Chapter 1

Some Useful Substitutions

Let us first recall the classical substitutions that will be used in the following problems. All of these are discussed in detail in [**?**], chapter 1 and the reader is invited to take a closer look there.

Consider three positive real numbers a, b, c. If $abc = 1$, a classical substitution is

$$a = \frac{x}{y}, \quad b = \frac{y}{z}, \quad c = \frac{z}{x}.$$

A less classical one is

$$a = \frac{x}{y+z}, \quad b = \frac{y}{z+x}, \quad c = \frac{z}{x+y}$$

(for some positive real numbers x, y, z) whenever $ab + bc + ca + 2abc = 1$, or its equivalent version

$$a = \frac{y+z}{x}, \quad b = \frac{z+x}{y}, \quad c = \frac{x+y}{z}$$

when $abc = a + b + c + 2$ (the equivalence between the two relations follows from the substitution $(a, b, c) \to \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$). Two other very useful substitutions concern the relations $a^2 + b^2 + c^2 = abc + 4$ and $a^2 + b^2 + c^2 + 2abc = 1$. In the first case, with the extra assumption $\max(a, b, c) \geq 2$, one can find positive numbers x, y, z such that $xyz = 1$ and

$$a = x + \frac{1}{x}, \quad b = y + \frac{1}{y}, \quad c = z + \frac{1}{z}.$$

In the second case one can find an acute-angled triangle ABC such that

$$a = \cos(A), \quad b = \cos(B), \quad c = \cos(C).$$

Of course, in practice one often needs to use a mixture of these substitutions and to be rather familiar with classical identities and inequalities. But experience comes with practice, so let us delve into some exercises and problems to see how things really work.

1.1 The relation $a^2 + b^2 + c^2 = abc + 4$

We start with an easy exercise, based on the resolution of a quadratic equation.

1. Prove that if $a, b, c \geq 0$ satisfy $|a^2 + b^2 + c^2 - 4| = abc$, then

$$(a-2)(b-2) + (b-2)(c-2) + (c-2)(a-2) \geq 0.$$

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Proof. If $\max(a, b, c) < 2$, then everything is clear, so assume that

$$\max(a, b, c) \geq 2.$$

Then $a^2 + b^2 + c^2 - abc = 4$, so there exist positive numbers x, y, z such that $xyz = 1$ and

$$a = x + \frac{1}{x}, \quad b = y + \frac{1}{y}, \quad c = z + \frac{1}{z}.$$

But then $a, b, c \geq 2$ and we are done again. □

Proof. The most natural idea is to consider the hypothesis as a quadratic equation in a, for instance. It becomes $a^2 \pm abc + b^2 + c^2 - 4 = 0$, and solving the equation yields

$$a = \frac{\mp bc \pm \sqrt{(b^2-4)(c^2-4)}}{2}.$$

Thus $(b^2 - 4)(c^2 - 4) = (bc \pm 2a)^2$, which can also be written as

$$(b-2)(c-2) = \frac{(bc \pm 2a)^2}{(b+2)(c+2)} \geq 0.$$

Writing similar expressions for the other two variables, we are done. □

The following exercise is trickier and one needs some algebraic skills in order to solve it. We present two solutions, neither of which is really easy.

2. Find all triples x, y, z of positive real numbers such that

$$\begin{cases} x^2 + y^2 + z^2 = xyz + 4 \\ xy + yz + zx = 2(x + y + z) \end{cases}$$

Proof. By the second equation we have $\max(x, y, z) \geq 2$ and so the first equation yields the existence of positive real numbers a, b, c such that

$$x = a + \frac{1}{a}, \quad y = b + \frac{1}{b}, \quad z = c + \frac{1}{c}$$

and $abc = 1$.

$$\sum \left(ab + \frac{1}{ab} + \frac{a}{b} + \frac{b}{a} \right) = 2 \sum \left(a + \frac{1}{a} \right).$$

