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Introduction

Let us start with a simple (?) question. In a chess tournament there are n players who play eliminatory games until only one winner remains. Thus, in the first round, the players are arbitrarily playing games (chosen by drawing lots), and only the winners of these games go to the second round. If the number of players is odd there is one player staying aside, but he/she (and all winners) will take part to the drawing for the second round. The process repeats in the second and all the following rounds until, as we said, finally, only one winner remains (and he is declared the winner of the championship). The question is: how many games are necessary in order to establish the champion?
Well, you might need a moment of thinking, and we strongly advise you to take it (or, maybe, you already got the answer, which is great). You will immediately see that if $n = 2^m$ is a power of 2, then in the first round there are 2^{m-1} games (and 2^{m-1} winners from these games accede to the second round), and the process goes on and on so that there will be 2^{m-j} games in the jth round. The total number of games will then be

$$2^{m-1} + 2^{m-2} + \cdots + 2 + 1 = 2^m - 1 = n - 1.$$

Although we cannot quite use this reasoning in the general case, the answer $n - 1$ is correct for each and every value of n, because in each game precisely one player is eliminated and, in order to arrive to the situation when only one player still stands, $n - 1$ players must be eliminated, so, $n - 1$ games are needed to see who the champion is.
The problem is solved, but we won't stop here. This is because the reasoning in the particular case of $n = 2^m$ furnishes a hint for the general case. Namely,

in the first round the number of games is k if $n = 2k$ is an even number, and it is also k if $n = 2k + 1$ is odd (and a player is forced – also by drawing, to make the competition fair – to stay aside). This number can be expressed (for both cases of even and odd n) as $\lfloor n/2 \rfloor$ where $\lfloor x \rfloor$ denotes the integer part of x (or the floor function of x) – the largest integer which is not greater than x (that is, $\lfloor x \rfloor = p$ if and only if p is the only integer such that $p \leq x < p + 1$). Thus in the first round there are $\lfloor n/2 \rfloor$ games and in the second round

$$n_2 = n - \left\lfloor \frac{n}{2} \right\rfloor$$

players participate. Then the same pattern repeats: $\lfloor n_2/2 \rfloor$ games are played in the second round, and

$$n_3 = n_2 - \left\lfloor \frac{n_2}{2} \right\rfloor$$

players enter the third round, and so on. Thus, if we define the function f by

$$f(x) = x - \left\lfloor \frac{x}{2} \right\rfloor$$

the total number of games played is

$$\left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \cdots$$

where the sequence $(n_k)_{k \geq 1}$ is defined by $n_1 = n$ and the recurrence $n_k = f(n_{k-1})$ for every $k \geq 2$. One can easily see that, starting with every positive integer n the terms of the sequence $(n_k)_{k \geq 1}$ eventually become equal to 1, because this is a sequence of positive integers that strictly decreases as long as its terms are greater than 1 (and thus, at some moment, a term equals 1, and then all the terms that follow are also equal to 1). Thus the above sum is actually a finite one (as we have already seen in the particular case of n being a power of 2), as the integer parts $\lfloor n_k/2 \rfloor$ are 0 as soon as n_k becomes 1. Actually, we can find a formula for n_k, namely

$$n_k = \left\lceil \frac{n}{2^{k-1}} \right\rceil,$$

where the ceiling function of the real number x is defined by $\lceil x \rceil = q$ if and only if q is the unique integer such that $q - 1 < x \leq q$, and this formula allows us to see that n_k becomes 1 as soon as $2^{k-1} \geq n$ (and the first such k is $\lceil \log_2 n \rceil + 1$) – but this is not our point of interest here. Nevertheless, we are interested in the fact that

$$\left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \cdots = n - 1,$$

since we know now that the total number of games is $n-1$. For instance, with $n_1 = n = 7$, we have

