \angle ACB$ and $\angle p'PR = \angle ABC$. Then

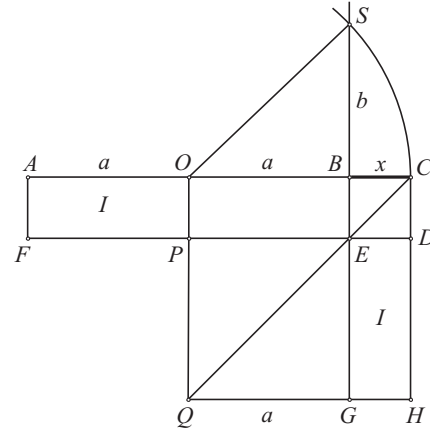


Figure 5. Geometrical solving of the quadratic equation $x^2 + 2ax = b^2$

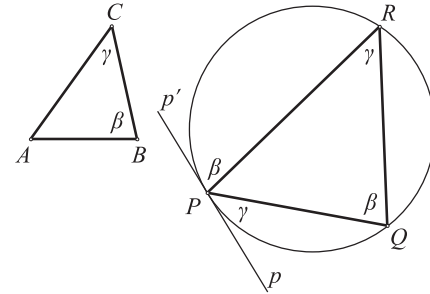


Figure 6. Construction for IV.2

based on theorem III.32 the angle formed by the tangent and the chord is equal to the peripheral angle on the arc of the circle on the opposite side of the chord from the side of the angle on the tangent (p), and so we obtain $\angle pPQ = \angle ACB = \angle PRQ$, and similarly $\angle p'PR = \angle ABC = \angle PQR$.

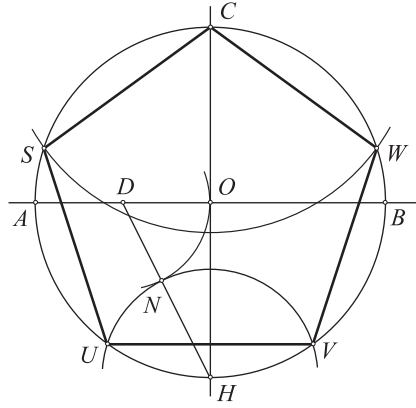


Figure 7. Construction of the regular pentagon using the side of a decagon

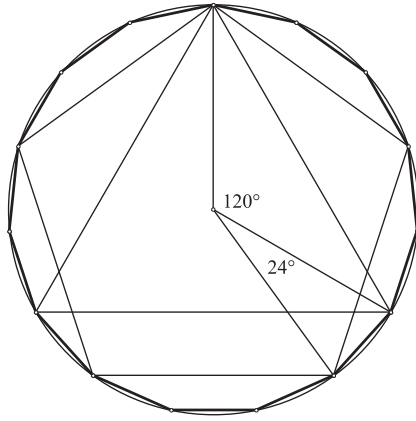


Figure 8. Construction of the regular pentadecagon
Then $CSUVW$ is an inscribed regular pentagon.

In Euclid's Elements, the regular pentadecagon is constructed in IV.16. He constructs it by combining the constructions of the regular pentagon and the equilateral

In this way, we obtain three vertices of a regular pentagon inscribed in a given circle, and then we construct bisectors of the angles on the base of the triangle, and prove that the intersections of these bisectors and the circle determine the two remaining vertices of the regular pentagon.

Using this idea the side of a regular pentagon can be constructed in a simpler manner. E.g. the chord which makes an angle of 36° with the tangent at one of its end-points, is a side of a inscribed regular pentagon. Moreover, the intersection points of the circle and legs of any angle of 36° with the vertex at an arbitrary point on the circumference if both its legs are at the same side of the tangent at that point, determine the side of a inscribed regular pentagon.

Alternatively, using the side of a regular decagon constructed using

$$s_{10} = \sqrt{a^2 + \left(\frac{a}{2}\right)^2} - \frac{a}{2},$$

(a is the radius of the circle), three consecutive vertices of a decagon are drawn, and then the shortest diagonal is constructed. This construction is shown in figure 7. In the circle with centre at O , we construct two perpendicular diameters AB and CH . If a circle with centre at D , the midpoint of AO , of radius $DO = a/2$, intersects DH at N , then $HN = s_{10}$. Let the circle with centre at H of radius HN intersect the initial circle at U and V . Then $UV = s_5$. Let the circle with centre at C of radius UV intersect the initial circle at S and W .