Since $abc = 1$, we have

$$\sum \frac{1}{a} = \sum ab, \quad \sum \frac{1}{ab} = \sum a,$$

so the second equation can be written

$$\sum \left(\frac{a}{b} + \frac{b}{a} \right) = \sum a + \sum ab.$$

The left-hand side is also equal to

$$\sum c(a^2 + b^2) = \left(\sum a \right) \left(\sum ab \right) - 3,$$

because
$$\frac{a}{b}+\frac{b}{a}=\frac{a^2+b^2}{ab}=c(a^2+b^2).$$
We deduce that $\left(\sum a-1\right)\left(\sum ab-1\right)=4$. Since $\sum a\geq 3$ and $\sum ab\geq 3$ (by the AM-GM inequality and the fact that $abc=1$), this can only happen if $a=b=c=1$ and thus when $x=y=z=2$. □

Proof. If $x+y=2$, the second equation yields $xy=4$, so that $(x-y)^2=-12$ which is a contradiction. Thus $x+y\neq 2$ and similarly $y+z\neq 2$, $z+x\neq 2$. The second equation yields
$$z=2+\frac{4-xy}{x+y-2},$$
and a rather brutal insertion of this expression in the first equation gives
$$(x-y)^2+\left(\frac{4-xy}{x+y-2}\right)^2=\frac{(4-xy)^2}{2-x-y}.$$
Unless $x=y=2$, this implies the inequality $2>x+y$. If two of the numbers x,y,z are equal to 2, then trivially so is the third one. If not, the previous argument shows that $2>x+y, 2>y+z$ and $2>x+z$. But then the second equation yields
$$x+y+z>\sum_{cyc}x\left(\frac{y+z}{2}\right)=xy+yz+zx=2(x+y+z),$$
a contradiction. Thus, the only solution is $x=y=z=2$. □

The following problem hides under a clever algebraic manipulation a very simple AM-GM argument. The inequality is quite strong, as the reader can easily see by trying a brute-force approach.

3. Prove that if $a,b,c\geq 2$ satisfy $a^2+b^2+c^2=abc+4$, then
$$a+b+c+ab+ac+bc\geq 2\sqrt{(a+b+c+3)(a^2+b^2+c^2-3)}.$$

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Proof. The hypothesis yields the existence of positive real numbers x, y, z such that

$$a=\frac{x}{y}+\frac{y}{x},\quad b=\frac{y}{z}+\frac{z}{y},\quad c=\frac{x}{z}+\frac{z}{x}.$$

The miracle is that both sides of the inequality have very nice factorizations. For the right-hand side, this is easy to observe, since

$$a+b+c+3=(xy+yz+zx)\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)$$

and

$$a^2+b^2+c^2-3=\sum\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)+3=\left(\sum x^2\right)\left(\sum\frac{1}{x^2}\right).$$

For the left-hand side, things are more subtle, but one finally reaches the identity

$$a+b+c+ab+bc+ca=\left(\sum x^2\right)\left(\sum\frac{1}{xy}\right)+\left(\sum\frac{1}{x^2}\right)\left(\sum xy\right).$$

The desired inequality becomes simply the AM-GM inequality for two numbers! □

The following problems are rather tricky mixtures of algebraic manipulations and elementary number theory.

4. Find all triplets of positive integers (k, l, m) with sum 2002 and for which the system

$$\begin{cases}\dfrac{x}{y}+\dfrac{y}{x}=k\\[2ex] \dfrac{y}{z}+\dfrac{z}{y}=l\\[2ex] \dfrac{z}{x}+\dfrac{x}{z}=m\end{cases}$$

has real solutions.

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Proof. The system has solutions if and only if

$$k^2 + l^2 + m^2 = lkm + 4.$$

An easy computation shows that this relation is equivalent to

$$(k+2)(l+2)(m+2) = (k+l+m+2)^2.$$

As $k + l + m = 2002$, we deduce that any solution of the problem satisfies $k + l + m = 2002$ and

$$(k+2)(l+2)(m+2) = (k+l+m+2)^2 = 2004^2 = 2^4 \cdot 3^2 \cdot 167^2.$$

A simple case analysis shows that the only solutions are $k = l = 1000, m = 2$ and its permutations. The result follows. □