$$\begin{aligned} n_2 &= n_1 - \left\lfloor \frac{n_1}{2} \right\rfloor = 7 - 3 = 4, \\ n_3 &= n_2 - \left\lfloor \frac{n_2}{2} \right\rfloor = 4 - 2 = 2, \\ n_4 &= n_3 - \left\lfloor \frac{n_3}{2} \right\rfloor = 2 - 1 = 1, \end{aligned}$$

and, consequently, $n_k = 1$ for all $k \geq 4$. Thus the number of games measured as a sum is

$$\left\lfloor \frac{7}{2} \right\rfloor + \left\lfloor \frac{4}{2} \right\rfloor + \left\lfloor \frac{2}{2} \right\rfloor + \left\lfloor \frac{1}{2} \right\rfloor + \cdots = \left\lfloor \frac{7}{2} \right\rfloor + \left\lfloor \frac{4}{2} \right\rfloor + \left\lfloor \frac{2}{2} \right\rfloor.$$

Of course, this is $3 + 2 + 1 = 6$ and corresponds to the result $7 - 1$ given by the (let us call it) global reasoning from the beginning.
To summarize: we defined a function

$$f(x) = x - \left\lfloor \frac{x}{2} \right\rfloor$$

and a sequence $(n_k)_{k \geq 1}$ starting with some arbitrary positive integer $n_1 = n$ and satisfying the recurrence relation $n_k = f(n_{k-1})$ for $k \geq 2$, and we obtained

$$\left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor + \cdots = n - 1.$$

This may be a somehow unexpected equation (although it is very clear in the particular case $n = 2^m$) and illustrates a simple principle in mathematics: do something in two different ways (in this case "do something" is "count"), then equate the two results (they must be equal, because, in the end, they represent the same thing – in our case the total number of games). You will be amazed of what can be obtained by using this very simple (because fundamental) principle.

Anyway, it is not our purpose here to examine such reasonings. Instead, we are dealing throughout this book with sums and products (as the title says). Sums and products are everywhere in mathematics. Probably the first matters that a man (usually a child) learns in mathematics are addition and multiplication (of natural numbers, then of integers, and so on) – operations for which the results are called sum and product respectively. Clearly, we do not intend to take it all from zero, but rather we will try to calculate as many sums as possible of the form

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

and products of the form

$$\prod_{k=1}^{n} a_k = a_1 a_2 \cdots a_n$$

(that is, sums and products of an arbitrary number of terms, respectively factors). Also, we will enter a little in the more advanced topic of infinite sums and products, defined as limits of corresponding finite partial sums and products, respectively:

$$\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=1}^{n} a_k$$

and

$$\prod_{k=1}^{\infty} a_k = \lim_{n \to \infty} \prod_{k=1}^{n} a_k.$$

We assume the reader to be familiar with the basic concepts of limit (and to have the knowledge of elementary limits), and (more rarely) of derivative and Riemann integral. There are, however, only few examples of this kind, and the reader who is not familiar with these concepts can skip them without losing the rest of the book. This clearly means that we expect the reader to know basic arithmetic, algebra, and trigonometry (complex numbers in algebraic

and trigonometric form included). Also, some combinatorial problems, and a few problems of number theory will be encountered.
Thus, we hope the reader understands a few basic properties of the symbols defined above, such as

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k,$$

or, more generally,

$$\sum_{k=1}^{n}(\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^{n} a_k + \beta \sum_{k=1}^{n} b_k,$$

where all of α, β, a_k, and b_k are (in general) complex numbers, with α and β being, of course, independent of k – the so called index of summation. By the way, this index can (in the same problem, or along the same computation) be denoted by different letters; thus

$$\sum_{k=1}^{n} a_k = \sum_{l=1}^{n} a_l \quad \text{or} \quad \prod_{i=1}^{n} a_i = \prod_{j=1}^{n} a_j.$$

We also have

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k \quad \text{and} \quad \prod_{k=1}^{n} a_k = \prod_{k=1}^{m} a_k \cdot \prod_{k=m+1}^{n} a_k$$

for $1 \le m \le n$. Maybe wording this as

$$\sum_{k=1}^{n} a_k = \left(\sum_{k=1}^{m} a_k\right) + \left(\sum_{k=m+1}^{n} a_k\right) \quad \text{and} \quad \prod_{k=1}^{n} a_k = \left(\prod_{k=1}^{m} a_k\right) \cdot \left(\prod_{k=m+1}^{n} a_k\right)$$

would be more accurate, but we prefer the first form, apart from the situation when we desperately need to avoid confusion.
All the above are clear consequences of the properties of addition and multiplication (commutativity, associativity, distributivity of multiplication over

addition). The same is true for

$$\sum_{k=1}^{n}(b_{k+1}-b_k)=b_{n+1}-b_1 \quad \text{and} \quad \prod_{k=1}^{n}\frac{b_{k+1}}{b_k}=\frac{b_{n+1}}{b_1}.$$

Understanding these equalities and learning to work with them (or with similar ones) is very important because they represent a powerful tool for evaluating sums and products with an arbitrary number of terms. More precisely, when we have to calculate a sum

$$\sum_{k=1}^{n} a_k,$$

expressing the general term a_k in the form

$$a_k = b_{k+1} - b_k$$

is very effective, due to the above formula: the numerous cancellations allow us to find a simple closed formula for the given sum. We call such a sum *telescopic* (or we say that the sum telescopes, etc.). This is because we can write

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(b_{k+1}-b_k) = -b_1 - b_2 - \cdots - b_n + b_2 + \cdots + b_n + b_{n+1} = b_{n+1} - b_1.$$

One of the simplest examples is the one that appeared in the beginning, namely

$$1 + 2 + \cdots + 2^{n-1} = \sum_{k=1}^{n} 2^{k-1}.$$

Thus we have $a_k = 2^{k-1} = 2^k - 2^{k-1} = b_{k+1} - b_k$ for $k = 1, 2, \ldots, n$, and we can consider $b_k = 2^{k-1}$ (the b_k appear here to be equal to the a_k; of course, this does not usually happen). Consequently,

$$\begin{aligned}1 + 2 + \cdots + 2^{n-1} &= \sum_{k=1}^{n} 2^{k-1} = \sum_{k=1}^{n}(2^k - 2^{k-1}) \\ &= \sum_{k=1}^{n}(b_{k+1} - b_k) = b_{n+1} - b_n \\ &= 2^n - 1.\end{aligned}$$

Throughout the book we will simply write this as

$$\sum_{k=1}^{n} 2^{k-1} = \sum_{k=1}^{n} (2^k - 2^{k-1}) = 2^n - 1.$$

As we said, we expect the reader to be familiar with some simple computations (and we think that this first sum that we just evaluated does not represent a mystery for our readers; we just used it as an example). Also, we hope that the fact that we replaced m by n does not represent an issue (the formula $\sum_{k=1}^{m} 2^{k-1} = 2^m - 1$ is the same as $\sum_{k=1}^{n} 2^{k-1} = 2^n - 1$, isn't it?). We preferred n to be more in the vein of what we just discussed about telescoping sums. Yet another simple example is the sum of the first n odd positive integers, that is

$$1 + 3 + \cdots + (2n - 1), \quad \text{or} \quad \sum_{k=1}^{n} (2k - 1).$$

Can you see the telescope? We have

$$\sum_{k=1}^{n} (2k - 1) = \sum_{k=1}^{n} (k^2 - (k-1)^2) = n^2 - 0^2 = n^2,$$

yielding a beautiful formula: the sum of the first n odd positive integers equals the square of n. In order to do this computation you only need to know the elementary algebraic formulae

$$(a \pm b)^2 = a^2 \pm 2ab + b^2.$$

More specific, we need $(k-1)^2 = k^2 - 2k + 1$, but we use it in the form

$$2k - 1 = k^2 - (k-1)^2.$$

This is the main difficulty when we try to evaluate a sum (or a product) by the telescoping method: how to find the numbers b_k? Of course, this depends on the skills and the experience of each solver. If one can find a closed form for a sum, then it is always possible to evaluate that sum by telescoping. Indeed, if we have