5. Solve in positive integers the equation

$$(x+2)(y+2)(z+2) = (x+y+z+2)^2.$$

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Proof. A simple algebraic manipulation shows that the equation is equivalent to $x^2+y^2+z^2 = xyz+4$ and, seeing this as a quadratic equation in z, we obtain the equivalent form $(x^2-4)(y^2-4) = (xy-2z)^2$. If $x^2 < 4$, then $y^2 < 4$ as well and so $x = y = 1$, yielding $z = 2$. If $x = 2$, then $y = z$. In all other cases, we can find a positive square-free integer D (which is easily seen to be different from 1) and positive integers u, v such that $x^2 - 4 = Du^2$ and $y^2 - 4 = Dv^2$. Thus, solving the problem comes down to solving the generalized Pell equation $a^2 - Db^2 = 4$, which is a classical topic: this equation always has nontrivial integer solutions and if (a_0, b_0) is the smallest solution with $a_0, b_0 > 0$, then all solutions are given by

$$a_n = \left(\frac{a_0 + b_0\sqrt{D}}{2}\right)^n + \left(\frac{a_0 - b_0\sqrt{D}}{2}\right)^n,$$

$$b_n = \frac{1}{\sqrt{D}}\left[\left(\frac{a_0 + b_0\sqrt{D}}{2}\right)^n - \left(\frac{a_0 - b_0\sqrt{D}}{2}\right)^n\right]. \qquad \square$$

Part of the following problem can be dealt in a classical way, but we do not know how to solve it entirely without using the trick of substitutions.

6. The sequence $(a_n)_{n\geq 0}$ is defined by $a_0 = a_1 = 97$ and

$$a_{n+1} = a_n a_{n-1} + \sqrt{(a_n^2 - 1)(a_{n-1}^2 - 1)}$$

for all $n \geq 1$. Prove that $2 + \sqrt{2 + 2a_n}$ is a perfect square for all n.

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Proof. Writing

$$(a_{n+1} - a_n a_{n-1})^2 = (a_n^2 - 1)(a_{n-1}^2 - 1)$$

and simplifying this expression yields

$$a_{n-1}^2 + a_n^2 + a_{n+1}^2 - 2a_n a_{n-1} a_{n+1} = 1,$$

thus

$$(2a_{n-1})^2 + (2a_n)^2 + (2a_{n+1})^2 - (2a_{n-1})(2a_n)(2a_{n+1}) = 4.$$

Since we clearly have $a_n > 2$ for all n, this implies the existence of a sequence $x_n > 1$ such that $2a_n = x_n + x_n^{-1}$ and such that $x_{n+1} = x_n x_{n-1}$. Thus $\log x_n$ satisfies a Fibonacci-type recursive relation and so we can immediately find out the general term of the sequence $(a_n)_n$. Namely, a small computation shows that if we define $\alpha = 2 + \sqrt{3}$, then $x_n = \alpha^{4F_n}$, where F_n is the nth Fibonacci number. Thus

$$a_n = \frac{1}{2}\left(\alpha^{4F_n} + \frac{1}{\alpha^{4F_n}}\right)$$

and so

$$2 + \sqrt{2 + 2a_n} = 2 + \left(\alpha^{2F_n} + \frac{1}{\alpha^{2F_n}}\right) = \left(\alpha^{F_n} + \frac{1}{\alpha^{F_n}}\right)^2.$$

The result follows, since $\alpha^n + \alpha^{-n} \in \mathbb{Z}$ for all n, by the binomial formula. $\square$

Remark 1.1. Here is an alternative proof of the fact that all terms of the sequence are integers, without the use of substitutions. The method that we will use for this problem appears in many other problems. As we saw in the previous solution, the sequence satisfies the recursive relation

$$a_{n+1}^2 + a_n^2 + a_{n-1}^2 - 2a_{n+1}a_n a_{n-1} = 1.$$

Writing the same relation for $n+1$ instead of n and subtracting the two yields the identity

$$a_{n+2}^2 - a_{n-1}^2 = 2a_n a_{n+1}(a_{n+2} - a_{n-1}).$$

Note that $(a_n)_n$ is an increasing sequence (this follows trivially by induction from the recursive relation), so that we can divide by $a_{n+2} - a_{n-1} \neq 0$ in the previous relation and get $a_{n+2} = 2a_n a_{n+1} - a_{n-1}$. The last relation clearly implies that all terms of the sequence are integers (since one can immediately check that this is the case with the first three terms of the sequence). Note however that it does not seem to follow easily that $2 + \sqrt{2 + 2a_n}$ is a perfect square using this method.

Remark 1.2. There are a lot of examples of very complicated recurrence relations that rather unexpectedly yield integers. For instance, the reader can try to prove the following result concerning Somos-5 sequences: let $a_1 = a_2 = \cdots = a_5 = 1$ and let

$$a_{n+5} = \frac{a_{n+1}a_{n+4} + a_{n+2}a_{n+3}}{a_n}$$

for $n \geq 0$. Then a_n is an integer for all n. Similarly one defines Somos-6, Somos-7, etc sequences by the formulas

$$a_0 = a_1 = \cdots = a_5 = 1, \quad a_{n+6} = \frac{a_{n+1}a_{n+5} + a_{n+2}a_{n+4} + a_{n+3}^2}{a_n},$$

$$a_0 = a_1 = \cdots = a_6 = 1, \quad a_{n+7} = \frac{a_{n+1}a_{n+6} + a_{n+2}a_{n+5} + a_{n+3}a_{n+4}}{a_n},$$

etc. One can prove (though this is not easy) that all terms of Somos-6 and Somos-7 sequences are integers. Surprisingly, this fails for Somos-8 sequences (in which case a_{17} is no longer an integer!).

1.2 The relations $abc = a + b + c + 2$ and $ab + bc + ca + 2abc = 1$

The first inequality in the following problem is very useful in practice and we will meet it very often in the following problems.

7. Prove that if $x, y, z > 0$ and $xyz = x + y + z + 2$, then

$$xy + yz + zx \geq 2(x + y + z) \text{ and } \sqrt{x} + \sqrt{y} + \sqrt{z} \leq \frac{3}{2}\sqrt{xyz}.$$

Proof. With the usual substitutions

$$x = \frac{b+c}{a}, \quad y = \frac{c+a}{b}, \quad z = \frac{a+b}{c},$$

the first inequality comes down (after clearing denominators and canceling out similar terms) to Schur's inequality

$$a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b) \geq 0,$$

while the second one follows by adding up the inequalities

$$\sqrt{\frac{1}{xy}} = \sqrt{\frac{a}{a+c} \cdot \frac{b}{b+c}} \leq \frac{1}{2}\left(\frac{a}{c+a} + \frac{b}{b+c}\right). \qquad \square$$

The reader may find a bit strange the first method of proof of the following problem, but it is actually a quite powerful one. We will use again this kind of argument, see problem 11 for instance. Also, the third solution uses a very useful technique.

8. Let $x, y, z > 0$ be such that $xy + yz + zx = 2(x + y + z)$. Prove that

$$xyz \leq x + y + z + 2.$$

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Proof. We will argue by contradiction, assuming that $xyz > x+y+z+2$. We claim that we can find $0 < r < 1$ such that $X = rx, Y = ry, Z = rz$ satisfy $XYZ = X + Y + Z + 2$. Indeed, this comes down to the vanishing of

$$f(r) = r^3xyz - r(x + y + z) - 2$$

between 0 and 1, and this is clear, since $f(0) < 0$ and $f(1) > 0$. Next, the condition $xy + yz + zx = 2(x + y + z)$ yields

$$XY + YZ + ZX = 2r(X + Y + Z) < 2(X + Y + Z).$$

This contradicts the first inequality of problem 7. □

Proof. The condition can also be rewritten in the form

$$(x - 1)(y - 1) + (y - 1)(z - 1) + (z - 1)(x - 1) = 3$$

or in the form

$$xyz - x - y - z - 1 = (x - 1)(y - 1)(z - 1).$$

We will discuss several cases. If $x, y, z \geq 1$, then by the AM-GM inequality and the first identity, we get

$$1 \geq \sqrt[3]{(x-1)^2(y-1)^2(z-1)^2},$$

which yields, thanks to the second identity, the desired estimate.