$$\sum_{k=1}^{n} a_k = S_n$$

for any positive integer n, then we also have

$$a_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k = S_n - S_{n-1}$$

for any n (where we define $S_0 = 0$). Thus we have $a_k = b_{k+1} - b_k$ for $b_k = S_{k-1}$. So, if we have the result, we can also telescope (but we prefer to be able to find the telescope ourselves). For instance, we have the (again well-known) identity

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

It is often said that, asked – when he was a little boy – by his teacher to sum the first 100 positive integers, Gauss did the job immediately, to the great surprise of the teacher, who had no idea about this method. His approach uses again fundamental properties of addition. First, we have

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a_{n-k+1}$$

(since addition is commutative, we can reverse the order of summation), then

$$\sum_{k=1}^{n} a_k = \frac{1}{2}\left(\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} a_{n-k+1}\right) = \sum_{k=1}^{n} \frac{1}{2}(a_k + a_{n-k+1})$$

(because if $A = B$, then $A = (A+B)/2$, too). In our case,

$$\sum_{k=1}^{n} k = \sum_{k=1}^{n} \frac{1}{2}(k + n - k + 1) = \sum_{k=1}^{n} \frac{n+1}{2} = \frac{n(n+1)}{2}.$$

(Note that, in general, $\sum_{k=1}^{n} a = na$ when a does not depend on the summation index; for example, $\sum_{k=1}^{n} 1 = n$.)

Basically, Gauss observed that the sums of the kth term and $(n-k+1)$th term of the given sum are all equal (to $n+1$), and he paired terms having the same sum. This can be done in general for an arithmetic progression $(a_n)_{n\geq 1}$ (that is, a sequence for which the differences $a_{k+1} - a_k$ are all equal to the same number d, called the common difference of the progression) in order to get the sum of the first n terms; as above, we have

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{1}{2}(a_k + a_{n-k+1}) = \frac{n(a_1 + a_n)}{2} = \frac{n(2a_1 + (n-1)d)}{2},$$

because $a_k + a_{n-k+1} = a_1 + a_n$ for every $k = 1, 2, \ldots, n$. Note that if we use the formula $a_k = a_1 + (k-1)d$ for the general term of the progression (with d the common difference) and the above formula for the sum of the first n (actually, here, the first $n-1$) positive integers, we can also evaluate this sum as

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(a_1 + (k-1)d) = \sum_{k=1}^{n} a_1 + d\sum_{k=1}^{n}(k-1) = na_1 + \frac{(n-1)n}{2}d.$$

Going back to Gauss's sum, now that we have the formula, we can also prove it by mathematical induction. To verify it for $n = 1$ is immediate, so we still need to show that if it is true for n, then it also works for $n+1$. Indeed, if

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

then

$$\begin{aligned}\sum_{k=1}^{n+1} k &= \sum_{k=1}^{n} k + (n+1) = (1 + 2 + \cdots + n) + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}.\end{aligned}$$

Also, we can prove it by telescoping:

$$\sum_{k=1}^{n} k = \sum_{k=1}^{n} \left(\frac{k(k+1)}{2} - \frac{(k-1)k}{2} \right) = \frac{n(n+1)}{2}.$$