If $x, y, z \leq 1$ or if only one of the numbers x, y, z is smaller than or equal to 1, then $(x - 1)(y - 1)(z - 1) \leq 0$ and so $xyz \leq x + y + z + 1$ in this case. Finally, if two of the numbers are smaller than 1, say $x, y \leq 1$, the desired inequality can be written in the form $0 \leq x + y + z(1 - xy) + 2$, which is obvious. □

Proof. For three positive real numbers x, y, z consider fixing the first two elementary symmetric polynomials $\sigma_1 = x + y + z$ and $\sigma_2 = xy + yz + zx$ and letting $\sigma_3 = xyz$ vary. This amounts to varying only the constant term in the polynomial

$$p(t) = t^3 - \sigma_1 t^2 + \sigma_2 t - \sigma_3 = (t - x)(t - y)(t - z)$$

and defining x, y, z to be the three roots of this polynomial (in some order). Increasing σ_3, i.e. lowering the constant term, corresponds geometrically to lowering the graph. As we lower the graph, the smallest root increases, thus we maintain three positive real roots until the smallest root becomes a double root. If the double root is at $t = a$ and the larger root at $t = b$, then we have $\sigma_1 = 2a + b$, $\sigma_2 = a^2 + 2ab$ and $\sigma_3 = a^2 b$.

If we fix σ_1 and σ_2 with $\sigma_2 = 2\sigma_1$ as hypothesized, then we find $b = \frac{(4-a)a}{2(a-1)}$ and because $0 < a \le b$, we see that $1 < a \le 2$. By the discussion above $xyz = \sigma_3 \le a^2 b$, so it suffices to show that

$$a^2 b = \frac{(4-a)a^3}{2(a-1)} \le 2a + b + 2 = 2a + \frac{(4-a)a}{2(a-1)} + 2$$

for $1 < a \le 2$. But this rearranges to $(a-2)^2(a^2-1) \ge 0$ and we are done. □

The technique used in the first solution of the next problem is rather versatile and the reader is invited to read the addendum **??** for more examples.

9. Let $x, y, z > 0$ be such that $xy + yz + zx + xyz = 4$. Prove that

$$3\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}}\right)^2 \ge (x+2)(y+2)(z+2).$$

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Proof. Using the usual substitution

$$x = \frac{2a}{b+c}, \quad y = \frac{2b}{c+a}, \quad z = \frac{2c}{a+b},$$

the problem reduces to proving the inequality

$$3\left(\sum \sqrt{\frac{b+c}{a}}\right)^2 \ge 16\frac{(a+b+c)^3}{(a+b)(b+c)(c+a)}.$$

This is a quite strong inequality and it is easy to convince oneself that most applications of classical techniques fail. However, the following smart application of Hölder's inequality does the job:

$$\left(\sum \sqrt{\frac{b+c}{a}}\right)^2 \cdot \left(\sum a(b+c)^2\right) \geq \left(\sum (b+c)\right)^3,$$

so it is enough to prove that

$$3\prod(a+b) \geq 2\sum a(b+c)^2.$$

This reduces after expanding to $\sum a(b-c)^2 \geq 0$, which is clear. □

Proof. First, we get rid of those nasty square roots, via the substitution

$$xy = 4c^2, \quad yz = 4a^2, \quad zx = 4b^2.$$

Then

$$x = \frac{2bc}{a}, \quad y = \frac{2ca}{b}, \quad z = \frac{2ab}{c}$$

and replacing these values in the inequality yields the equivalent form

$$3(a+b+c)^2 \geq 16(a+bc)(b+ca)(c+ab).$$

The hypothesis becomes $a^2+b^2+c^2+2abc = 1$, so that there exists an acute-angled triangle ABC such that $a = \cos A, \quad b = \cos B, \quad c = \cos C$. Next, observe that

$$c + ab = \cos C + \cos A \cdot \cos B = -\cos(A+B) + \cos A \cdot \cos B = \sin A \cdot \sin B.$$

Using this (and similar identities obtained by permuting the variables), the desired inequality becomes

$$\sqrt{3}\sum \cos A \geq 4\prod \sin A.$$

Using the well-known identities

$$s = 4R\prod \cos\frac{A}{2}, \quad r = 4R\prod \sin\frac{A}{2}, \quad \sum \cos A = 1 + \frac{r}{R},$$