Another famous example is the sum of a geometric progression

$$S = 1 + x + x^2 + \cdots + x^N = \sum_{n=0}^{N} x^n.$$

If $x \neq 1$, we can find a telescope by first multiplying through by $1 - x$ giving

$$(1-x)S = \sum_{n=0}^{N} (1-x)x^n = \sum_{n=0}^{N} (x^n - x^{n+1}) = 1 - x^{N+1},$$

and hence

$$S = \frac{1 - x^{N+1}}{1 - x}.$$

(Of course we could have also telescoped by noting that $x^n = \frac{x^n}{1-x} - \frac{x^{n+1}}{1-x}$.) A few other simple telescoping sums include (think for yourself before reading the solution)

$$\sum_{k=1}^{n} k \cdot k! = \sum_{k=1}^{n} ((k+1)! - k!) = (n+1)! - 1,$$

or

$$\begin{aligned} \sum_{k=1}^{n} k^3 = \sum_{k=1}^{n} k^2 \cdot k &= \sum_{k=1}^{n} k^2 \cdot \frac{(k+1)^2 - (k-1)^2}{4} \\ &= \sum_{k=1}^{n} \left(\left(\frac{k(k+1)}{2} \right)^2 - \left(\frac{(k-1)k}{2} \right)^2 \right) \\ &= \left(\frac{n(n+1)}{2} \right)^2. \end{aligned}$$

(Notice the beautiful result

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 \Leftrightarrow 1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2,$$

which is a rarity in the world of sums of powers of the first n integers.)
Also we have

$$\sum_{k+1}^{n} k(k+1) = \sum_{k=1}^{n} \left(\frac{k(k+1)(k+2)}{3} - \frac{(k-1)k(k+1)}{3}\right) = \frac{n(n+1)(n+2)}{3},$$

which permits us to evaluate

$$\begin{aligned}\sum_{k=1}^{n} k^2 &= \sum_{k=1}^{n} k(k+1) - \sum_{k=1}^{n} k \\ &= \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2} \\ &= \frac{n(n+1)(2n+1)}{6}.\end{aligned}$$

It would be hard to notice that this sum can be telescoped by using

$$k^2 = \frac{k(k+1)(2k+1)}{6} - \frac{(k-1)k(2k-1)}{6},$$

wouldn't it?
Now let us find a closed form for the (very important, as we will see) sum

$$S_n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

where x is an arbitrary number, and the *binomial coefficients* $\binom{n}{k}$ are defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

for integers $n \geq k \geq 0$. One also calls $\binom{n}{k}$ "n choose k" because this number counts all possibilities of arbitrarily choosing k objects from n given objects, disregarding their order. The equality

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

can be immediately verified for integers n and k with $1 \leq k \leq n-1$ by direct computation. It is called the recursive formula of the binomial coefficients. Sometimes we will use the convention $\binom{n}{k} = 0$ for $k > n$, or for $k < 0$; with this in mind the recursive formula holds for $k = 0$ and for $k = n$, too.
Using the equality $0! = 1$ (again, a convention) one finds immediately $S_0 = 1$ (and $S_1 = 1 + x$, and $S_2 = 1 + 2x + x^2 = (1+x)^2$). Then we have, for $n \geq 2$,

$$\begin{aligned} S_n - S_{n-1} &= x^n + \sum_{k=1}^{n-1} \left(\binom{n}{k} - \binom{n-1}{k} \right) x^k \\ &= x^n + x \sum_{k=1}^{n-1} \binom{n-1}{k-1} x^{k-1} \\ &= x^n + x \sum_{j=0}^{n-2} \binom{n-1}{j} x^j \\ &= x^n + x(S_{n-1} - x^{n-1}) = xS_{n-1}, \end{aligned}$$

that is,

$$S_n = (1+x)S_{n-1}.$$

(Notice the changing of the summation index with $k - 1 = j$; when k runs from 1 to $n-1$, j runs from 0 to $n-2$.) This leads to

$$\sum_{k=0}^{n} \binom{n}{k} x^k = S_n = \prod_{k=1}^{n} \frac{S_k}{S_{k-1}} = \prod_{k=1}^{n} (1+x) = (1+x)^n$$

(see below how to telescope a product; do not forget $S_0 = 1$). Or, if we want to avoid the situation when some S_k is zero, we just use induction based on the recurrence formula that we found. Induction can also be used if we add all equalities $S_k - S_{k-1} = xS_{k-1}$ for $k = 1, 2, \ldots, n$ in order to get

$$S_n - 1 = \sum_{k=1}^{n} (S_k - S_{k-1}) = x \sum_{k=1}^{n} S_{k-1},$$

hence another recurrence relation for the sums S_n:

$$S_n = 1 + x(S_0 + S_1 + \cdots + S_{n-1}),\ n \geq 1.$$

Note that if we replace $x = \frac{b}{a}$ in the formula $S_n = (1+x)^n$ and then multiply by a^n we find the binomial formula (or theorem) for the expansion of the binomial $a + b$ raised to the nth power:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

which clearly holds for $a = 0$, too, although $a = 0$ is not allowed when considering $x = \frac{b}{a}$. The appearance of the numbers $\binom{n}{k}$ in the binomial formula explains why they are called binomial coefficients.

Again, we used the telescoping method (for a product, or for a sum) in a way that seems not to be very obvious. That is why we tried to illustrate a few more methods for evaluating sums, as induction and the use of simple algebraic rules. We will see other sums (some more general than some of those presented above) and other methods in the following chapters of the book.

Before we go on, we give a few more examples; we have

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1} = \frac{n}{n+1},$$

and

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} &= \frac{1}{2} \sum_{k=1}^{n} \left(\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)} \right) \\ &= \frac{1}{2} \left(\frac{1}{2} - \frac{1}{(n+1)(n+2)} \right) \\ &= \frac{n(n+3)}{4(n+1)(n+2)}. \end{aligned}$$

And here is our first example of infinite sum:

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k(k+1)} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

Please verify that

$$\sum_{k=m}^{n} \frac{1}{k(k+1)} = \frac{1}{m} - \frac{1}{n+1} \quad (m \leq n),$$

$$\sum_{k=m}^{\infty} \frac{1}{k(k+1)} = \frac{1}{m},$$

and that

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)} = \frac{1}{4}.$$

What is the value of

$$\sum_{k=m}^{\infty} \frac{1}{k(k+1)(k+2)}?$$

Notice also that we used a slightly different (but not *essentially* different) telescoping formula, namely

$$\sum_{k=1}^{n} (b_k - b_{k+1}) = b_1 - b_{n+1}.$$

There are, of course, many possibilities for telescoping. For example, check that

$$\sum_{k=1}^{n} (b_k - b_{k+2}) = b_1 + b_2 - b_{n+1} - b_{n+2}.$$

Finally in this introduction we will see a few products that telescope.

For telescoping a product $\prod_{k=1}^{n} a_k$ we would like to have $a_k = b_{k+1}/b_k$ for every

$k = 1, 2, \ldots, n$, with nonzero numbers $b_1, b_2, \ldots, b_n$, then use the formula that we have already seen

$$\prod_{k=1}^{n} a_k = \prod_{k=1}^{n} \frac{b_{k+1}}{b_k} = \frac{b_{n+1}}{b_1}.$$

Of course, in some situations, we can also use

$$\prod_{k=1}^{n} \frac{b_k}{b_{k+1}} = \frac{b_1}{b_{n+1}},$$

or other similar formulae.
We have, for example,

$$\prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) = \prod_{k=1}^{n} \frac{k+1}{k} = n+1,$$

where $a_k = 1 + 1/k$ and $b_k = k$ for all k. Also

$$\prod_{k=2}^{n} \left(1 - \frac{1}{k}\right) = \prod_{k=2}^{n} \frac{k-1}{k} = \frac{1}{n}$$

(if we allow $k = 1$, the product is trivially 0, because its first factor is 0), and, consequently,

$$\prod_{k=2}^{\infty} \left(1 - \frac{1}{k}\right) = \lim_{n\to\infty} \prod_{k=2}^{n} \left(1 - \frac{1}{k}\right) = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Further we can find

$$\begin{aligned} \prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right) &= \prod_{k=2}^{n} \left(\left(1 - \frac{1}{k}\right)\left(1 + \frac{1}{k}\right)\right) \\ &= \prod_{k=2}^{n} \left(1 - \frac{1}{k}\right) \prod_{k=2}^{n} \left(1 + \frac{1}{k}\right) \\ &= \frac{1}{n} \cdot \frac{n+1}{2} = \frac{n+1}{2n}, \end{aligned}$$

yielding

$$\prod_{k=2}^{\infty}\left(1-\frac{1}{k^2}\right)=\frac{1}{2}.$$

We invite the reader who is less familiar to the material to do all the computations that we omit as being "evident", or "obvious", or "clear" and so on (and, in general, to do all the computations). Also, we strongly advise the reader to repeat things (which we will also do, from time to time).
A very interesting example for the beginners is

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{a^{2^k}}\right),$$

where a is a real (or even complex) number of absolute value greater than 1 (therefore a is nonzero).
Of course, we look first at the finite case

$$\prod_{k=1}^{n}\left(1+\frac{1}{a^{2^k}}\right).$$

Again, a simple formula is of real help, namely $a^2-b^2=(a-b)(a+b)$ but we use it in a particular case, and in a slightly different form, more precisely we use

$$1+a=\frac{1-a^2}{1-a},\quad a\neq 1.$$

Thus we have

$$\prod_{k=1}^{n}\left(1+\frac{1}{a^{2^k}}\right)=\prod_{k=1}^{n}\frac{1-\dfrac{1}{a^{2^{k+1}}}}{1-\dfrac{1}{a^{2^k}}}=\frac{1-\dfrac{1}{a^{2^{n+1}}}}{1-\dfrac{1}{a^{2^1}}}.$$

Now we see why the condition $|a|>1$ is given: it ensures $|1/a|<1$, hence

$$\lim_{n\to\infty}\left(\frac{1}{a}\right)^{x_n}=0$$

whenever $(x_n)_{n\geq 1}$ is a sequence of real numbers with limit ∞. In particular

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{a^{2^k}}\right)=\lim_{n\to\infty}\frac{1-\dfrac{1}{a^{2^{n+1}}}}{1-\dfrac{1}{a^{2^1}}}=\frac{1}{1-\dfrac{1}{a^2}}=\frac{a^2}{a^2-1}.$$

We end this introductory part with a question related to the problem from which we started. For

$$f(x)=x-\left\lfloor\frac{x}{2}\right\rfloor$$

and n_k defined by $n_1 = n$ (an arbitrary positive integer) and $n_k = f(n_{k-1})$ for $k \geq 2$ we have seen that

$$\left\lfloor\frac{n_1}{2}\right\rfloor+\left\lfloor\frac{n_2}{2}\right\rfloor+\cdots=n-1.$$

Can we compute this sum by telescoping? (The answer is yes. So, find how.)

Chapter 1

Telescoping Sums and Products in Algebra

One of the most useful techniques for computing sums and products is the use of the identity

$$(a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{n+1} - a_n) = a_{n+1} - a_1,$$

valid for any complex numbers $a_1, \ldots, a_n$. Therefore, if we need to compute a sum $\sum_{k=1}^{n} b_k$, we might try to find numbers $a_1, \ldots, a_{n+1}$ such that

$$b_1 = a_2 - a_1,\ b_2 = a_3 - a_2, \ldots, b_n = a_{n+1} - a_n$$

and then apply the previous identity to deduce that the sum we are looking for is simply $a_{n+1} - a_1$. If we can do that, we say that the sum is telescopic, or that it telescopes (as shown in the introduction). Finding the numbers $a_1, \ldots, a_n$ is the hard part of the game and lots of practice is certainly helpful! Note that it is always possible to find $a_1, \ldots, a_{n+1}$ as above, namely choose $a_1 = 0$, then $a_2 = b_1$, $a_3 = b_1 + b_2, \ldots$, $a_{n+1} = b_1 + \cdots + b_n$. Of course, this is not very satisfying for our needs...

Let us start with a few classical examples. You certainly know the following